Path (or cycle)-trees with Graph Equations involving Line and Split Graphs

H. P. Patil* and V. Raja†

Abstract

$H$-trees generalize the existing notions of trees, higher dimensional trees and $k$-ctrees. The characterizations and properties of both $P_k$-trees for $k \geq 4$ and $C_n$-trees for $n \geq 5$ and their hamiltonian property, dominations, planarity, chromatic and $b$-chromatic numbers are established. The conditions under which $P_k$-trees for $k \geq 3$ (resp. $C_n$-trees for $n \geq 4$), are the line graphs are determined. The relationship between path-trees and split graphs are developed.

Keywords: Cycle, Path, Tree, Connected graph, Coloring, Line graph, Split graph

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1. Introduction

We follow Harary[5] for all terminologies related to graphs. Given a graph $G$, $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively and $\bar{G}$ denotes the complement of $G$. $P_n$ and $C_n$ denote a path of $n$ vertices and cycle of $n$ vertices, respectively. For any connected graph $G$, $nG$ denotes the graph with $n$ components, each being isomorphic to $G$. For any two disjoint graphs $G$ and $H$, $G+H$ denotes the join of $G$ and $H$. [5] A tree is a connected graph without cycles. A star is a tree $K_{1,n}$ for $n \geq 1$. A graph $G$ is $n$-connected if the removal of any $m$ vertices for $0 \leq m < n$, from $G$ results in neither a disconnected graph nor a trivial graph. A graph $G$ is triangulated if every cycle of length strictly greater than 3 possesses a chord; that is, an edge joining two nonconsecutive vertices of the cycle. Equivalently, $G$ does not
contain an induced subgraph isomorphic to $C_n$ for $n > 3$. A graph $G$ is $n$-degenerate for $n \geq 0$ if every induced subgraph of $G$ has a vertex of degree at most $n$.

2. Structure of $H$-trees

Notice that trees are equivalently defined by the following recursive construction rule:

**Step 1.** A single vertex $K_1$ is a tree.

**Step 2.** Any tree of order $n \geq 2$, can be constructed from a tree $Q$ of order $n - 1$ by inserting an $n^{th}$-vertex and joining it to any vertex of $Q$.

In [10], the above tree-construction procedure is extended by allowing the base to be any graph. It is natural that a connected graph, which is not a tree possesses a structure that reflects like a tree and its recursive growth starts from any graph. In other words, for any given graph $H$, there is associated another graph, we call $H$-tree that is constructed as follows.

**Definition 2.1.** Let $H$ be any graph of order $k$. An $H$-tree, denoted by $G(H)$, is a graph that can be obtained by the following recursive construction rule:

**Step 1.** $H$ is the smallest $H$-tree.

**Step 2.** To an $H$-tree $G(H)$ of order $n \geq k$, insert an $(n + 1)^{th}$-vertex and join it to any set of $k$ distinct vertices: $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ of $G(H)$, so that the induced subgraph $\langle v_{i_1}, v_{i_2}, \ldots, v_{i_k}\rangle$ is isomorphic to $H$.

For example, $K_{1,3}$-tree of order 8 is shown in Figure 1.

**Remark 2.2.** 1. The notion of $K_1$-trees is the usual concept of trees.
2. The notion of $K_2$-trees is equivalent to the notion of 2-trees, which is studied in [7]. Actually, they form a special subclass of planar graphs. In fact, the maximal outerplanar graphs are the only outerplanar $K_2$-trees.
3. The notion of $K_k$-trees is equivalent to the notion of $k$-trees[2, 7] and they form actually a family of $k$-connected, triangulated and $K_{k+2}$-free graphs of order $\geq k + 1$.
4. The notion of $K_k$-trees is equivalent to the concept of $k$-ctrees[9] and they form a family of $k$-degenerate and triangle-free graphs of order $p \geq 2k$ and size $k(p - k)$.

The development in the class of $H$-trees is motivated by the notion of $k$-trees[2, 7] or $k$-ctrees[9] and their applications in the area of reliability of communication networks, have generated much interest from an algorithmic (or theoretical) point of view.
Definition 2.3. A graph $F$ is called a $H$-tree if there exists a graph $H$ such that $F$ is isomorphic to $G(H)$.

Equivalently, a $H$-tree $G(H)$ of order $\geq k + 1$, (where $|H| = k$) can be reduced to $H$ by sequentially removing the vertices of degree $k$ from $G(H)$.

For a vertex $v$ of a graph $G$, a neighbour of $v$ is a vertex adjacent to $v$ in $G$. The neighbourhood $N(v)$ of $v$ is the set of all neighbours of $v$.

The following result is a simple characterization of $H$-trees involving their hereditary subgraphs and is simply the restatement of Definition 2.1.

Proposition 2.4. Let $H$ be any graph of order $k$. Then $G$ is a $H$-tree of order $\geq k + 1$ if and only if $G$ contains a vertex $v$ of degree $k$ such that $N(v)$ induces $H$ in $G$ and $G - v$ is a $H$-tree.

An immediate consequence of the above result is the following corollary.

Corollary 2.5. For any graph $H$ of order $k$ and size $m$, let $G$ be a $H$-tree of order $p \geq k$. Then

1. $|E(G)| = m + k(p - k)$.
2. $G$ contains a subgraph isomorphic to $H + 2K_1$, provided $p \geq k + 2$.
3. If $H$ has $t$ triangles, then the number of triangles in $G$ is $t + m(p - k)$.
3. Properties and Characterizations

**Definition 3.1.** A graph $F$ is called a $P_k$-tree (or path-tree) if there exists a path $P_k$ of order $k$ such that $F$ is isomorphic to $G(P_k)$.

We define similarly, a $C_k$-tree (or cycle-tree). Generally speaking, every $P_k$ (resp. $C_k$)-tree of order $\geq k + 1$, can be reduced to $P_k$ (resp. $C_k$) by sequentially removing the vertices of degree $k$ from $P_k$ (resp. $C_k$)-tree.

In [10], the following general open-problem is proposed for further research.

**Open Problem 1.** Characterize the class of star-trees $G(K_{1,n})$ for $n \geq 2$.

We now characterize path-trees $G(P_k)$ for $k \geq 4$.

**Theorem 3.2.** A graph $G$ of order $p \geq k + 1$, is a $P_k$-tree if and only if $G$ is isomorphic to $P_k + (p-k)K_1$.

**Proof.** Suppose that $G$ is isomorphic to $P_k + (p-k)K_1$. Then $G$ contains the vertices $v_1, v_2, \ldots, v_{p-k}$, each of degree $k$ such that $N(v_i)$ induces $P_k$ in $G$ for $1 \leq i \leq p-k$. By repeated removal of each vertex $v_i$ from $G$ reduces to $P_k$. Hence, $G$ is a $P_k$-tree.

We prove the converse by induction on $p$.

If $p = k + 1$, then by the recursive definition, a $P_k$-tree $G$ of order $k + 1$, is isomorphic to $P_k + K_1$, which is obviously true.

Assume that the result is true for any positive integer $m < p$. Next, we consider a $P_k$-tree of order $p$. By Proposition 2.4 with $H = P_k$, $G$ contains a vertex $v$ of degree $k$ such that $N(v)$ induces $P_k$ in $G$ and $G - v$ is again a $P_k$-tree of order $p - 1$. By induction hypothesis, $G - v$ is isomorphic to $P_k + (p-k-1)K_1$. Consequently, $G - v$ is the join of two disjoint graphs: $P_k$ and $I = (p-k-1)K_1$.

Suppose that $v$ is adjacent to each vertex of $P_k$ in $G$. Then the result follows immediately. Otherwise, $v$ is adjacent to at least one vertex of $I$ in $G$. Moreover, $deg(v) = k$ in $G$. There exist two disjoint nonempty sets: $A$ and $B$ such that $A \subseteq P_k$; $B \subseteq I$ with $A \cup B = N(v)$ and $|A| + |B| = k$. (Figure 2) We discuss four cases, depending on the cardinalities of $A$ and $B$:

**Case 1.** $|A| = k - 1$ and $|B| = 1$. Since $k \geq 4$, $\langle A \rangle$ contains at least one edge, say $e = xy$. Then for any vertex $u$ of $B$, we have a triangle $uxyu$ in $N(v)$, which is not possible.

**Case 2.** $|A| = k - 2$ and $|B| = 2$. Immediately, we have $|A| \geq 2$ (because $k \geq 4$ ).

There are two possibilities for discussion.

**2.1.** Suppose that $A$ is independent. Certainly, there are two non-adjacent vertices $x$ and $y$ in $A$. Let us consider $B = \{a, b\}$. Immediately,
\[ \langle x, y, a, b \rangle \] is isomorphic to \( C_4 \) and it appears in \( \langle N(v) \rangle \). This is impossible.

2.2. Suppose that \( A \) is non-independent. Then \( \langle A \rangle \) contains at least one edge. In this situation, Case 1 repeats.

Case 3. \( |A| = 1 \) and \( |B| = k - 1 \). It is easy to see that \( \langle N(v) \rangle \) is a star \( K_1 + \overline{K}_{k-1} \) and this is not possible.

Case 4. \( |A| \geq 2 \) and \( |B| \geq 3 \).

We discuss two possibilities, depending on \( A \):

4.1. Suppose that \( A \) is non-independent. Then Case 1 repeats.

4.2. Suppose that \( A \) is independent. Then Case 2 repeats.

In each of the above cases, we see that \( \langle N(v) \rangle \) is not isomorphic to \( P_k \). This is a contradiction.

In [7], it is shown that the notion of \( C_3 \)-trees are equivalent to the family of 3-trees and it is also proved that this class of graphs are equivalent to the family of 3-connected, triangulated and \( K_5 \)-free graphs of order \( \geq 4 \). Further, it is noticed that the graphs in the class of \( C_4 \)-trees have highly irregular structure. In fact, it is hard to find a characterization of \( C_4 \)-trees. We first propose the following problem for further research.

**Open Problem 2.** Characterize the class of \( C_4 \)-trees.

The following theorem is a characterization of \( C_k \)-trees for \( k \geq 5 \) and its proof is quite similar to that of Theorem 3.2, with the replacement of \( P_k \) by \( C_k \).

**Theorem 3.3.** A graph \( G \) of order \( p \geq k + 1 \), is a \( C_k \)-tree if and only if \( G \) is isomorphic to \( C_k + (p-k)K_1 \).
The immediate consequence of theorems 3.2 and 3.3 is the following corollary.

**Corollary 3.4.**

1. \( \chi(G\langle P_k \rangle) = 3 \) for \( k \geq 4 \).
2. \( \chi(G\langle C_k \rangle) = \begin{cases} 3 & \text{if } k \geq 6 \text{ and is even.} \\ 4 & \text{if } k \geq 5 \text{ and is odd.} \end{cases} \)

**Proposition 3.5.** Let \( G\langle H \rangle \) be a \( H \)-tree of order \( p \geq k + 1 \), where \( H \) is either \( P_k \), \( k \geq 4 \) or \( H \) is \( C_k \), \( k \geq 5 \).

1. \( G\langle H \rangle \) is hamiltonian if and only if \( p \geq 2k \).
2. \( G\langle H \rangle \) is planar if and only if \( p \leq k + 2 \).

**Proof.** By theorems 3.2. and 3.3, \( G\langle H \rangle \) is isomorphic to \( H + (p - k)K_1 \).

1. Assume that \( G\langle H \rangle \) is hamiltonian and on contrary, \( p \geq 2k + 1 \). Since \( |V(H)| = k \), we have \( |(p - k)K_1| = k + 1 \). Consider \( S = V(H) \). Then \( G - S \) is isomorphic to \( (p - k)K_1 \) and hence the number of components of \( (G - S) \geq k + 1 \). This implies that \( G\langle H \rangle \) is not hamiltonian. So, \( p \leq 2k \).

To prove the converse, it is sufficient to obtain a Hamilton-cycle in \( G\langle H \rangle \), where \( G\langle H \rangle \) is isomorphic to \( H + tK_1 \) for \( 1 \leq t \leq k \). Let \( V(H) = \{u_1, u_2, \ldots, u_k\} \) and \( V(tK_1) = \{v_1, v_2, \ldots, v_t\} \). Since \( k \geq t \), we have \( (k - t) = m \geq 0 \). Immediately, a Hamilton cycle: \( u_1, u_2, \ldots, u_{m+1}, v_1, u_{m+2}, v_2, u_{m+3}, \ldots, v_{t-1}, u_k, v_t, u_1 \) appears in \( G\langle H \rangle \) (Figure 3). Hence, \( H \)-tree is hamiltonian.

2. Assume that \( G\langle H \rangle \) is planar and on contrary, \( p \geq k + 3 \). Immediately, we observe that \( (H + 3K_1) \subseteq G\langle H \rangle \). Since \( K_{3,3} \) appears as an induced subgraph in \( (H + 3K_1) \), it follows that \( K_{3,3} \) appears as a forbidden subgraph in \( G\langle H \rangle \) and hence by Kuratowski theorem, \( G\langle H \rangle \) is not planar. This is a contradiction to our assumption. Hence, \( p \leq k + 2 \).

It is easy to prove the converse. \( \square \)
4. Dominations and $b$-coloring

For any graph $G$, $\gamma(G)$ denotes the domination number of $G$. A Roman domination function (in short, RDF) on $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of this function is $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on $G$ is the Roman domination number of $G$ and is denoted by $\gamma_R(G)$.[3] The following result gives both the domination and Roman domination numbers of path-trees and cycle-trees and its proof is obvious.

**Proposition 4.1.** Let $G(H)$ be a $H$-tree of order $p \geq k + 1$, where $H$ is either $P_k$ for $k \geq 4$ or $C_k$ for $k \geq 5$. Then

1. $\gamma(G(H)) = \begin{cases} 1 & \text{if } p = k + 1. \\ 2 & \text{otherwise}. \end{cases}$

2. $\gamma_R(G(H)) = \begin{cases} 2 & \text{if } p = k + 1. \\ 3 & \text{if } p = k + 2. \\ 4 & \text{otherwise}. \end{cases}$

The $b$-chromatic number $b(G)$ of a graph $G$ is the largest integer $k$ such that $G$ admits a proper $k$-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring of $G$ is a $b$-coloring of $G$[6] It is shown in [6] that for any path $P_k$ and a cycle $C_k$ for $k \geq 5$, $b(P_k) = b(C_k) = 3$.

Next, we determine the $b$-chromatic number of the path-trees and cycle-trees. For this, we establish the following lemma.

**Lemma 4.2.** In any $b$-coloring of a graph $H + (p - k)K_1$, where $H$ is any graph of order $k$ and $p \geq k + 1$, all the vertices of $(p - k)K_1$ receive the same color.

**Proof.** Let $u_1, u_2, \ldots, u_k$, be the vertices of $H$ and let $v_1, v_2, \ldots, v_{p-k}$ be the vertices of $I$, where $I = (p - k)K_1$. If $p = k + 1$, then $|I| = 1$. The result obvious.

If $p \geq k + 2$, then $|I| \geq 2$. If possible, then assume that in some $b$-coloring of $H + I$, the vertices of $I$ receive $q \geq 2$ different colors, say $c_1, c_2, \ldots, c_q$. Since $I$ is independent and each vertex of $I$ is adjacent to all vertices of $H$, it follows that there is no color dominating vertex corresponding to the colors $c_i \ (1 \leq i \leq q)$ in $H + I$. This is not possible in any $b$-coloring of $H + I$, because each color class has at least one color dominating vertex. \qed

**Theorem 4.3.** Let $G(H)$ be a $H$-tree of order $p \geq k + 1$, where $H$ is either a path $P_k$ for $k \geq 4$ or a cycle $C_k$ for $k \geq 5$. Then
1. \( b(G(P_k)) = \begin{cases} 3 & \text{if } k = 4, \\ 4 & \text{otherwise}. \end{cases} \)

2. \( b(G(C_k)) = 4. \)

**Proof.** By theorems 3.2, and 3.3, we have \( G(H) \) is isomorphic to \( H + I \), where \( I = (p - k)K_1 \). We discuss two cases depending on \( k \) in (1):

**Case 1.** Assume that \( k = 4 \). Since \( b(P_k) = 2 \) and from Lemma 4.2, all the vertices of \( I \) receive a single color, it follows that \( b(G(P_k)) \leq 3 \). To achieve the lower bound, color \( P_k \) properly by using the colors 1 and 2 and next, assign the color 3 to each vertex of \( I \). Thus, we have \( b(G(P_k)) = 3 \).

**Case 2.** Assume that \( k \geq 5 \). Since it is shown in [6] that \( b(P_k) = 3 \) and all the vertices of \( I \) receive a single color, it follows that \( b(G(P_k)) \leq 4 \). To achieve the lower bound, color \( P_k \) properly by all three colors 1, 2 and 3 and next, assign color 4 to each vertex of \( I \). Thus, we have \( b(G(P_k)) = 4 \).

For (2), since \( b(C_k) = 3 \) and all the vertices of \( I \) receive a single color, it follows that \( b(G(C_k)) \leq 4 \). To achieve the lower bound, color \( C_k \) properly by using all three colors 1, 2, 3, and next, assign the color 4 to each vertex of \( I \). Thus, we have \( b(G(C_k)) = 4 \).

**5. Line graphs and path (or cycle)-trees**

In this section, we determine all the graphs, whose line graphs are either \( P_k \)-trees or \( C_k \)-trees for \( k \geq 3 \). We begin with the definition of line graph. The line graph \( L(G) \) of a graph \( G \), is the graph whose vertex set is the edge set of \( G \) and in which two vertices are adjacent, if the corresponding edges are adjacent in \( G \).[5] Beineke [5, p.75] has shown that a graph is a line graph if and only if it has none of nine specified graphs as induced subgraphs, including \( K_1; 3 \), \( (K_1 + 2K_2) + 2K_1 \) and \( (C_5 + K_1) \). The problem of obtaining all the graphs, whose line graphs are \( P_k \)-trees for \( 1 \leq k \leq 2 \), is already done in [8, 9] and therefore, we solve the problem for \( k \geq 3 \).

**Proposition 5.1.** A \( P_k \)-tree of order \( p \geq k + 1 \); \( k \geq 3 \), is the line graph of a graph \( G \) if and only if both the following conditions hold:

1. \( k = 3 \); \( G \) is either \( (K_2 + 2K_1) \) or a triangle with exactly one end-edge at some vertex.
2. \( k = 4 \); \( G \) is a triangle with exactly two end-edges, one at some vertex.

**Proof.** We first show that \( G \) is connected. If not, then \( L(G) \) is disconnected and by Definition 2.1 with \( H = P_k, L(G) \) is not a \( P_k \)-tree. This is a contradiction. Since \( L(G) \) is a \( P_k \)-tree of order \( p \geq k + 1 \) and \( k \geq 3 \), by Theorem 3.2, \( P_k \)-tree \( T \) is isomorphic to \( P_k + (p - k)K_1 \). Suppose \( k \geq 5 \). Then \( T \) contains a subgraph \( F \) isomorphic to \( P_5 + K_1 \). Since
$F \subseteq T$ and $T$ is $L(G)$, immediately a forbidden subgraph isomorphic to $K_{1,3}$ appears in $L(G)$. This is impossible and it shows that $k \leq 4$. Thus, either $k = 3$ or $k = 4$.

**Case 1.** Assume that $k = 3$. Further, we observe that $p \leq 5$; since otherwise, $P_3 + 3K_1$ appears in $T$ and $L(G)$ contains a forbidden subgraph $K_{1,3}$.

We discuss two possibilities depending on $p$.

1.1. If $p = 4$, then $L(G) = P_3 + K_1$ and hence $G$ is isomorphic to triangle with exactly one end-edge at some vertex.

1.2. If $p = 5$, then $L(G) = P_3 + 2K_1$ and therefore, $G$ is isomorphic to $K_2 + 2K_1$.

**Case 2.** Assume that $k = 4$. Moreover, we observe that $p = 5$; since otherwise, $P_4 + 2K_1$ appears in $T$ and $L(G)$ contains a forbidden subgraph isomorphic to $(K_1 \cup K_2) + 2K_1$. Since $k = 4$ and $p = 5$, it follows that $L(G) = P_4 + K_1$ and hence $G$ is isomorphic to a triangle with exactly two end-edges, one at some vertex.

It is easy to prove the converse. \qed

Finally, we determine all the graphs whose line graphs are $C_k$-trees for $k \geq 3$. However for $k = 3$, this problem is solved in [8] and now we solve this problem, for $k \geq 4$.

**Proposition 5.2.** There are only two graphs whose line graphs are $C_k$-trees for $k \geq 4$. These graphs are $K_2 + 2K_1$ and $K_4$.

**Proof.** Suppose that $L(G)$ is a $C_k$-tree of order $p \geq k+1$ ; $k \geq 4$. Clearly, $G$ is connected. Assume that $k \geq 5$. Then $p \geq 6$ and immediately, $L(G)$ contains a subgraph $F$ isomorphic to $C_k + K_1$. There are two possibilities, depending on $k$ :

1. If $k = 5$, then $F = C_5 + K_1$ is a forbidden subgraph of $L(G)$.

2. If $k \geq 6$, then $F$ contains a forbidden subgraph isomorphic to $K_{1,3}$.

In either case, we arrive at a contradiction. Hence, $k = 4$. Furthermore, we observe that $p \leq 6$ ; since otherwise, $L(G)$ contains a subgraph $F$ isomorphic to $C_4 + 3K_1$. It is easy to check that a forbidden subgraph isomorphic to $K_{1,3}$ appears in $F$ and hence in $L(G)$.

Next, we discuss two possibilities depending on $p$.

1. If $p = 5$, then $L(G) = C_4 + K_1$ and hence $G = K_2 + 2K_1$.

2. If $p = 6$, then $L(G) = C_4 + 2K_1$ and hence $G = K_4$. \qed

6. Relation between $P_k$-trees and split graphs

A nonempty subset $S$ of $V(G)$ is an independent set $I(G)$ in a graph $G$ if no two vertices of $S$ are adjacent in $G$. A nonempty subset $K$ of $V(G)$ is a complete set $K(G)$ in $G$ if every two vertices of $K$ are adjacent in $G$.

The concept of a split graph appears in [4]. A split graph is defined to be a graph $G$, whose vertex set $V(G)$ can be partitioned into a
complete set \( K \) and an independent set \( I \) such that \( G = (K \cup I \cup (K, I)) \), where \( (K, I) \) denotes a set of edges \( xy \) for \( x \in K \) and \( y \in I \). Notice that the partition \( V(G) = K \cup I \) of a split graph \( G \) will not be unique always. Let us denote a split graph \( G \) with its bipartition \( (K, I) \) by \( G(K, I) \). In [4, Theorem 6.3], it is proved that a graph \( G \) is a split graph if and only if \( G \) contains no induced subgraph isomorphic to \( 2K_2, C_4 \) or \( C_5 \).

Now, we obtain the conditions under which \( P_1 \)-trees to be the split graphs. We begin with the following definitions.

**Definition 6.1.** A double-star \( D(m,n) \) for \( m, n \geq 1 \) is a tree, obtained from a complete graph \( K_2 \), by joining \( m \) isolated vertices to one end of \( K_2 \) and \( n \) isolated vertices to the other end of \( K_2 \).

**Definition 6.2.** For any triangle \( K_3 \) with vertices \( a, b \) and \( c \), there are three special families of \( K_2 \)-trees as follows:

1. A \( m \)-graph for \( m \geq 1 \), denoted by \( T(m) \), is a \( K_2 \)-tree, obtained from \( K_3 \), by joining \( m \) isolated vertices to both \( a \) and \( b \) of \( K_3 \).

2. A \( (m,n) \)-graph for \( m,n \geq 1 \), denoted by \( T(m,n) \), is a \( K_2 \)-tree, obtained from \( T(m) \), by joining \( n \) isolated vertices to both \( b \) and \( c \) of \( K_3 \) in \( T(m) \).

3. A \( (m,n,k) \)-graph for \( m,n,k \geq 1 \), denoted by \( T(m,n,k) \), is a \( K_2 \)-tree, obtained from \( T(m,n) \), by joining \( k \) isolated vertices to both \( a \) and \( c \) of \( K_3 \) in \( T(m,n) \).

**Proposition 6.3.** A \( P_1 \)-tree of order \( p \geq k + 1 \), is a split graph if and only if the following statements hold:

1. \( k = 1 \). There are only two split graphs:
   a) \( G(K_1, \bar{K}_{p-1}) \) is a star \( K_1 + \bar{K}_{p-1} \).
   b) \( G(K_2, \bar{K}_{p-2}) \) is a double-star \( D(m,n) \), where \( (m+n+2) = p \); \( m, n \geq 1 \).

2. \( k = 2 \). There are only two split graphs:
   a) \( G(K_2, \bar{K}_{p-2}) \) is a \( K_2 \)-tree \( K_2 + \bar{K}_{p-2} \).
   b) \( G(K_3, \bar{K}_{p-3}) \) is one of the following three \( K_2 \)-trees: \( T(n_1) \) for \( n_1 + 3 = p \); \( T(n_1,n_2) \) for \( n_1 + n_2 + 3 = p \) and \( T(n_1,n_2,n_3) \) for \( n_1 + n_2 + n_3 + 3 = p \).

3. \( k = 3 \). Either \( G(K_2, \bar{K}_2) \) or \( G(K_3, K_1) \) is a \( P_3 \)-tree \( P_3 + K_1 \).

4. \( k = 4 \). \( G(K_3, \bar{K}_2) \) is a \( P_4 \)-tree \( P_4 + K_1 \).

**Proof.** Suppose that a \( P_1 \)-tree of order \( p \geq k + 1 \), is a split graph of the form \( G(K, I) \). Immediately, \( k \leq 4 \); since otherwise, \( 2K_2 \) appears as a forbidden subgraph in \( P_k \).

We discuss three cases, depending on \( k \).

**Case 1.** Assume that \( k = 1 \). Then \( P_k \) is \( K_1 \). Clearly, a \( P_k \)-tree \( T \) is a nontrivial tree. In this case, the star \( K_1 + \bar{K}_{p-1} \) and double-stars \( D(m,n) \) with \( (m+n = p-2) ; \ m, n \geq 1 \), are the only split graphs of the
form: \(G(K_1, \bar{K}_{p-1})\) and \(G(K_2, \bar{K}_{p-2})\), respectively; since otherwise, \(2K_2\) appears immediately as a forbidden subgraph in \(T\).

**Case 2.** Assume that \(k = 2\). Then \(P_k\) is \(K_2\). Clearly, the notion of \(K_2\)-tree is equivalent to the notion of 2-tree.[7] By (3) of Remark 2.2 (with \(k = 2\)), a \(K_2\)-tree \(T\) is 2-connected, triangulated and \(K_4\)-free. Consequently, the complete sets \(K\) in \(T\) are the only \(K_2\) and \(K_3\).

Next, there are two possibilities to discuss on \(K\).

2.1. If \(K = K_2\), then \(T\) is isomorphic to \(K_2 + \bar{K}_{p-2}\), is the split graph of the type:

\[G(K_2; \bar{K}_{p-2})\]

2.2. If \(K = K_3\), then one of the following types of \(K_2\)-trees:

\[T(n_1), \text{with } n_1 + 3 = p; T(n_1, n_2) \text{ with } (n_1 + n_2 + 3) = p \text{ and } T(n_1, n_2, n_3) \text{ with } (n_1 + n_2 + n_3 + 3) = p,\]

is a split graph of the form: \(G(K_3, \bar{K}_{p-3})\).

**Case 3.** Assume \(k\) such that \((3 \leq k \leq 4)\). Since \(k \geq 3\), \(P_k\) contains \(P_3\) as an induced subgraph. By (2) of Corollary 2.5, a \(P_k\)-tree of order \(p \geq k + 2\), contains a subgraph isomorphic to \(P_k + \bar{K}_2\). Immediately, a forbidden subgraph \(C_4\) appears in \(P_3 + \bar{K}_2\) and hence, in \(P_k + \bar{K}_2\). This is a contradiction and hence proves that \(p = k + 1\). Now, we discuss two possibilities.

3.1. \(k = 3\). Then both \(K_3\) and \(K_4\) are the complete sets in a \(P_3\)-tree of order 4. This shows that \(P_3\)-tree \(P_3 + K_1\) is a split graph either of the type: \(G(K_2, \bar{K}_2)\) or \(G(K_3, K_1)\).

3.2. \(k = 4\). Then \(K_3\) is the only complete set in a \(P_4\)-tree of order 5. This shows that \(P_4\)-tree \(P_4 + K_1\) is a split graph of the type \(G(K_3, \bar{K}_2)\).

It is easy to prove the converse. \(\square\)

**Open Problem 3.** Determine the conditions under which the \(C_k\)-trees for \(k \geq 3\), are the split graphs.

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