Further characterizations and Helly-property in $k$-trees

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Abstract

The purpose of this paper is to obtain a characterization of $k$-trees in terms of $k$-connectivity and forbidden subgraphs. Also, we present the other characterizations of $k$-trees containing the full vertices by using the join operation. Further, we establish the property of $k$-trees dealing with the degrees and formulate the Helly-property for a family of nontrivial $k$-paths in a $k$-tree. We study the planarity of $k$-trees and express the maximal outerplanar graphs in terms of 2-trees and $K_2$-neighbourhoods. Finally, the similar type of results for the maximal planar graphs are obtained.

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1. Introduction

All graphs considered here are finite and simple. For any graph $G$, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. The order of $G$ is $|V(G)|$ and its size is $|E(G)|$. A graph of order $p$ and size $q$ is a $(p,q)$-graph. For any two disjoint graphs $G$ and $H$, $G + H$ denotes the join of $G$ and $H$. All definitions and notations are not given here may be found in Harary[4]. A graph $G$ is $n$-connected if the removal of any $m$ vertices for $0 \leq m < n$, from $G$ results in neither a disconnected graph nor a trivial graph. 1-connected graphs are simply the connected graphs. A graph $G$ is triangulated if every cycle of length strictly greater than 3 possesses a chord. Any $n$ mutually adjacent vertices i.e., $K_n$ in a graph is $n$-clique. For any set $S$ of vertices of a graph $G$, $\langle S \rangle$ denotes the induced subgraph of $G$ induced by $S$. For

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any connected graph $G$, $nG$ denotes the graph with $n$ components, each being isomorphic to $G$.

A family of trees, which are connected and acyclic, can be equivalently defined by the following recursive construction rule:

**Step 1.** A single vertex $K_1$ is a tree.

**Step 2.** Any tree of order $n \geq 2$, can be constructed from a tree $T$ of order $(n - 1)$ by inserting an $n^{th}$-vertex and joining it to any vertex of $T$.

More general, the multidimensional-trees can be constructed from the above tree-construction procedure by allowing the base of the recursive growth to be any clique. Notice that a connected graph, which is not a tree possesses a tree-like structure, which is actually reflected by constructing the new family of graphs, whose recursive growth just starts from any given clique $K_k$. This family of graphs are generally known as $k$-trees or $K_k$-trees or $k$-dimensional trees.[1, 5, 7, 8]

**Definition 1.1.** The family of $k$-trees (or $K_k$-trees) is the set of all graphs that can be obtained by the following recursive construction procedure:

1. A clique-$K_k$ is the smallest $k$-tree.
2. To a $k$-tree $G$ with $n - 1$ vertices for $n \geq k + 1$, add a new vertex and make it adjacent to any $k$ mutually adjacent vertices of $G$, so that the resulting $k$-tree is of order $n$.

![Figure 1](image)

Figure 1 gives the example of a 3-tree of order 6. Generally speaking, every $k$-tree $G$ of order $\geq k + 1$, can be reduced to a clique $K_k$, by sequentially removing the vertices of degree $k$ from $G$.

2. Properties and Characterizations

We need the following characterization theorem for later use.

**Theorem 2.1.** [5] Let $G$ be a $(p, q)$-graph with $p \geq k + 1$. Then $G$ is a $k$-tree if and only if $G$ is $k$-connected, triangulated and either $G$ is $K_{k+2}$-free or $q = (kp - \frac{k(k+1)}{2})$. 

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The immediate consequence of Theorem 2.1 is another characterization of \( k \)-trees in terms of forbidden subgraphs and \( k \)-connectivity.

**Corollary 2.2.** Let \( G \) be a graph of order at least \( k + 1 \). Then \( G \) is a \( k \)-tree if and only if \( G \) is \( k \)-connected and has no induced subgraph isomorphic to either \( C_n \) for \( n \geq 4 \) or \( K_{k+2} \).

We first obtain the basic property of \( k \)-trees dealing with degrees. For this, we need to establish the following lemma.

**Lemma 2.3.** Every \( k \)-connected, \((p, q)\)-graph \( G \) with \( p \geq k + 1 \) and \( q = (kp - \frac{k(k+1)}{2}) \), has at least \( k + 1 \) vertices, whose degrees do not exceed \( 2k - 1 \).

**Proof.** Since \( G \) is \( k \)-connected, \( \deg v_i \geq k \) for all \( v_i \) in \( V(G) \). Let \( t \) be the number of vertices in \( G \), whose degrees are at most \( 2k - 1 \). Consequently, \( G \) contains \( p - t \) vertices of degrees at least \( 2k \). Immediately, we have

\[
\sum_{i=1}^{p} \deg v_i \geq tk + (p - t)2k. \tag{1}
\]

On the other hand, by the handshaking theorem, we have

\[
\sum_{i=1}^{p} \deg v_i = 2q = 2(kp - \frac{k(k+1)}{2}). \tag{2}
\]

From equations (1) and (2), we have

\[
2kp - k(k+1) \geq tk + (p - t)2k.
\]

This shows that \( t \geq k + 1 \) and hence, \( G \) contains at least \( k + 1 \) vertices, whose degrees do not exceed \( 2k - 1 \). \( \square \)

The direct consequence of Lemma 2.3 is the following result. Moreover, for \( k = 1 \), this result extends the property of trees (**Corollary 4.1 (a) p.34, [4])**.

**Corollary 2.4.** Every \( k \)-tree of order at least \( k + 1 \), has at least \( k + 1 \) vertices, whose degrees do not exceed \( 2k - 1 \).

**Proof.** Let \( G \) be a \( k \)-tree of order \( p \geq k + 1 \). By Theorem 2.1, \( G \) is a triangulated, \( k \)-connected graph of size \((kp - \frac{k(k+1)}{2})\). From Lemma 2.3, the result follows. \( \square \)

Next, we show that the bound given in **Corollary 2.4**, is the best possible by constructing below a \( k \)-tree \( G \) with exactly \( k + 1 \) vertices, whose degrees do not exceed \( 2k - 1 \). Let \( G \) be a graph consists of \( K_{k+1} \cup \overline{K_{k+1}} \), with all the possible additional edges \( u_iv_j \) for \( i \neq j \), where \( u_i \) and \( v_j \) are the vertices in \( K_{k+1} \) and \( \overline{K_{k+1}} \), respectively (for \( 1 \leq i, j \leq k+1 \)). Now, we observe that \( G \) is a \( k \)-tree of order \( 2k + 2 \) and it contains \( k + 1 \) vertices of degree \( k \) and \( k + 1 \) vertices of degree \( 2k \).
**Definition 2.5.** Let $G$ be a graph of order $p$. A vertex $v$ in $G$ is called a full-vertex if $\deg v = p - 1$.

For example, $K_k + \overline{K}_{p-k}$ (for $k < p$), is a $k$-tree of order $p$, containing exactly $k$ full-vertices. We now obtain a characterization of $k$-trees containing at least one full-vertex.

**Theorem 2.6.** Let $G$ be a graph of order $p \geq k + 1$. Then $G$ is a $k$-tree containing a full-vertex if and only if $G$ is isomorphic to $K_1 + H$, where $H$ is a $(k - 1)$-tree of order $p - 1$.

**Proof.** Suppose that $G$ is a $k$-tree, containing a full-vertex $v$. By Theorem 2.1, $G$ is a $k$-connected, triangulated graph of size $(kp - \frac{k(k+1)}{2})$. Let $\langle |v| \rangle \cong K_1$. Since $\deg v = p - 1$ in $G$, the removal of $v$ from $G$ certainly reduces its connectivity by one, without affecting its triangularity property and further, we have

$$|E(G-v)| = (kp - \frac{k(k+1)}{2}) - (p - 1) = (k - 1)(p - 1) - \frac{k(k-1)}{2}.$$ 

From Theorem 2.1, $G - v$ is a $(k - 1)$-tree of order $p - 1$. However, we see that $G$ is isomorphic to $K_1 + (G - v)$.

Conversely, assume that $G$ is isomorphic to $K_1 + H$, where $H$ is a $(k - 1)$-tree of order $p - 1$. Since $\deg v = p - 1$ in $G$, it follows that $H$ is isomorphic to $G - v$. Consequently, $G = K_1 + (G - v)$ is a $k$-connected, triangulated graph of size $(kp - \frac{k(k+1)}{2})$. By Theorem 2.1, $G$ is a $k$-tree. \qed

Repeated application of Theorem 2.6, yields the general criterion for $k$-trees containing at most $k$ full-vertices.

**Corollary 2.7.** Let $G$ be a graph of order $p \geq k + 1$. Then $G$ is a $k$-tree containing $t$ full-vertices $(1 \leq t \leq k)$ if and only if $G$ is isomorphic to $K_t + T_{p-t}$, where $T_{p-t}$ is a $(k-t)$-tree of order $p - t$ and $T_{p-k}$ is a forest.

### 3. Helly-property on $k$-paths

We begin with the notion of $m$-walk for $m \geq 2$, which extends the concept of a walk (i.e., 1-walk) introduced by Beineke and Pippert.[1]

**Definition 3.1.** (1). A $m$-walk for $m \geq 1$, in a graph $G$, denoted by $W(K_m^0, K_m^n); \ n \geq 0$, is an alternating finite sequence of its distinct cliques $K_m$ and $K_{m+1}$ of the form:

$(K_m^0, K_{m+1}^1, K_m^1, K_{m+1}^2, \ldots, K_m^{n-1}, K_{m+1}^n, K_m^n)$, beginning and ending with the cliques $K_m^0$ and $K_m^n$, respectively such that for each $i$ $(1 \leq i \leq n)$, $K_{m+1}^i = K_m^{i-1} \cup K_m^i$ and $K_m^{i-1} \cap K_m^i = K_{m-1}^i$.

(2). A $m$-walk $W(K_m^0, K_m^n); \ n \geq 0$, is called a $m$-path if all its cliques
$K^0_m, K^1_m, \ldots, K^n_m$ and $K^1_{m+1}, K^2_{m+1}, \ldots, K^n_{m+1}$ are distinct. The length of a $m$-path, is the number of occurrences of cliques $K^1_{m+1}$ in it. For example, any clique $K^1_m$ is a trivial $m$-path; $K^1_{m+1}$ is a nontrivial $m$-path of length 1; $K^1_m + K^1_2$ is a nontrivial $m$-path of length 2.

In Figure 2, the anatomy of a 2-path is shown.

Let $\Pi = \{J_i : i \in I\}$ be a family of subsets of a finite set $S$ (where $I$ denotes the index set). Then $\Pi$ is said to satisfy the Helly-property if $J_i \cap J_j \neq \emptyset$ for all $i, j$ in $I$, implies that $\cap_{k \in I} J_k \neq \emptyset$.

For example, $\Pi = \{J_1, J_2, J_3\}$, where the nontrivial paths: $J_1 = abc$; $J_2 = cba$; $J_3 = adb$, of the tree $K^1_{1,3}$ as shown in Figure 3.

Notice that every two paths in $\Pi$ have a nontrivial intersection, but there is no common nontrivial path for all three paths in $\Pi$.

We now establish the Helly-property for a family of nontrivial $k$-paths of a $k$-tree.

**Proposition 3.2.** Let $\Pi = \{J_i : i \in I\}$ be a finite family of nontrivial $k$-paths of a $k$-tree. If every three $k$-paths $J_i, J_j, J_k$ for $i, j, k \in I$, have a nontrivial intersection, then $\cap_{n \in I} J_n$ is a nontrivial intersection.

**Proof.** Let $G$ be a $k$-tree. We prove the result by induction on the number of nontrivial $k$-paths of $G$. Assume that $\cap_{n \in I} J_n$ is isomorphic to $W$, 5
where \(|J| = t < |I|\); \(J\) is an index set, is a nontrivial \(k\)-path of \(G\).

If \(J_{t+1}\) has no nontrivial intersection with \(W\), then there exist always three \(k\)-paths \(J_{t+1}, J_t, \) and \(J_{t-1}\) of \(G\), which have no nontrivial intersection. (In fact, for \(k = 1\), this fact is illustrated in Figure 4). This is a contradiction to the hypothesis. Hence, the desired property is proved.

\[ \]

4. Planarity and Clique-neighbourhoods

The *neighbourhood* of a vertex \(u\) in a graph \(G\) is the set \(N(u)\) consisting of all the vertices, which are adjacent to \(u\). A vertex \(u\) is *simplicial* if \(N(u)\) induces a clique in \(G\).

**Definition 4.1.** For any clique \(K_p\) of a graph \(G\) with vertices \(u_1, u_2, u_3, \ldots, u_p\), the \(K_p\)-neighbourhood, denoted by \(N(K_p)\) is \(\cap_{i=1}^{p} N(u_i)\).

Notice that 1-trees (i.e., trees) are obviously planar. The maximal outerplanar graphs are the special class of 2-trees. The triangulated, maximal planar graphs are restricted family of 3-trees. All nontrivial 4-trees (other than \(K_4\)) and \(k\)-trees \((k \geq 5)\) are nonplanar. To study (outer)planarity, let us first establish the following lemma.

**Lemma 4.2.** Let \(G\) be a \(k\)-tree of order \(\geq k + 1\). For any clique \(K_k\) in \(G\),

\(a\). \(N(K_k) \neq \emptyset\).

\(b\). \(N(K_k)\) is an independent set.
Proof. To prove (a), we use the induction on order \( p \geq k + 1 \) of \( G \). If \( p = k + 1 \), then \( G = K_{k+1} \). Obviously, \( |N(K_k)| = 1 \) for any clique \( K_k \) in \( G \) and hence the result is obvious. We assume that the result holds for any \( p : k + 2 \leq p \leq n \). Let \( G \) be a \( k \)-tree with \( p = n + 1 \). Then by Definition 1.1, \( G \) contains a simplicial vertex \( u \) of degree \( k \) and \( G - u \) is a \( k \)-tree of order \( n \). By induction hypothesis, \( N(K_k) \neq \emptyset \) for any clique \( K_k \) in \( G - u \). Let \( N(u) = \{u_1, u_2, \ldots, u_k\} \) and \( N(u) \) is isomorphic to \( K_k \). Consider any clique \( K'_k \) of \( G \) with \( V(K'_k) = \{u\} \cup (N(u) - \{u_i\}) \) for \( 1 \leq i \leq k \). Immediately, we observe that \( N(K'_k) = \{u_i\} \). Thus, \( N(K'_k) \neq \emptyset \). By induction, the result follows for all \( p \geq k + 1 \).

To prove (b), if possible, we assume that for some clique \( K_k \) in \( G \), \( N(K_k) \) is not independent. Then \( G \) contains at least two vertices \( u \) and \( v \) in \( N(K_k) \) such that \( u \) and \( v \) are adjacent in \( G \). This shows that \( \langle N(u) \cup \{u, v\} \rangle \) is isomorphic to \( K_{k+2} \) in \( G \). This is not possible (by Theorem 2.1), because \( G \) is a \( k \)-tree. \qed

In [5], it is proved that any graph \( G \) of order \( \geq 3 \), is maximal outerplanar if and only if \( G \) is 2-connected, triangulated and outerplanar. Next, we present another characterization of a maximal outerplanar graph involving 2-trees and \( K_2 \)-neighbourhoods.

**Proposition 4.3.** Let \( G \) be a graph of order \( \geq 3 \). Then \( G \) is maximal outerplanar if and only if \( G \) is a 2-tree and for any complete graph \( K_2 \) of \( G \), \( \langle N(K_2) \rangle \) is either \( K_1 \) or \( 2K_1 \).

*Proof.* Suppose that \( G \) is maximal outerplanar. Immediately, \( G \) is 2-connected, triangulated and outerplanar. Since \( G \) is outerplanar, \( G \) is \( K_4 \)-free. By Theorem 2.1 with \( k = 2 \), \( G \) is a 2-tree. On contrary, assume that \( |N(K_2)| \geq 3 \) for some complete graph \( K_2 \) of \( G \). Let \( x, y \) and \( z \) be the vertices in \( N(K_2) \). Consequently, \( \langle u, v, x, y, z \rangle \) isomorphic to \( K_2 + 3K_1 \) appears in \( G \). But \( K_2 + 3K_1 \) contains a subgraph isomorphic to \( K_{2,3} \) and hence \( G \) is not outerplanar. This leads to a contradiction. So, \( |N(K_2)| \leq 2 \) for each complete graph \( K_2 \) of \( G \). From Lemma 4.1 with \( k = 2 \), we have \( |N(K_2)| \geq 1 \) and \( \langle N(K_2) \rangle \) is either \( K_1 \) or \( 2K_1 \). Necessity is thus proved.

It is easy to prove the converse. \qed

The immediate consequence of the above proposition is Corollary 11.9 (a) of [4, p. 107]. Certainly, this bound can be improved for nonouterplanar, 2-trees.

**Corollary 4.4.** Every 2-tree other than maximal outerplanar, has at least three vertices of degree 2.

*Proof.* Follows from the immediate consequence of Proposition 4.3. \qed
Notice that a maximal planar graph need not be triangulated. For example, \( C_4 + 2K_1 \) is maximal planar but not triangulated.

**Proposition 4.5.** Let \( G \) be a triangulated graph of order \( \geq 4 \). Then \( G \) is maximal planar if and only if \( G \) is a 3-tree and for any triangle \( K_3 \) in \( G \), \( \langle N(K_3) \rangle \) is either \( K_1 \) or \( 2K_1 \).

The proof follows on the similar arguments as used in the proof of Proposition 4.3, by using Theorem 2.1 with \( k = 3 \).

The following corollary is the immediate consequence of the above result.

**Corollary 4.6.** Every nonplanar 3-tree, has at least three vertices of degree 3.

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**References**


