The Graphs Whose Sum of Global Connected Domination Number and Chromatic Number is 2n-5

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Abstract

A subset S of vertices in a graph G = (V,E) is a dominating set if every vertex in V-S is adjacent to at least one vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced sub graph < S > is connected. A set S is called a global dominating set of G if S is a dominating set of both G and \( \overline{G} \). A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by \( \gamma_{gc}(G) \). In this paper we characterize the classes of graphs for which \( \gamma_{gc}(G) + \chi(G) = 2n-5 \) and 2n-6 of global connected domination number and chromatic number and characterize the corresponding extremal graphs.

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1. Introduction

Graphs discussed in this paper are simple, finite and undirected graphs. A subset S of vertices in a graph G = (V,E) is a dominating set if every vertex in V-S is adjacent to atleast one vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced sub graph < S > is connected. A set S is called a global dominating set of G if S is a dominating set of both G and G. A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by γgc(G). Note that any global connected dominating set of a graph G has to be connected in G (but not necessarily in G ). Here global connected domination number γgc is well defined for any connected graph. For a cycle C_n of order n ≥ 6, γgc(C_n) = \left\lceil n/3 \right\rceil while γgc(C_n) = n-2 for n ≥ 4 and γg(Kn) = 1,while γg(K_n) = n. The chromatic number χ(G) is defined as the minimum number of colors required to color all the vertices such that adjacent vertices do not receive the same color.

Notations

K_n (P_k ) is the graph obtained from K_n by attaching the end vertex of P_k to any one vertices of K_n. K_n(mP_k) is the graph obtained from K_n by attaching the end vertices of m copies of P_k to any one vertices of K_n. The graph (m_1,m_2,………, m_n) denote the graph obtained from K_n by pasting m_1 edges to any one vertex u_i of K_n m_2 edges to any vertex u_j of K_n for i ≠ j m_3 edges to any vertex u_k of K_n for i ≠ j ≠ k ≠1, ............ m_n edges to all the distinct vertices of K_n. C_n (P_k ) is the graph obtained from C_n by attaching the end vertex of P_k to any one vertices of C_n. S*(K_1,n) is a graph obtained from K_{1,n} by subdividing n-1 edges.

2. Preliminary Results

Theorem 2.1 [1] Let G be a graph of order n ≥ 2. Then, (i) 2 ≤ γgc(G) ≤ n (ii) γgc(G) = n if and only if G ≅ K_n.

Corollary 2.2 [1] For all positive integers p and q, γgc(K_{p,q}) = 2.
**Theorem 2.3** [1] For any graph $G$ of order $n \geq 3$, $\gamma_{gc}(G) = n-1$ if and only if $G \cong K_{n-e}$, where $e$ is an edge of $K_n$.

**Theorem 2.4** For any connected graph $G$ of order $n \geq 1$, $\gamma_{gc}(G) + \chi(G) < 2n-1$.

**Theorem 2.5** For any connected graph $G$ of order $n \geq 3$, $\gamma_{gc}(G) + \chi(G) = 2n-2$ if and only if $G \cong K_{n-e}$, where $e$ is any edge of $K_n$.

**Theorem 2.6** For any connected graph $G$ of order $n > 2$, $\gamma_{gc}(G) + \chi(G) = 2n-3$ if and only if $G \cong K_3(P_2), K_n - \{e_1,e_2\}$, where $e$ is an edge in outside the cycle of graph, $n \geq 5$.

**Theorem 2.7** For any connected graph $G$ of order $n \geq 3$, $\gamma_{gc}(G) + \chi(G) = 2n-4$ if and only if $G \cong P_4, C_4, K_2(2P_2), K_4(P_2), K_n - \{e_1,e_2,e_3\}$ for $n \geq 6$ and $e$ is an edge in outside the cycle of graph.

### 3. Main Result

**Theorem 3.1** For any connected graph $G$ for $n \geq 3$, $\gamma_{gc}(G) + \chi(G) = 2n-5$ if and only if $G \cong G_1, G_2, G_3, G_4, G_5, G_6, P_5, K_3(P_3), K_3(2P_2), K_4(P_2,P_2,0), K_5(P_2)$ and $K_n - \{e_1,e_2,e_3,e_4\}$, where $e_1, e_2, e_3, e_4$ are consecutive edges in outside the cycle of $K_n$ of order $n \geq 7$. 
Proof: Assume that $\gamma_{g^c}(G) + \chi(G) = 2n-5$. This is possible only if $\gamma_{g^c}(G) = n$ and $\chi(G) = n-5$ (or) $\gamma_{g^c}(G) = n-1$ and $\chi(G) = n-4$ (or) $\gamma_{g^c}(G) = n-2$ and $\chi(G) = n-3$ (or) $\gamma_{g^c}(G) = n-3$ and $\chi(G) = n-2$ (or) $\gamma_{g^c}(G) = n-4$ and $\chi(G) = n-1$ (or) $\gamma_{g^c}(G) = n-5$ and $\chi(G) = n$.

Case 1: Let $\gamma_{g^c}(G) = n$ and $\chi(G) = n-5$. Since $\chi(G) = n-5$, $G$ contains a clique $K$ on $n-5$ vertices or does not contain a clique $K$ on $n-5$ vertices. Let $G$ contains a clique $K$ on $n-5$ vertices. Let $S = \{v_1, v_2, v_3, v_4, v_5\} \in V-K$. Then the induced sub graph $<S>$ has the following possible cases. $<S> = K_5, \overline{K}_5, C_3(C_3), K_4 \cup K_1, K_3(P_3), C_5, C_3(P_2, P_2, 0), C_3(C_3), C_4(P_2), P_5, K_3 \cup K_2, K_1, C_4 \cup K_1, K_3(P_2) \cup K_1, P_3 \cup K_2, K_1 \cup K_2 \cup \overline{K}_2, K_1 \cup K_2 \cup \overline{K}_2, K_3 \cup \overline{K}_2, K_2 \cup \overline{K}_3, K_2 \cup K_2 \cup K_1, P_3 \cup \overline{K}_2, C_3(2P_2), K_4(P_2), K_4-e \cup K_1$, Petersen graph, $K_4-e(P_2)$, $K_5-e$, $K_5-2e$, where $e$ is any edge on the cycle of $K_5$. 

![Graph G5](image1.png)

![Graph G6](image2.png)
It can be verified that for all the above cases no graph exist.

If $G$ does not contain the clique $K$ on n-5 vertices, then it can be verified that no new graph exists.

**Case 2:** Let $\gamma_{gc}(G) = n-1$ and $\chi(G) = n-4$. Since $\gamma_{gc}(G) = n-1$, then by theorem 2.3, $G \cong K_{n-e}$. But for $K_{n-e}$, $\chi(G) = n-1$, which is a contradiction.

**Case 3:** Let $\gamma_{gc}(G) = n-2$ and $\chi(G) = n-3$. Since $\chi(G) = n-3$, $G$ contains a clique $K$ on n-3 vertices or does not contain a clique $K$ on n-3 vertices. Let $G$ contains a clique $K$ on n-3 vertices. Let
S = \{v_1, v_2, v_3\} \in V-K. Then the induced sub graph < S > has the following possible cases: < S > = K_3, \overline{K}_3, P_3, K_2 \cup K_1.

**Subcase (i):** Let < S > = K_3. Since G is connected, there exist a vertex u_i of K_{n-3} adjacent to anyone of \{v_1, v_2, v_3\}. Without loss of generality, let v_1 be adjacent to u_i. Then \{v_1, u_i\} is a global connected dominating set, hence \gamma_{gc}(G) = 2 so that K \cong K_1 which is a contradiction.

**Subcase (ii):** Let < S > = \overline{K}_3. Since G is connected, let all the vertices of \overline{K}_3 be adjacent to vertex u_i. Then u_i and anyone of the vertices of \overline{K}_3 forms a global connected dominating set. Without loss of generality v_1 and u_i forms a global connected dominating set. Hence \gamma_{gc}(G) = 2, which is a contradiction. If two vertices of \overline{K}_3 are adjacent to u_i and the third vertex adjacent to u_i for some i \neq j, Then \{u_i, u_j\} forms a global connected dominating set. Hence \gamma_{gc}(G) = 2, which is a contradiction. If all the three vertices of \overline{K}_3 are adjacent to three distinct vertices of K_{n-3} say \{u_i, u_j, u_k\} for some i \neq j \neq k. Then \{u_i, u_j, u_k\} forms a global connected dominating set in G. Hence \gamma_{gc}(G) = 3, then n=5, which is a contradiction.

**Subcase (iii):** Let < S > = P_3. Since G is connected there exist a vertex u_i of K_{n-3} which is adjacent to any one of the pendent vertices of P_3 say v_1 or v_3. Without loss of generality let v_1 be adjacent to u_i. Then \{v_1, v_2, u_i\} forms global connected dominating set. Hence \gamma_{gc}(G) = 3, so that K = K_2 then G \cong P_5. On the increasing the degree of u_i, \gamma_{gc}(G) = 2, which is a contradiction. Let there exist a vertex u_i of K_{n-3} be adjacent to v_2 then \{v_2, u_i\} forms global connected dominating set. Hence \gamma_{gc}(G) = 2 which is a contradiction.

**Subcase (iv):** Let < S > = K_2 \cup K_1. Since G is connected, there exist a vertex u_i of K_{n-3} which is adjacent to anyone of \{v_1, v_2\} and v_3. Without loss of generality let v_1 be adjacent to u_i. Then \{v_1, u_i\} forms a global connected dominating set in G. Hence \gamma_{gc}(G) = 2 so that K = K_1 which is a contradiction. Let there exist a vertex u_i of K_{n-3} be adjacent to anyone of \{v_1, v_2\} and u_i for some i \neq j in K_{n-3} adjacent to v_3. Without loss of generality let u_i be adjacent to v_1. Then \{v_1, u_i, u_j\} forms a global connected dominating set in G. Hence \gamma_{gc}(G) = 3 so that K \cong K_2, then G \cong P_5. On increasing the degree of u_i, K \cong K_3, which is a contradiction.
If $G$ does not contain the clique $K$ on $n$-3 vertices, then it can be verified that no new graph exists.

**Case 4:** Let $\gamma_{gc}(G) = n-3$ and $\chi(G) = n-2$. Since $\chi(G) = n-2$, $G$ contains a clique $K$ on $n$-2 vertices or does not contain a clique $K$ on $n$-2 vertices. Let $S = \{v_1, v_2\} \in V-K$. Then the induced subgraph $< S >$ has the following possible cases $<S> = K_2$ and $K_2$.

**Subcase (i):** Let $<S> = K_2$. Since $G$ is connected, there exist a vertex $u_i$ of $K_{n-2}$ which is adjacent to anyone of $\{v_1, v_2\}$. Without loss of generality let $v_1$ be adjacent to $u_i$. Then $\{v_1, u_i\}$ forms a global connected dominating set in $G$ so that $\gamma_{gc}(G) = 2$ hence $K \cong K_3$. Then $G \cong K_3(P_3)$. On increasing the degree, $G \cong G_1, G_2$.

**Subcase (ii):** Let $< S > = K_2$. Since $G$ is connected. Let both the vertices of $K_{n-2}$ be adjacent to vertex $u_i$ for some $i$ in $K_{n-2}$. Then anyone of the vertices of $K_2$ and $u_i$ forms a global connected dominating set in $G$. Hence $\gamma_{gc}(G) = 2$ so that $K \cong K_3$. Then $G \cong K_3(P_2)$. On increasing the degree of $u_i$, $G \cong G_3$. If both the vertices of $K_2$ are adjacent to two distinct vertices of $K_{n-2}$ say $u_i$ and $u_j$ for $i \neq j$ in $K_{n-2}$. $\{v_1, u_i, u_j\}$ forms a global connected dominating set in $G$. Hence $\gamma_{gc}(G) = 3$. Then $K \cong K_4$ hence $G \cong K_4(P_2,P_2,0,0)$. On increasing the degree, $G \cong G_4, G_5, G_6$.

If $G$ does not contain the clique $K$ on $n$-2 vertices, then it can be verified that no new graph exist.

**Case 5:** Let $\gamma_{gc}(G) = n-4$ and $\chi(G) = n-1$. Since $\chi(G) = n-1$, $G$ contains a clique $K$ on $n$-1 vertices or does not contain a clique $K$ on $n$-1 vertices. Let $v$ be the vertex not in $K_{n-1}$. Since $G$ is connected the vertex $v$ is adjacent to vertex $u_i$ of $K_{n-1}$. Then $\{v_1, u_i\}$ forms a global connected dominating set in $G$. Then $\gamma_{gc}(G) = 2$, so that $K \cong K_5$ hence $G \cong K_5(P_2)$. On increasing the degree, $G \cong K_{n-1}\{e_1, e_2, e_3, e_4\}$ where $e_1, e_2, e_3, e_4$ are consecutive edges in outside the cycle of $K_n$ of order $n \geq 7$.

If $G$ does not contain the clique $K$ on $n$-1 vertices, then it can be verified that no new graph exists.

**Case 6** Let $\gamma_{gc}(G) = n-5$ and $\chi(G) = n$. Since $\chi(G) = n$, $G \cong K_n$. But for $K_n$, $\gamma_{gc}(G) = n$, which is a contradiction.
Conversely if G is anyone of the graph $G_1, G_2, G_3, G_4, G_5, G_6, P_5, K_3(P_3), K_3(2P_2), K_4(P_2, P_2, 0), K_5(P_2)$ and $K_n\{e_1, e_2, e_3, e_4\}$ where $e_1, e_2, e_3, e_4$ are edges in outside the cycle of $K_n$ of order $n \geq 7$, then it can be verified that $\gamma_{gc}(G) + \chi(G) = 2n-5$ for which $\gamma_{gc}(G) + \chi(G) = 2n-7, 2n-8$, which will be reported later.

References


