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Propagation Characteristics of Acoustic Wave in Non Isothermal Earth's Atmosphere

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Abstract

Acoustic waves are those waves which travel with the speed of sound through a medium. H. Lamb (1909, 1910) had derived a cutoff frequency for stratified and isothermal medium for the propagation of acoustic waves. In order to find the cutoff frequency many methods were introduced after Lamb's work. In this paper, we have chosen the turning point frequency method following Musielak *et al.* (2006) and Routh *et al.* (2014) to determine cutoff frequencies for acoustic waves propagating in non-isothermal medium which can be applied to various atmospheres like solar atmosphere, stellar atmosphere, earth's atmosphere etc. Here, we have analytically derived the cutoff frequency and have analyzed and compared with the Lamb's cut-off frequency for earth's troposphere.

Keywords: Acoustic waves, Troposphere, Atmosphere

1. Introduction

Cutoff frequency was introduced, about a century ago, for linear acoustic waves propagating in isothermal and stratified medium [12]. The cutoff frequency, due to Lamb, is defined as the ratio of sound speed to twice density (pressure) scale height. We compute such frequency by solving the acoustic wave equation for displacement in the vertical direction. If the background medium is isothermal,that renders

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globality to the cutoff frequency implying the cutoff frequency remains unaffected throughout the medium. This cutoff will be designated as the global acoustic cutoff frequency, for the remainder of the manuscript. Non-isothermal atmosphere has also been considered where the temperature diminishes linearly with height. Lamb's original treatment of acoustic waves was expanded to vertical and horizontal dimensions by assuming existence of a uniform vertical temperature gradient. The manuscript details and exploits analytical results to attain theoretical insights into the range of frequencies related to the propagating acoustic waves in the model [13, 11]. Subsequent studies in propagation of acoustic waves revealed different aspects of the wave propagation. These studies are based on investigations using a variety of methods including global and local dispersion relations, the WKB approximation, analytical or numerical solutions to acoustic wave equations. If the background medium is homogeneous [36, 16], we may obtain the global dispersion relation for acoustic waves. The dispersion relation can be constructed even when the speed of sound is not adversely impacted by gradients of the physical parameters of the medium , as observed in Lamb's isothermal atmosphere [17, 21, 4, 15, 25, 26]. Several authors attempted to justify "local dispersion relation" approach, which requires shorter acoustic wavelength compared to characteristic scales. It is worth mentioning that the basic physical parameters in the medium vary over these wavelengths [36, 4, 5]. This is known as the WKB approximation used to investigate acoustic (and other) waves [5, 24, 28, 27] Analytical and numerical solutions to acoustic wave equations, applicable to different physical scenario, are used to ascertain conditions of the wave propagation [36, 17, 21, 4, 22, 3].

Lamb exhibited that the cutoff is the innate frequency of the atmosphere. This implies that atmospheric oscillations are triggered by acoustic waves which propagate through the medium, endowed with frequency identical to the natural frequency [12, 9]. The acoustic cutoff frequency plays an important role in Helioseismology and Asteroseismology. Helioseismology is the study of solar oscillations responsible for establishing the internal structure of the Sun [20, 23]. Asteroseismology deals with oscillations of different stars [2, 10, 7]. The cutoff has also been used to study free atmospheric oscillations of the Earth [23] and other planets [10] and acoustic oscillations of Jupiter [7]. The acoustic cutoff frequency is a global quantity as discussed and therefore cannot be computed formally for the entire atmosphere. This is because planetary and stellar atmospheres are not isothermal. Hence, classic approaches are adopted that evaluate the cutoff at each atmospheric height by leveraging the neighborhood values of the temperature [20, 7]. The approximation is rather crude, in the presence of steep temperature gradients in atmosphere.

This paper puts forward the concept of turning-point frequency method [8, 18, 19, 31, 33, 34] to compute the cutoff frequency in Earth's non -isothermal atmosphere (troposphere). We obtain two important results from the proposed approach:

- derived cutoff frequency exhibits natural locality i.e. value at a given atmospheric height determines the frequency characteristic of acoustic waves enabling propagation at this height.
- cutoff frequence attains a large value at the bottom of Troposphere and falls off rapidly with the height and attains an almost constant value.

Therefore, required cut-off frequency for earth's troposphere has been derived based on local cut-off frequency results. A visual comparison with Lamb's solutions is accomplished. These results are obtained with the help of ISA (International Standard Atmosphere) model [1, 6]. The remainder of the paper is organized as follows:

Section 2. covers derivation of acoustic waves from basic hydrodynamic equations.

Section 3. discusses theoretical foundation and proof of concept of local cut-off frequency.

Section 4. details analytical calculations of earth's troposphere.

2. Acoustic Wave Equations

Routh *et al.* [32] has been imitated to derive cutoff frequency of acoustic wave for non-isothermal atmosphere. Let us consider the 1-D atmospheric model with density gradients, temperature and pressure along the Z-axis. In terms of basic 1-D hydrodynamic equations [30], propagation of linear and adiabatic acoustic waves are described and given by,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho_0 u)}{\partial z} = 0 , \tag{1}$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} + \rho g = 0 , \tag{2}$$

$$\frac{\partial p}{\partial t} + u\frac{dp_0}{dz} - c_s^2 \left(\frac{\partial \rho}{\partial t} + u\frac{d\rho_0}{dz} \right) = 0 , \tag{3}$$

where u, p and ρ are the perturbed velocity, pressure and density respectively. Also, $\vec{g}$ is gravity, c_s is the speed of sound, ρ_0 and p_0 are the background gas density and pressure respectively. The background medium is assumed to be in hydrostatic equilibrium. This indicates, $\frac{dp_0}{dz} = -\rho_0 g$.

The speed of sound is formulated as $c_s = [\gamma p_0/\rho_0]^{1/2} = [\gamma R T_0/\mu]^{1/2}$, where γ is the ratio of specific heat, R is the universal gas constant, μ is the mean molecular weight and T_0 is the background temperature.

For stratified and non-isothermal medium, $T_0 = T_0(z)$, $c_s = c_s(z)$, with density H_ρ and pressure scale heights H_p being functions of z. Define, q_i, where $i = 1$, 2 and 3, with $q_1 = u$, $q_2 = p$ and $q_3 = \rho$, and combining the linearized and 1-D hydrodynamic equations yields following wave equations

$$\hat{L}_i\left[\frac{\partial^2}{\partial t^2} - c_s^2(z)\,\frac{\partial^2}{\partial z^2} + \frac{c_s^2(z)}{H_i(z)}\,\frac{\partial}{\partial z}\right]\hat{L}_i^{-1} q_i \;=\; 0\,, \tag{4}$$

The above equation describes the propagation of linear and adiabatic acoustic waves in a non-isothermal atmosphere.
Here

$$\hat{L}_1 = \hat{1}, \hat{L}_2 = \hat{1} - g\left(\frac{\partial}{\partial t}\right)^{-2}\frac{\partial}{\partial z} \qquad \text{and} \qquad \hat{L}_3 = \frac{\partial^2}{\partial z^2}\,. \tag{5}$$

Now, $H_1(z) = H_p(z)$ and $H_2(z) = H_3(z) = -H_\rho(z)$ with

$$H_p(z) = \frac{1}{p_0(z)}\frac{dp_0(z)}{dz} \;\; \text{and} \;\; H_\rho(z) = \frac{1}{\rho_0(z)}\frac{d\rho_0(z)}{dz}\,, \tag{6}$$

W.K.T $H_\rho(z) \neq H_p(z)$ and $\frac{1}{H_\rho} = \frac{1}{H_p} + \frac{H_p\prime}{H_p}$.
.

On transformation of $q_i = \hat{L}_i q_{1i}$, Eq. (4) becomes

$$\left[\frac{\partial^2}{\partial t^2} - c_s^2(z)\,\frac{\partial^2}{\partial z^2} + \frac{c_s^2(z)}{H_i(z)}\,\frac{\partial}{\partial z}\right] q_{1i} \;=\; 0\,, \tag{7}$$

3. Local acoustic cutoff frequency

Let us consider the transformation [1],

$$d\tau = \frac{dz}{c_s(z)}\,, \tag{8}$$

Following Routh *et al.* [32], we can write the critical frequencies as functions of z in the following way

$$\Omega^2_{cr,u}(z) = (\omega_{ac} + \omega_{as})^2 + 2\omega_{ac}\omega_{as} - c_s\omega'_{as}\,, \tag{9}$$

and

$$\Omega^2_{cr,p}(z) = (\omega_{ac} + \omega_{as})^2 - 2\omega_{ac}\omega_{as} + c_s\omega'_{as}\,. \tag{10}$$

where $\omega_{ac} = \frac{\gamma g}{2c_s} = \frac{c_s}{2H}$ is the original Lamb acoustic cutoff frequency and $\omega_{as} = \frac{1}{2}\frac{dc_s}{dz}$.

It is worth mentioning that, the isothermal atmosphere of both the critical frequencies can be reduce to Lamb's cutoff $\omega_a c$.

Using the Fourier transform in time and applying the oscillation to turning-point theorems [8, 18, 19, 31, 33, 34], the following turning-point frequencies are obtained

$$\Omega^2_{tp,u,p}(\tau) = \Omega^2_{cr,u,p}(\tau) + \frac{1}{4\tau^2} \tag{11}$$

The turning-point frequencies have two separate solutions, propagating and non-propagating (evanescent) waves. However, each wave variable is endowed with a turning-point frequency out of which only one may be the cutoff frequency. Therefore, we follow Musielak *et al.* [18] and Routh *et al.* [8, 18, 19, 31, 33, 34], and identify the largest turning-point frequency as the cutoff frequency. (i.e., to check whether $\Omega^2_{cr,p}(z) > \Omega^2_{cr,u}(z)$ or vice versa). By using similar conversion for the turning-point frequencies $\Omega^2_{tp,u}(\tau)$ and $\Omega^2_{tp,p}(\tau)$. we get,

$$\Omega^2_{tp,u,p}(z) = \Omega^2_{cr,u,p}(z) + \frac{1}{4}\left[\int^z \frac{d\tilde{z}}{c_s(\tilde{z})} + \tau_C\right]^{-2}, \tag{12}$$

and the cutoff frequency given by

$$\Omega_{cut,}(z) = \max[\Omega_{tp,u}(z), \Omega_{tp,p}(z)] . \tag{13}$$

So the condition for propagation of waves is $\omega > \Omega_{cut}$ and similarly for non-propagating waves is $\omega \leq \Omega_{cut}$.[8]

Here the cutoff frequency is a local quantity, which describes the relation between height with respect to the frequency of acoustic waves.

4. Cutoff Frequency for Troposphere

To derive the analytical cutoff frequency, consider the variation of temperature T with respect to the height [1, 6] and is given by the formula

$$T = T_c - Cz \tag{14}$$

where T_c is background temperature =288.15k, C=$\frac{6.5}{1000}$ in terms of meters for height. Thus, the expression of sound speed assumes the following form:

$$C_s = [\gamma R(T_c - Cz)/\mu]^{1/2} \tag{15}$$

The above expression can be simplified further in the following manner,

$$C_s = C_{so}[1 - az]^{1/2} \tag{16}$$

Where, $C_{so} = [\gamma RT_c/\mu]^{1/2}$ and a=$\frac{C}{T_c}$

Now,

$$\omega_{ac} = \frac{\gamma g}{2C_{so}[1 - az]^{1/2}} \tag{17}$$

Again,

$$\omega_{as} = -\frac{ac_{so}}{4[1 - az]^{1/2}} \tag{18}$$

On differentiation,

$$\omega'_{as} = -\frac{a^2 c_{so}}{8[1 - az]^{3/2}} \tag{19}$$

Substituting the above results in equations(14) and (15) ,

$$\Omega^2_{cr,u}(z) = \frac{\gamma^2 g^2}{4C^2_{so}[1 - az]} + \frac{3a^2 C^2_{so}}{16[1 - az]} - \frac{a\gamma g}{2[1 - az]} \tag{20}$$

$$\Omega^2_{cr,p}(z) = \frac{\gamma^2 g^2}{4C^2_{so}[1 - az]} - \frac{a^2 C^2_{so}}{16[1 - az]} \tag{21}$$

$$\tau = \int^z \frac{d\tilde{z}}{c_s(\tilde{z})} + \tau_C = \tau_C - \frac{2}{ac_{so}}[(1 - az)^{1/2} - 1] \tag{22}$$

We get $\Omega^2_{cr,p}(z) > \Omega^2_{cr,u}(z)$.

Hence the cutoff frequency of equation (17) takes the form,

$$\Omega^2_{tp,p}(z) = \frac{\gamma^2 g^2}{4C^2_{so}[1 - az]} - \frac{a^2 C^2_{so}}{16(1 - az)} + \frac{1}{4(\tau_C - \frac{2}{ac_{so}}[(1 - az)^{1/2} - 1])^2} \tag{23}$$

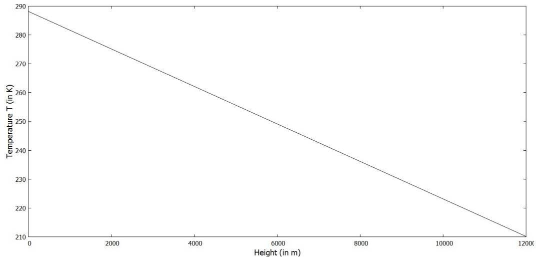

Figure 1. Temperature v/s height: Temperature ranges from 288.2 K to 210 K and Height ranges from 0-12 Km

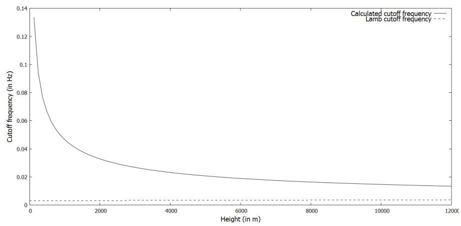

Figure 2. Comparison of Cutoff frequency and Lamb's cutoff v/s height in Troposphere:The calculated cutoff starts from a very large value at base of troposphere and drops rapidly to 0.138 Hz at height 115m following inverse law characteristic of plot and then attains an almost constant value of 0.0134 Hz at height 12 km. Lamb's frequency is almost constant (0.0032 Hz)

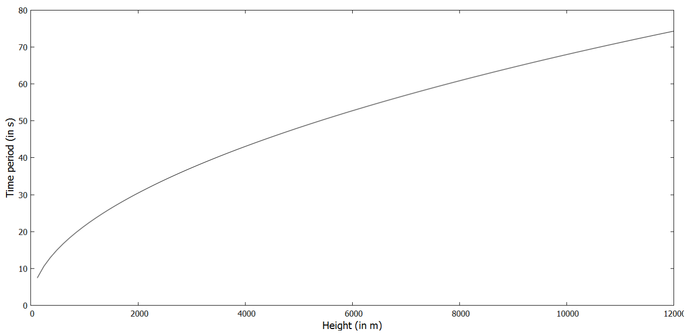

Figure 3. Time period v/s height: Time period ranges from a very low value (close to 0s) at base of troposphere to 74.3 s at 12 km height.

5. Results and Discussion

The fig (1) showing the plot of temperature v/s height describes the variation of temperature in isothermal medium of the troposphere.

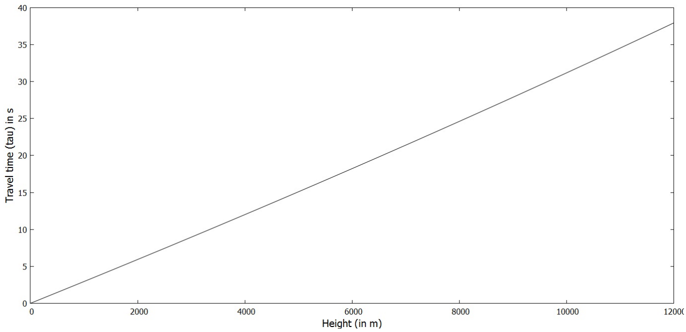

Figure 4. Travel time (τ) v/s height: Travel time increases linearly from 0 to 37.9 s at height 12 km.

This graph was obtained on the basis of analytical solution of the turning point frequency method (refer equation (4.19)). The temperature decreases linearly with height; temperature decreases linearly from 288.2 (k) to 210 (k) at height 12 km (troposphere). The fig (2) showing the plot of cut off frequency v/s height describes the variation of cutoff frequency in non-isothermal medium of the troposphere. This graph was obtained on the basis of analytical solution of the turning point frequency method [32]. The frequency decreases following inverse square law characteristic. The calculated cutoff starts from a very large value at base of troposphere and drops rapidly to 0.138 Hz at height 115m and then attains an almost constant value of 0.0134 Hz at height 12 km. Lamb's frequency is almost constant (0.0032 Hz) throughout the troposphere. The plots (3) and (4) show acoustic waves having period from few seconds at the base to 74.3 seconds at the top of the troposphere and the travel time to reach the top of troposphere is 38 seconds approximately.

6. Conclusion

We followed Routh *et al.* [32] to determine the cutoff frequency of acoustic waves in non-isothermal media. The novel technique makes use of integral transformations to cast the wave equations for both wave variables in their standard forms. Consequently, turning point frequencies are computed for each wave variable by using oscillation theorem. The cutoff frequency is the larger of the two turning point frequencies. We used the temperature variation of Earth's Troposphere to study the effects of the temperature gradients on the cutoff frequency. As the temperature increases linearly with height, the cutoff frequency is observed to be large at the bottom of the Troposphere and diminishes with height following inverse square law characteristic thereafter. We also compared the calculated cutoff to the Lamb cutoff frequency, treated as a height-dependent quantity. The comparison shows that they are significantly different from each other as Lamb's cutoff maintains an approximate constant value (0.0032 Hz) throughout the troposphere. The travel time for the acoustic wave linearly increases with height following the linear nature of temperature variation of Troposphere.

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