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Shuffle Operations on Euler Graphs

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Abstract

The shuffle operation on strings is a fundamental operation, well studied in the theory of formal languages. Shuffle on trajectories yields a flexible method to handle the shuffle operation on two strings. In this paper, the shuffle on trajectories is extended to the string representations of Euler graphs and interesting results are obtained. Some algebraic properties such as completeness, determinism and commutativity of the trajectories involved in this study are provided.

Keywords: Euler Graph, shuffle operation, literal shuffle, balanced literal shuffle

1. Preliminaries

Parallel composition of words and languages appears as a fundamental operation in parallel computation and in the theory of concurrency. Usually, this operation is modeled by the shuffle operation or restrictions of this operation, such as literal shuffle, balanced literal shuffle, insertion etc. A trajectory is a segment of a line in plane, starting in the origin of axes and continuing parallel with the Ox or Oy. The line can change its direction only in points of nonnegative integer coordinates. A trajectory defines how to switch from a word to another word during the shuffle operation. Shuffle on trajectories provides a method of great flexibility to

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handle the operation of parallel composition of processes: from the catenation to the usual shuffle of processes. It is natural to consider this operation on multi-dimensional structures or more complex objects like graphs and networks. In this section, we review the notions required for the development of our study [1, 2, 3].

Definition 1.1 A *linear graph* or *graph* $G = (V, E)$ consists of a set $V = \{v_1, v_2, \ldots\}$ called *vertices* and another set $E = \{e_1, e_2, \ldots\}$, whose elements are called *edges*, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i, v_j associated with edge e_k are called the *end vertices* of e_k. An edge having the same vertex as both its end vertices is called a *self-loop*. More than one edge associated with a given pair of vertices is referred as *parallel edges*. A graph that has neither self-loops nor parallel edges is called a *simple graph.*

Definition 1.2 A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. A vertex, however, may appear more than once. A walk is closed if it begins and ends at the same vertex. A walk that is not closed is called an open walk. An open walk in which no vertex appears more than once is called a *path*. A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a *circuit* or *cycle.*

Definition 1.3 A closed walk in a graph containing all the edges of the graph is called an *Euler line* and if such a walk exists in a graph, then the graph is called an *Euler graph.*

Definition 1.4 [1] A graph G is said to be in *Pseudo-Linear Form (PLF)* if the ordered vertices $\{v_1, v_2, \ldots, v_n\}$ are positioned as per the order, as if they lie along a line and the edges of the graph drawn accordingly.

Example 1.5 Let G be the graph in figure 1.

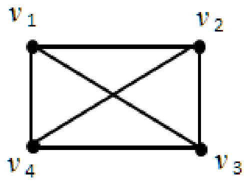


Figure 1

If the vertices of G are ordered as {v_1, v_2, v_3, v_4}, the corresponding PLF(G) is in figure 2.

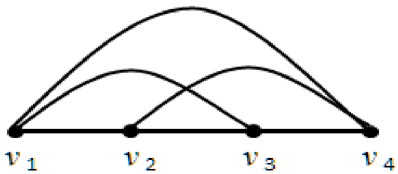


Figure 2

Note 1.1 A graph in PLF will look like a path graph with edges going above or below the linear path.

Notations

The set of nonnegative integers is denoted by N. Let Σ be a finite and nonempty set of symbols called the alphabet and Σ^* denotes the set of all words over Σ, including the empty word λ.

If $w \in \Sigma^*$, then $|w|$ denotes the length of w. Note that $|\lambda| = 0$. If $a \in \Sigma$ and $w \in \Sigma^*$ then $|w|_a$ denotes the number of occurrences of the symbol a in w. If $\Delta \subseteq \Sigma$ then $|w|_\Delta = \sum_{a \in \Delta} |w|_a$.

Definition 1.6 [1] The *shuffle* operation, denoted by Ш, is defined recursively by

$(au Ш bv) = a(u Ш bv) \cup b(au Ш v)$ and $(u Ш \lambda) = (\lambda Ш u) = \{u\}$, where $u, v \in \Sigma^*$ and $a, b \in \Sigma$.

Example 1.7 *ab*Ш*bc* = {*abbc*, *abcb*, *babc*, *bacb*, *bcab*}.

Definition 1.8 [1] The *literal shuffle*, denoted by $Ш_l$, is defined as

$$a_1 a_2 \ldots a_n \,Ш_l\, b_1 b_2 \ldots b_m = \begin{cases} a_1 b_1 a_2 b_2 \ldots a_n b_n b_{n+1} \ldots b_m & if \\ a_1 b_1 a_2 b_2 \ldots a_m b_m a_{m+1} \ldots a_n & if \end{cases}$$

$n \le m,$
$m < n,$ where $a_i, b_j \in \Sigma,$

and $(u \,Ш_l\, \lambda) = (\lambda \,Ш_l\, u) = \{u\}$, where $u \in \Sigma^*$.

Example 1.9 *aba* $Ш_l$ *bc* = {*abbca*}.

Definition 1.10 [1] The *balanced literal shuffle*, denoted by $Ш_{bl}$, is defined as

$$a_1 a_2 \ldots a_n \,Ш_{bl}\, b_1 b_2 \ldots b_m = \begin{cases} a_1 b_1 a_2 b_2 \ldots a_n b_n & if \;\; n = m, \\ \phi & if \;\; n \ne m, \end{cases}$$

where $a_i, b_j \in \Sigma.$

Definition 1.11 [1] The *insertion operation* denoted by ←, is defined as

$$u \leftarrow v = \{\alpha v \beta \,/\, \alpha\beta = u, \; where \; \alpha, \beta \in \Sigma^*\}.$$

Example 1.12 $ab \leftarrow bc = \{bcab, abcb, abbc\}$.

Definition 1.13 [1] The *balanced insertion*, denoted by $\leftarrow_b$, is defined as $\alpha\beta \leftarrow_b uv = \{u\alpha\beta v \,/\, |\alpha| = |\beta| \; and \; |u| = |v|, \; where \; \alpha, \beta, u, v \in \Sigma^*\}$. In a balanced insertion $u \leftarrow_b v$, both *u* and *v* have to be of even length.

Example 1.14 $ab \leftarrow_b bc = \{abcb\}$.

Definition 1.15 [1] Consider the alphabet $V = \{r, u\}$. It can be said that *r* and *u* are *versors* in the plane, *r* stands for the *right* direction, and *u* stands for the *up* direction. A *trajectory* is an element $t \in V^*$. Sets *T* of trajectories, $T \subseteq V^*$ are also considered.

Let Σ be an alphabet and let t be a trajectory, $t = t_1 t_2 \ldots t_n$, where $t_i \in V$, $1 \le i \le n$. Let α, β be two words over Σ, $\alpha = a_1 a_2 \ldots a_p$, $\beta = b_1 b_2 \ldots b_q$, where $a_i, b_j \in \Sigma$, $1 \le i \le p$ and $1 \le j \le q$.

Definition 1.16 [1] The shuffle of α with β on the trajectory *t*, denoted by $\alpha ш_t \beta$, is defined as follows: if $|\alpha| \ne |t|_r$ or $|\beta| \ne |t|_u$, then $\alpha ш_t \beta = \phi$, else

$\alpha ш_t \beta = c_1 c_2 \ldots c_{p+q}$, where, if $|t_1 t_2 \ldots t_{i-1}|_r = k_1$ and $|t_1 t_2 \ldots t_{i-1}|_u = k_2$, then

$$c_i = \begin{cases} a_{k_1+1} & if\ t_i = r, \\ b_{k_2+1} & if\ t_i = u. \end{cases}$$

If $T \subseteq V^*$ is a set of trajectories, the *shuffle of* α *with* β *on the set T of trajectories*, [1] denoted $\alpha ш_T \beta$, is $\alpha ш_T \beta = \bigcup_{t \in T} \alpha ш_t \beta$.

Example 1.17 Let α and β be the words $\alpha = a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8$, $\beta = b_1 b_2 b_3 b_4 b_5$ and assume that $t = r^3 u^2 r^3 ururu$. The shuffle of α with β on the trajectory *t* is

$$\alpha ш_t \beta = \{a_1 a_2 a_3 b_1 b_2 a_4 a_5 a_6 b_3 a_7 b_4 a_8 b_5\}.$$

The result has the following geometrical interpretation.

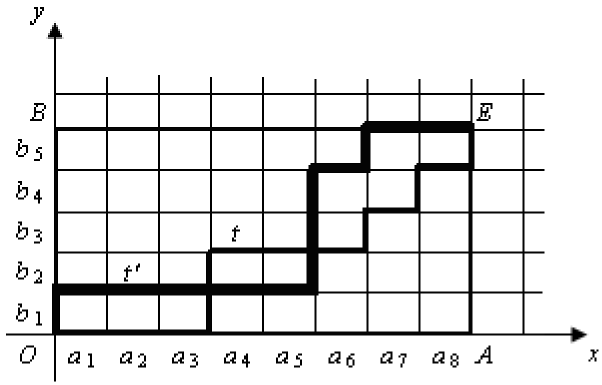


Figure 3

The trajectory *t* defines a line starting in the origin and continuing one unit to the right or up, depending on the definition of t. In our case, first there are three units right, then two units up, then three units right, etc. Assign α on the *Ox* axis and β on the *Oy* axis of the plane. Observe that the trajectory ends in the point with coordinates (8,5) denoted by E in figure 3, that is exactly the upper right corner of the rectangle defined by α and β, i.e., the rectangle *OAEB* in figure 3. Hence, the result of the shuffle of α with β on the trajectory *t* is nonempty.

The result can be read following the line defined by the trajectory *t*: that is, when being in a lattice point of the trajectory, with the trajectory going right, one should pick up the corresponding letter from α, otherwise, if the trajectory is going up, then one should add to the result the corresponding letter from β.

Hence, the trajectory *t* defines a line in the rectangle *OAEB*, on which one has "to walk" starting from the corner *O*, the origin, and ending in the corner *E*, the exit point. In each lattice point, one has to follow one of the versors *r* or *u*, according to the definition of *t*.

Assume now that *t′* is another trajectory, say $t' = ur^5u^3rur^2$. In figure 3, the trajectory *t′* is depicted by a much bolder line that the trajectory *t*. Observe that $\alpha \text{Ш}_{t'} \beta = \{b_1a_1a_2a_3a_4a_5b_2b_3b_4a_6b_5a_7a_8\}$.

Consider the set of trajectories, $T = \{t, t'\}$.

The shuffle of α with β on the set T of trajectories is

$$\alpha \text{Ш}_T \beta = \{a_1a_2a_3b_1b_2a_4a_5a_6b_3a_7b_4a_8b_5,\ b_1a_1a_2a_3a_4a_5b_2b_3b_4a_6b_5a_7a_8\}.$$

Definition 1.18 [1]

(i) A set T of trajectories is complete if and only if $\alpha \text{Ш}_T \beta \neq \phi$ for all $\alpha, \beta \in \Sigma^*$.

(ii) A set T of trajectories is deterministic if and only if $card(\alpha \text{Ш}_T \beta) \leq 1$, for all $\alpha, \beta \in \Sigma^*$.

(iii) A set T of trajectories is referred to as commutative if and only if the operation Ш_T is a commutative operation, i.e., $\alpha \text{Ш}_T \beta = \beta \text{Ш}_T \alpha$ for all $\alpha, \beta \in \Sigma^*$.

Example 1.19

(i) (a) Shuffle, Catenation, insertion are complete sets of trajectories.

(b) Non complete sets of trajectories are, for instance, balanced literal shuffle, balanced insertion, all finite sets of trajectories.

(ii) (a) Catenation, balanced literal shuffle, balanced insertion are deterministic sets of trajectories.

(b) Nondeterministic sets of trajectories are, for instance, shuffle and insertion.

(iii) (a) Shuffle is a commutative set of trajectories.

(b) Noncommutative sets of trajectories are, for instance, catenation and insertion.

2. Shuffle operations on Euler Graphs

In this section we define string Euler graph, standard form of string Euler graph and study the shuffle operation on the family of string Euler graphs.

Definition 2.1 Consider the Pseudo-Linear Form *PLF(G)* of an Euler graph *G*. Trace an Euler line in *PLF(G)* and name an edge from v_p to v_q as a_i if $p < q$ and $q - p = i$, as b_j if $p > q$ and $p - q = j$. The sequence of edges labeled by a_i and b_j, in tracing an Euler line is the string Euler graph of *G*, denoted by S_G.

It is clear that given an S_G, the corresponding Euler graph can be constructed upto isomorphism.

Definition 2.2 Consider an Euler graph G, label a vertex as v_1 arbitrarily. Trace an Euler line from v_1: $v_1e_1 \ldots v_q e_q v_1$ where $v_1, \ldots, v_q \in V$. Order the vertices of *G* in such a way that the vertex v_q receives the label v_n. Now the S_G corresponding to this ordering is said to be in standard form.

Observation 2.3 If $S_G = x_1\ x_2 \ldots x_q$, with $x_r \in \{a_i, b_j \ /\ 1 \le i, j \le q\}$, $1 \le r \le q$, then

$$\sum_{a_i} i = \sum_{b_j} j .$$

If $S_G = x_1\ x_2 \ldots x_q$, and if $x_l = b_j$ for some l, we have

$$j < \sum_{\substack{a_i = x_k \\ k < l}} i .$$

Example 2.4 Let G be an Euler graph in figure 4.

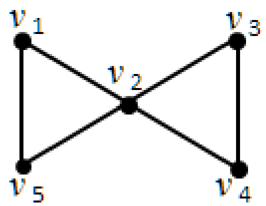


Figure 4

The PLF(G) is represented in figure 5.

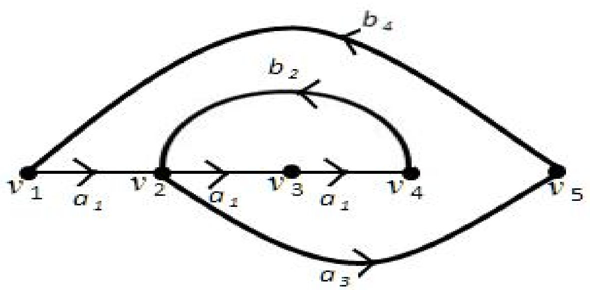


Figure 5

The corresponding string Euler graph $S_G = a_1a_1a_1b_2a_3b_4$.

Definition 2.5 Let G_1 and G_2 be two Euler Graphs and let their respective string Euler graphs be S_{G_1} and S_{G_2}.

i. The Graphs represented by each string in S_{G_1} Ш S_{G_2} are called as 'shuffled Euler graphs'.

ii. The graph represented by the single string in S_{G_1} Ш$_l$ S_{G_2} is called as 'literally shuffled Euler graph'.

iii. The graph represented by the single string in S_{G_1} Ш$_{bl}$ S_{G_2} is called as 'balanced literally shuffled Euler graph'.

Example 2.6 Let G_1 and G_2 be two Euler Graphs given in figure 6.

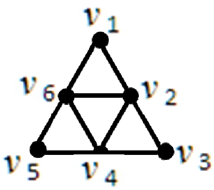


Figure 6

There corresponding PLF(G_1), PLF(G_2) are represented in figure 7.

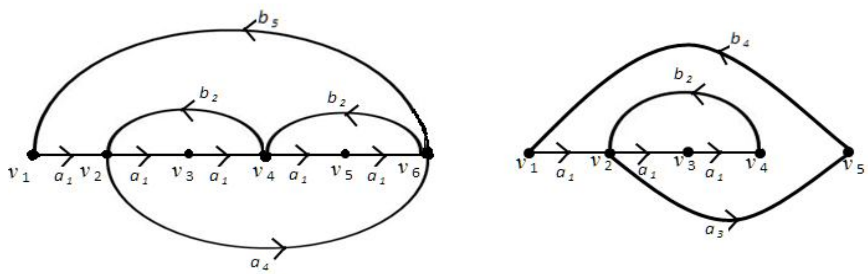


Figure 7

Here, S_{G_1} = $a_1a_1a_1a_1a_1b_2b_2a_4b_5$ and S_{G_2} = $a_1a_1a_1b_2a_3b_4$ with $\left|S_{G_1}\right| = 9$ and $\left|S_{G_2}\right| = 6$.

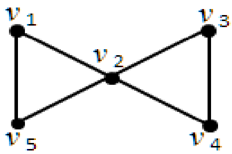
Therefore, S_{G_1} Ш S_{G_2} = $a_1a_1a_1a_1a_1b_2b_2a_4b_5$ Ш $a_1a_1a_1b_2a_3b_4$

$$= \begin{Bmatrix} a_1a_1a_1a_1a_1b_2b_2a_4b_5a_1a_1a_1b_2a_3b_4, & a_1a_1a_1a_1a_1a_1a_1b_2a_1a_3b_2b_4b_2a_4b_5, \\ a_1a_1a_1b_2a_3b_4a_1a_1a_1a_1a_1b_2b_2a_4b_5, & \cdots \end{Bmatrix}.$$

We notice that $n\ (S_{G_1}$ Ш $S_{G_2})$ = $15C_6$ = 5005 and for each x $\in S_{G_1}$ Ш$_{bl}$ S_{G_2}, $|x| = 15$.

Also, S_{G_1} Ш$_l$ S_{G_2} = $\{a_1a_1a_1a_1a_1a_1a_1b_2a_1a_3b_2b_4b_2a_4b_5\}$.

Example 2.7 Let G_1 and G_2 be two Euler Graphs given in figure 8.



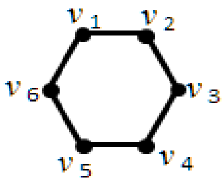


Figure 8

There corresponding PLF(G_1), PLF(G_2) are represented in figure 9.

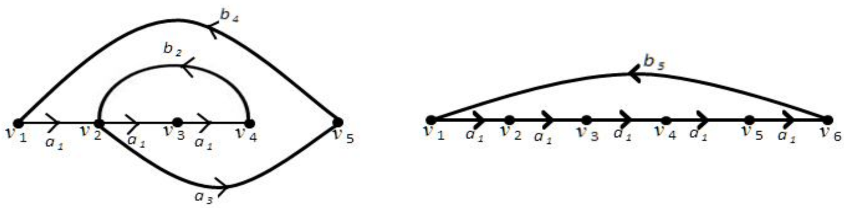


Figure 9

Here, S_{G_1} = $a_1a_1a_1b_2a_3b_4$ and S_{G_2} = $a_1a_1a_1\ a_1a_1b_5$ with $\left|S_{G_1}\right| = 6$ and $\left|S_{G_2}\right| = 6$.

Therefore, S_{G_1} Ш$_{bl}$ S_{G_2} = $\{a_1a_1a_1a_1a_1a_1b_2a_1a_3a_1b_4b_5\}$.

Theorem 2.8 Each element in the set S_{G_1} Ш S_{G_2} is an Euler graph.

Proof: Let G_1 and G_2 be two Euler graphs and let PLF(G_1), PLF(G_2) be their corresponding Pseudo-Linear Forms.

Trace an Euler line in PLF(G_1) and PLF(G_2). Obtain their respective string Euler graphs S_{G_1} and S_{G_2}. By observation 2.3, we have $\sum_{a_i} i = \sum_{b_j} j$ for both S_{G_1} and S_{G_2}.

Hence the shuffled string graphs will also preserves this property implying that the resultant graphs are Euler.

Therefore, each element in the set S_{G_1} Ш S_{G_2} is an Euler graph.

Remark 2.9

(i) The literal shuffle of S_{G_1} and S_{G_2}, S_{G_1} Ш S_{G_2} is an Euler graph

(ii) The balanced literal shuffle of S_{G_1} and S_{G_2}, S_{G_1} Ш$_{bl}$ S_{G_2} is an Euler graph.

3. Some algebraic properties

In this section, we investigate some of the algebraic properties of the trajectories associated with string Euler graphs.

Example 3.1 Let $S_{G_1} = a_1a_1a_1a_1a_1b_2b_2a_4b_5$ and $S_{G_2} = a_1a_1a_1b_2a_3b_4$ be two string Euler graphs. Assume a trajectory $t = r^3u^2r^4u^3r^2u$. The shuffle of S_{G_1} with S_{G_2} on the trajectory t is S_{G_1} Ш$_t$ $S_{G_2} = \{a_1a_1a_1a_1a_1a_1a_1b_2b_2a_1b_2a_3a_4b_5b_4\}$.

Theorem 3.2 Let G_1 and G_2 be two simple Euler graphs and their respective standard string Euler graphs be S_{G_1} and S_{G_2} with $|S_{G_1}| = p$ and $|S_{G_2}| = q$. The string Euler graphs in S_{G_1} Ш$_T$ S_{G_2} represent simple Euler graphs for $T = \{t, t'\}$, where $t = u^{q-1}r^p u$ and $t' = r^{p-1}u^q r$.

Proof: Let G_1 and G_2 be two simple Euler graphs, S_{G_1} and S_{G_2} be their corresponding standard string Euler graphs, with $|S_{G_1}| = p$ and $|S_{G_2}| = q$.

Since we consider only simple graphs which are Euler, there is no possibility of the occurrence of the pair (a_i, b_j) such that i and j have the same value in S_{G_1} and S_{G_2}.

Consider the two trajectories,

$$t = u^{q-1}r^p u,$$

$$t' = r^{p-1}u^q r.$$

Perform the shuffle using the above two trajectories. As observed earlier the two shuffled string graphs obtained have no possibility of the occurrence of the pair (a_i, b_j) such that i and j have the same value.

Thus, the graphs in S_{G_1} Ш$_T$ S_{G_2} are simple Euler graphs.

Theorem 3.3 If $T = \{t, t'\}$, as given in the above theorem, then

(i) T is complete.

(ii) T is not deterministic.

(iii) T is commutative.

Proof The proof of (i) follows from its definition. The set of trajectories $T = \{t, t'\}$ is not deterministic and commutative as observed in the below example which implies the proof of (ii) and (iii).

Example 3.4 Let G_1 and G_2 be two simple Euler Graphs given in figure 10.

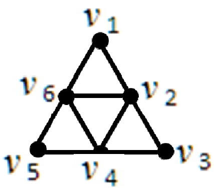


Figure 10

There corresponding PLF(G_1), PLF(G_2) are represented in figure 11.

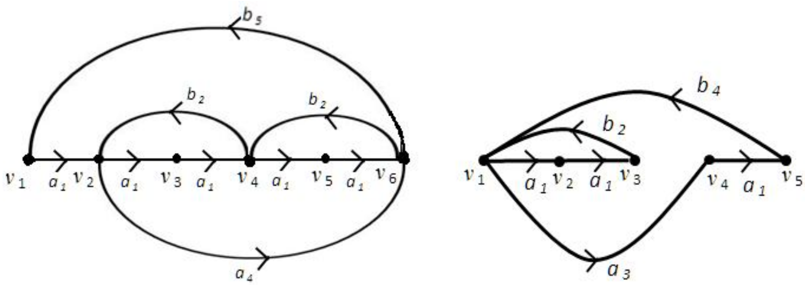


Figure 11

Here, $S_{G_1} = a_1a_1a_1a_1a_1b_2b_2a_4b_5$ and $S_{G_2} = a_1a_1b_2a_3a_1b_4$ with $\left|S_{G_1}\right| = 9$ and $\left|S_{G_2}\right| = 6$.

Let $T = \{t, t'\}$, as given in the theorem 3.2 that is $t = u^{q-1} r^{p} u$, $t' = r^{p-1} u^{q} r$ then $t = u^{5} r^{9} u$, $t' = r^{8} u^{6} r$.

$$S_{G_1} \text{Ш}_T S_{G_2} = \{a_1a_1b_2a_3a_1a_1a_1a_1a_1a_1b_2b_2a_4b_5b_4, a_1a_1a_1a_1a_1b_2b_2a_4a_1a_1b_2a_3a_1b_4b_5\}.$$

Similarly, for $S_{G_2} \text{Ш}_T S_{G_1}$ by theorem 3.2, $t = u^{8} r^{6} u$, $t' = r^{5} u^{9} r$.

$$S_{G_2} \text{Ш}_T S_{G_1} = \{a_1a_1a_1a_1a_1b_2b_2a_4a_1a_1b_2a_3a_1b_4b_5, a_1a_1b_2a_3a_1a_1a_1a_1a_1a_1b_2b_2a_4b_5b_4\}.$$

The graphs for the strings in $S_{G_1} \text{Ш}_T S_{G_2}$ = $S_{G_2} \text{Ш}_T S_{G_1}$ are given in the figures 12 and 13.

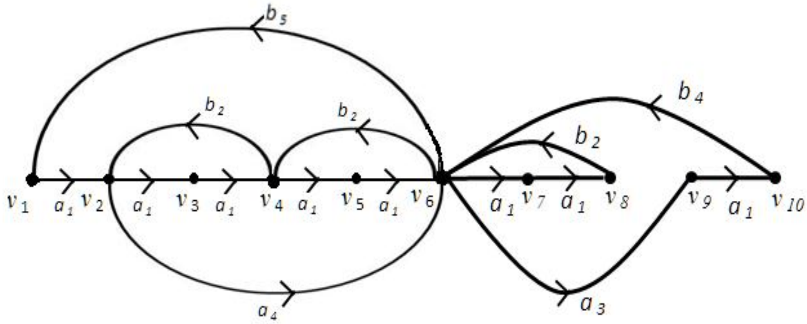


Figure 12

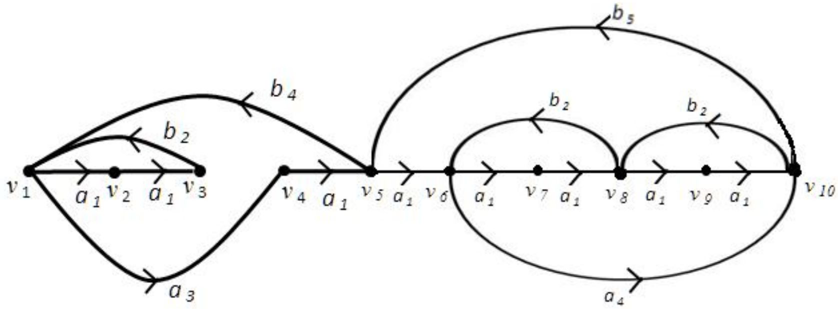


Figure 13

We observe that the above graphs are simple Euler graphs as given in the theorem 3.2.

References

[1] A. Mateescu, G. Rozenberg and A. Salomaa, Shuffle on trajectories: syntactic constraints, Theoretical Computer Science 197 (1998) 1-56.

[2] L. Jeganathan, K. Krithivasan and R. Rama, Graph Splicing System, in the proceedings of International Conference on Theoretical and Mathematical Foundation of Computer Science, (2008) 72-79.