ASSOCIATE RING GRAPHS

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ABSTRACT

R is a commutative ring with unity. The associate ring graph AG(R) is the graph with the vertex set \( V = R - \{0\} \) and edge set \( E = \{ (a, b) \mid a, b \text{ are associates and } a \neq b \} \). Since the relation of being associate is an equivalence relation, this graph is an undirected graph and also each component is complete. In this paper, I present some of the interesting results majority of which are for the ring of integers modulo \( n \), \( n \) is a positive integer.

1) \( AG(R) \) is an empty graph if \( R \) is a Boolean ring.
2) \( AG(\mathbb{Z}_n) \) is complete if and only if \( n \) is prime.
3) If \( n \) is even then \( AG(\mathbb{Z}_n) \) has an isolated vertex \( n/2 \).
4) If \( p \) is prime and \( p \neq 2 \), then \( AG(\mathbb{Z}_p) = K_1 \cup K_{p-1} \cup K_{p^2-1} \).
5) \( AG(\mathbb{Z}_{p^2}) = K_{p^2-1} \cup K_{p(p-1)} \).
6) \( AG(\mathbb{Z}_{pq}) = K_{p^2-1} \cup K_{q^2-1} \cup K_{pq-p-q+1} \).
7) A C-program to find the components of \( AG(\mathbb{Z}_n) \).

1. Introduction

The motivation for associate ring graphs is from zero-divisor graphs defined by I. Beck in the year 1988. He introduced the idea of these graphs for commutative rings \( R \) with unity \( 1 \). He defined \( \Gamma_0(R) \) to be the graph whose vertices are elements of \( R \) and in which two vertices \( x \) and \( y \) are adjacent if and only if \( xy = 0 \). Beck was
mostly concerned with coloring $\Gamma_0(R)$. In his paper [1] he studied the subgraph $\Gamma(R)$ whose set of vertices is $Z(R)^* = Z(R) - \{0\}$ where $Z(R)$ is the set of zero-divisors of $R$. $\Gamma(R)$ is non empty unless $R$ is an integral domain and, by a result of G. Ganesan, $Z(R)$ and hence $(R)$ is finite if and only if $R$ is finite. It is shown that $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \leq 3$. Lot of results were subsequently developed (Some of them can be seen in [2] and [3]) by several authors for zero-divisor graphs. If $R$ is a field then $(R)$ is empty or $(R)$ has no edges when all non-zero elements are used as vertices. Since a field is very rich with respect to algebraic structure, it is quite reasonable to associate a graph which is also rich graph theoretically. We know that complete graphs take this place. So I thought of defining a graph from a ring $R$ so that it is complete when $R$ is a field. This graph is nothing but the so called ASSOCIATE RING GRAPH.

2. Preliminaries

All the fundamental concepts of ALGEBRA are from [4] and of GRAPH THEORY are from [5].

3. Associate Ring Graphs

3.1 Associate ring graph: Let $R$ be a ring with unity 1 (not necessarily commutative). The associate ring graph of $R$ denoted by $AG(R)$ is the graph $(V,E)$ where the vertex set $V = R - \{0\}$ and the edge set $E = \{(a,b) / a \text{ is an associate of } b \text{ and } a \neq b\}$.  

Note: Throughout this paper a ring always means a ring with unity 1.

3.2 Orbit of an element of a ring: If $a$ is an element of a ring $R$ then the orbit of $a$ denoted by $Or(a)$ is defined as $Or(a) = \{a.u \mid u \text{ is a unit in } R\}$.  

3.3 Theorem: The orbits of elements of a ring are either identical or disjoint.  
Proof: Let $R$ be a ring and $a, b$ are two elements of $R$.  
If $Or(a)$ and $Or(b)$ are disjoint we have nothing to prove.  
Suppose that $Or(a) \cap Or(b) \neq \emptyset$.  
Let $c \in Or(a) \cap Or(b)$. Then $c = a.u$ and $c = b.v$ for some units $u, v$ in $R$.  
$\therefore a.u = b.v \Rightarrow a = b.(v.u^{-1})$ and $b = a.(u.v^{-1})$ and so $a$ and $b$ are associates.  
Let $x$ be an arbitrary element in $Or(a)$. Then $x = a.s$, $s$ is a unit in $R$.  
So $x = b.(v.u^{-1}).s$  
i.e., $x = b.(v.u^{-1}).s$  

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i.e., \( x = b \) (\( a \) unit in \( R \)).

i.e., \( x \in \text{Or}(b) \) and so \( \text{Or}(a) \subseteq \text{Or}(b) \). Similarly we can show that \( \text{Or}(b) \subseteq \text{Or}(a) \).

Thus \( \text{Or}(a) = \text{Or}(b) \).

Hence the Orbits of any two elements of a ring are either disjoint or identical.

3.4 Observation: Since the relation of being associative is an equivalence relation it partitions \( R \) into disjoint sets and it can be easily seen that the equivalence class containing an element \( a \) is nothing but \( \text{Or}(a) \). Thus our graph contains connected components equal in number to the number of disjoint equivalence classes except \( \{0\} \).

3.5 Example 1. Consider the ring \( (\mathbb{Z}, +, \cdot) \) of integers. We know that 1 and \(-1\) are the only units of \( \mathbb{Z} \). Therefore for any \( 0 \neq a \) in \( \mathbb{Z} \), \( \text{Or}(a) = \{ a, -a \} \). Hence \( \text{AG}(\mathbb{Z}) \) consists of infinite number of components each is a \( K_2 \).

\[
\text{AG}(\mathbb{Z}) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
-1 & -2 & -3 & -4
\end{array}
\]

Therefore \( \text{AG}(\mathbb{Z}) = K_2 \cup K_2 \cup K_2 \cup K_2 \ldots \).

Example 2. Consider \( (\mathbb{Z}_5, +_5, \cdot_5) \). This is a field. Every non zero element is a unit and so any two non-zero elements are associates. Hence the graph is a complete graph with four vertices \( 1, 2, 3, 4 \).

The graph \( \text{AG}(\mathbb{Z}_5) \) is

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 0 & 1
\end{array}
\]

Hence \( \text{AG}(\mathbb{Z}_5) = K_4 \).

Example 3. Consider \( (\mathbb{Z}_6, +_6, \cdot_6) \). Here \( \mathbb{Z}_6 = \{ 0, 1, 2, 3, 4, 5 \} \).

The Units of \( \mathbb{Z}_6 \) are \( 1, 5 \).

\( \text{Or}(1) = \{ 1 \cdot_6 1, 1 \cdot_6 5 \} = \{ 1, 5 \} \).
\begin{align*}
\text{Or}(2) &= \{2, 4\} \\
\text{Or}(3) &= \{3\} \\
\text{Or}(4) &= \{2, 4\} \\
\text{Or}(5) &= \{1, 5\}.
\end{align*}

The graph \( AG(Z_6) \) is

\[
\begin{array}{c}
\text{1} \\
\text{3} \\
\text{5} \\
\text{4}
\end{array}
\]

Hence \( AG(Z_6) = K_1 \cup K_2 \cup K_2 \).

3.6 Theorem: \( AG(R) \) is an empty graph (without edges) if \( R \) is a Boolean ring.

Proof: Let \( R \) be a Boolean ring with unity 1. We show that \( R \) has no units other than the unity 1.

Let \( 0 \neq a \) be a unit in \( R \). i.e., \( a.b = 1 \) for some \( 0 \neq b \) in \( R \). Since \( R \) is Boolean, \( a^2 = a \).

Now \( a.b = 1 \Rightarrow a.(a.b) = a.1 \Rightarrow a^2.b = a \Rightarrow a.b = a = 1 \Rightarrow a = 1 \).

Hence 1 is the only unit in \( R \). Therefore the orbit of every non-zero element of \( R \) contains only itself.

Hence \( AG(R) \) has no edges.

3.7 Theorem: \( AG(Z_n) = K_{n-1} \) (the complete graph with \( n-1 \) vertices) if and only if \( n \) is prime.

Proof: Suppose that \( AG(Z_n) \) is complete.

i.e., every pair of non-zero elements of \( Z_n \) are connected by an edge.

We know that \( Z_n \) is a commutative ring with unity 1.

If \( a \) is any non-zero element of \( Z_n \) then \( a \) and 1 are joined by an edge. i.e., \( a \) and 1 are associates.
i.e., \(1 = u \cdot a\) for some unit \(u\) in \(Z_n\).

i.e., \(a\) is an invertible element in \(Z_n\).

i.e., every non-zero element in \(Z_n\) is invertible.

Thus \(Z_n\) is a field and hence \(n\) is prime.

Conversely suppose that \(n\) is prime.

Therefore \(Z_n\) is a field.

Let \(x\) and \(y\) be two non-zero elements of \(Z_n\).

Since \(Z_n\) is a field \(x\) and \(y\) are units.

So \(x^{-1} \cdot y\) is also a unit in \(Z_n\).

We have \(x \cdot (x^{-1} \cdot y) = y\).

\(\Rightarrow x\) is an associate of \(y\).

\(\Rightarrow x\) and \(y\) are joined by an edge.

Thus every pair of non-zero elements of \(Z_n\) are joined by an edge.

Hence \(AG(Z_n)\) is complete.

3.8 Theorem: If \(n\) is even then \(AG(Z_n)\) has an isolated vertex namely \(n/2\).

Proof: Suppose \(n\) is even.

i.e., \(n = 2m\) for some \(m\) in \(N = \{ 1, 2, 3, \ldots \}\).

We show that \(m = n/2\) is an isolated vertex in \(AG(Z_n)\).

We know that the units of \(Z_n\) are the non-zero elements of \(Z_n\) which are relatively prime to \(n\). Since \(n\) is even these units must be odd.

Let \(a = 2k+1\) be a unit in \(Z_n\).

Then we have \(m \cdot a = m \cdot (2k+1) = 2mk + m = nk + m = m\) (Since \(nk = 0\) in \(Z_n\)).

Thus the only associate of \(m\) is \(m\) itself.

Since \(AG(Z_n)\) has no self loops \(m\) is an isolated vertex of \(AG(Z_n)\).
3.9 Theorem: If $n = 2p$ where $p$ is a prime ($\neq 2$) then $AG(Z_n) = K_1 \cup K_{p-1} \cup K_{p-1}$.

Proof: Let $n = 2p$. By 3.8, $AG(Z_n)$ has an isolated vertex $n/2 = p$. So $AG(Z_n)$ contains $K_1$. Also $AG(Z_n)$ has a component $K_{\phi(n)} = K_{\phi(2p)} = K_{\phi(2p)(p)} = K_{p-1}$.

It is enough to prove that the graph has only one component left and that is also $K_{p-1}$.

We show that the remaining vertices other than $p$ and the units in $K_{\phi(n)} = K_{p-1}$ form the vertices of the other $K_{p-1}$.

Clearly the number of vertices remaining are $[(n-1)-(p-1)-1] = p-1$.

We have $m$ is a unit if and only if $(m, 2p) = 1$.

If and only if $m$ is odd and not a multiple of $p$.

If and only if $m$ is odd and $m \neq p$.

If and only if $m = 1, 3, 5, \ldots, (p-2), (p+2), \ldots, (2p-1)$.

Therefore the set of remaining elements is $D = \{2, 4, \ldots, (p-1), (p+1), \ldots, (2p-2)\}$.

We show that the orbit of any general element $2k$ of $D$ is $D$. The associates of $2k$ are $2k(1), 2k(3), \ldots, 2k(p-2), 2k(p+2), \ldots, 2k(2p-1)$. These products are all even and so are elements of $D$. We show that that these products are distinct.

Suppose that $2k(2m-1) = 2k(2s-1)$ where $m \neq s$ and $m > s$.

So $2p$ divides $2k(2m-1) - 2k(2s-1) = 4k(m-s)$.

So $p$ divides $2k(m-s)$.

Since $p$ does not divide 2 and $k$, we must have $p \mid (m-s)$.

Since $(m-s) < p$ we must have $m = s$, a contradiction.

Thus the orbit of $2k$ is $D$. Therefore every element of $D$ is an associate to every other element of $D$. This shows that the elements in $D$ form the required $K_{p-1}$.

Hence $AG(Z_{2p}) = K_1 \cup K_{p-1} \cup K_{p-1}$.

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3.10 Theorem: \( \text{AG}(\mathbb{Z}_{p^2}) = K_{p+1} \cup K_{p(p-1)} \).

Proof: Let \( p \) be a prime number.

We have \( \mathbb{Z}_{p^2} = \{0, 1, 2, \ldots, (p^2-1)\} \).

For any \( 0 \neq a \) in \( \mathbb{Z}_{p^2} \), \((a, p^2) = 1\) if and only if \( p \) does not divide \( a \).

if and only if \( a \) is not a multiple of \( p \).

Hence \( \text{Or}(1) = \text{units of } \mathbb{Z}_{p^2} = \{1, 2, \ldots, (p-1), (p+1), \ldots, (2p-1), (2p+1), \ldots, (p^2-1)\} \).

The remaining non-zero elements of \( \mathbb{Z}_{p^2} \) are \( p, 2p, 3p, \ldots, (p-1)p \).

Obviously the number of elements in \( \text{Or}(1) = \text{number of units} = (p^2-1) - (p-1) = p(p-1) \).

Thus \( \text{AG}(\mathbb{Z}_{p^2}) \) has \( K_{p(p-1)} \) as a component.

To prove the theorem it is enough to show that the remaining \((p-1)\) non-units (zero-divisors) forms a \( K_{p+1} \).

Let \( D = \{p, 2p, \ldots, (p-1)p\} \).

We have \( \text{Or}(p) = \{p, 1, p, 2, \ldots, p, (p-1), p, (p+1), \ldots\} \).

Clearly the first \((p-1)\) elements of \( \text{Or}(p) \) are elements of \( D \).

So \( D \) is a subset of \( \text{Or}(p) \). \( \text{(1)} \)

Since \( p \) is a non-unit, all elements of \( \text{Or}(p) \) are non-units.

So \( \text{Or}(p) \cap \text{Or}(1) = \emptyset \)

Therefore \( \text{Or}(p) \) is a subset of \( \{\text{Or}(1)\}^c = D \). \( \text{(2)} \)

From \( \text{(1)} \) and \( \text{(2)} \) we get \( \text{Or}(p) = D \).

Thus the elements \( p, 2p, 3p, \ldots, (p-1)p \) of \( \text{Or}(p) \) forms the vertices of the required \( K_{p+1} \).

Hence \( \text{AG}(\mathbb{Z}_{p^2}) = K_{p+1} \cup K_{p(p-1)} \).

3.11 Theorem: \( \text{AG}(\mathbb{Z}_{pq}) = K_{(p-1)} \cup K_{(q-1)} \cup K_{pq \cdot (p-1)} \).

Proof: Without loss of generality we assume that \( p < q \). The cases when \( p = 2 \) and \( p = q \) are already dealt in 3.9 and 3.10 respectively.
Now \( n \) is a unit in \( \mathbb{Z}_{pq} \) if and only if \( (n, pq) = 1 \).

If and only if \( n \) is neither a multiple of \( p \) nor a multiple of \( q \).

Also \( n \) is not a unit if and only if \( n \) is either a multiple of \( p \) or a multiple of \( q \).

We have \( \text{Or}(1) = \{1, 2, \ldots, (p-1), (p+1), \ldots, (q-1), \ldots, (pq-1)\} \).

Obviously \( n[\text{Or}(1)] = i(pq) = i(p) j(p) = (p-1)(q-1) = pq - p - q + 1 \).

Thus \( K_{pq, p-q+1} \) is a component of \( AG(\mathbb{Z}_{pq}) \).

Since \( p, q \) are distinct primes they are not associates.

For let \( p = u.q \) where \( u \) is a unit in \( \mathbb{Z}_{pq} \).

i.e., \( p - u.q \) is divisible by \( pq \).

i.e., \( p - u.q = k.pq \) where \( k \) is an integer.

i.e., \( p = q(u + kp) \).

i.e., \( p \) is divisible by \( q \), a contradiction.

Hence \( \text{Or}(p) \cap \text{Or}(q) = \emptyset \).

Here \( (p + q) \) is neither a multiple of \( p \) nor a multiple of \( q \).

So \( (p + q) \) is a unit in \( \mathbb{Z}_{pq} \) and hence \( p(p + q) \) is an associate of \( p \).

But \( p(p + q) = p^2 + pq = p^2 \) (since \( pq \neq 0 \) in \( \mathbb{Z}_{pq} \)).

Thus \( p^2 \) is an associate of \( p \).

Similarly we can show that \( p^3, p^4, \ldots \) are associates of \( p \).

Thus \( 1, p, 2.p, \ldots, p.p, \ldots, (q-1).p \) are distinct elements in \( \text{Or}(p) \).

Therefore \( n[\text{Or}(p)] \geq q-1 \) and similarly \( n[\text{Or}(q)] \geq p-1 \).

We have \( \text{Or}(1) \subseteq \text{Or}(p) \subseteq \text{Or}(q) \) \( \subseteq \mathbb{Z}_{pq} \).

Also \( n[\text{Or}(1) \cap \text{Or}(p) \cap \text{Or}(q)] = n[\text{Or}(1)] + n[\text{Or}(p)] + n[\text{Or}(q)] \) (the union is disjoint)

\[ \geq (pq-p-q+1) + (p-1) + (q-1) \]

\[ = pq - 1 \]

\[ = n[\mathbb{Z}_{pq}] \]
Therefore $n[\text{Or}(1) \cup \text{Or}(p) \cup \text{Or}(q)] \geq n[Z_{pq}]$ \hspace{1cm} (2)

From (1) and (2) we get  $\text{Or}(1) \cup \text{Or}(p) \cup \text{Or}(q) = Z_{pq}$.

Now $\text{Or}(p)$ cannot contain more than $(q-1)$ elements otherwise $\text{Or}(q)$ contains less than $(p-1)$ elements which is not true. Thus $n[\text{Or}(p)] = (q-1)$ and so $n[\text{Or}(q)] = (p-1)$.

Hence $Z_{pq}$ has only three distinct orbits namely $\text{Or}(1)$, $\text{Or}(p)$ and $\text{Or}(q)$ with elements $(pq-p-q+1)$, $(q-1)$ and $(p-1)$ respectively.

Hence $AG(Z_{pq}) = K_{(p-1)} \cup K_{(q-1)} \cup K_{pq-p-q+1}$.

3.12 C-program to find the components of $AG(Z_p)$: A C-programming is prepared to find the components of $AG(Z_p)$ for a given positive integer $n$.

Example:

Enter 'n' value: 50

ORBIT 1: \{ 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 49 \}.

No. of elements is: 20

ORBIT 2: \{ 2, 4, 6, 8, 12, 14, 16, 18, 22, 24, 26, 28, 32, 34, 36, 38, 42, 44, 46, 48 \}.

No. of elements is: 20

ORBIT 5: \{ 5, 15, 35, 45 \}.

No. of elements is: 4

ORBIT 10: \{ 10, 20, 30, 40 \}.

No. of elements is: 4

ORBIT 25: \{ 25 \}.

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No. of elements is: 1

FINAL SET is: \{1, 4, 4, 20, 20\} = 49.

Thus \(AG(Z_{50}) = K_1 U K_4 U K_4 U K_{20} U K_{20}\).

References

3. V.K. Bhat and Ravi Raina, Neeraj Nehra, *A Note on Zero Divisor Graph Over*.