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Path (or cycle)-trees with Graph Equations involving Line and Split Graphs

H. P. Patil*and V. Raja†

Abstract

H-trees generalizes the existing notions of trees, higher dimensional trees and k-ctrees. The characterizations and properties of both P_k-trees for $k \geq 4$ and C_n-trees for $n \geq 5$ and their hamiltonian property, dominations, planarity, chromatic and b-chromatic numbers are established. The conditions under which P_k-trees for $k \geq 3$ (resp. C_n-trees for $n \geq 4$), are the line graphs are determined. The relationship between path-trees and split graphs are developed.

Keywords: Cycle, Path, Tree, Connected graph, Coloring, Line graph, Split graph

Mathematics Subject Classification (2010): 05C10

1. Introduction

We follow Harary[5] for all terminologies related to graphs. Given a graph G, $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively and $\bar{G}$ denotes the *complement* of G. P_n and C_n denote a *path* of n vertices and *cycle* of n vertices, respectively. For any connected graph G, nG denotes the graph with n components, each being isomorphic to G. For any two disjoint graphs G and H, $G + H$ denotes the *join* of G and H.[5] A *tree* is a connected graph without cycles. A star is a tree $K_{1,n}$ for $n \geq 1$. A graph G is *n-connected* if the removal of any m vertices for $0 \leq m < n$, from G results in neither a disconnected graph nor a trivial graph. A graph G is *triangulated* if every cycle of length strictly greater than 3 possesses a chord; that is, an edge joining two nonconsecutive vertices of the cycle. Equivalently, G does not

*Department of Mathematics, Pondicherry University; hpppondy@gmail.com
†Research Scholar, Pondicherry University; vraja.math@gmail.com

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contain an induced subgraph isomorphic to C_n for $n > 3$. A graph G is *n-degenerate* for $n \geq 0$ if every induced subgraph of G has a vertex of degree at most n.

2. Structure of H-trees

Notice that trees are equivalently defined by the following recursive construction rule:
Step 1. A single vertex K_1 is a tree.
Step 2. Any tree of order $n \geq 2$, can be constructed from a tree Q of order $n - 1$ by inserting an n^{th} - vertex and joining it to any vertex of Q.

In [10], the above tree-construction procedure is extended by allowing the base to be any graph. It is natural that a connected graph, which is not a tree possesses a structure that reflects like a tree and its recursive growth starts from any graph. In other words, for any given graph H, there is associated another graph, we call H-tree that is constructed as follows.

Definition 2.1. *Let H be any graph of order k. An H-*tree*, denoted by $G\langle H\rangle$, is a graph that can be obtained by the following recursive construction rule:*
Step 1. *H is the smallest H-tree.*
Step 2. *To an H-tree $G\langle H\rangle$ of order $n \geq k$, insert an $(n + 1)^{th}$-vertex and join it to any set of k distinct vertices: $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ of $G\langle H\rangle$, so that the induced subgraph $\langle\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}\rangle$ is isomorphic to H.*

For example, $K_{1,3}$-tree of order 8 is shown in Figure 1.

Remark 2.2. ***1.*** *The notion of K_1-trees is the usual concept of trees.*
2. *The notion of K_2-trees is equivalent to the notion of 2-trees, which is studied in [7]. Actually, they form a special subclass of planar graphs. In fact, the maximal outerplanar graphs are the only outerplanar K_2-trees.*
3. *The notion of K_k-trees is equivalent to the notion of k-trees[2, 7] and they form actually a family of k-connected, triangulated and K_{k+2}-free graphs of order $\geq k + 1$.*
4. *The notion of $\overline{K_k}$-trees is equivalent to the concept of k-ctrees[9] and they form a family of k-degenerate and triangle-free graphs of order $p \geq 2k$ and size $k(p - k)$.*

The development in the class of H-trees is motivated by the notion of k-trees[2, 7] or k-ctrees[9] and their applications in the area of reliability of communication networks, have generated much interest from an algorithmic (or theoretical) point of view.

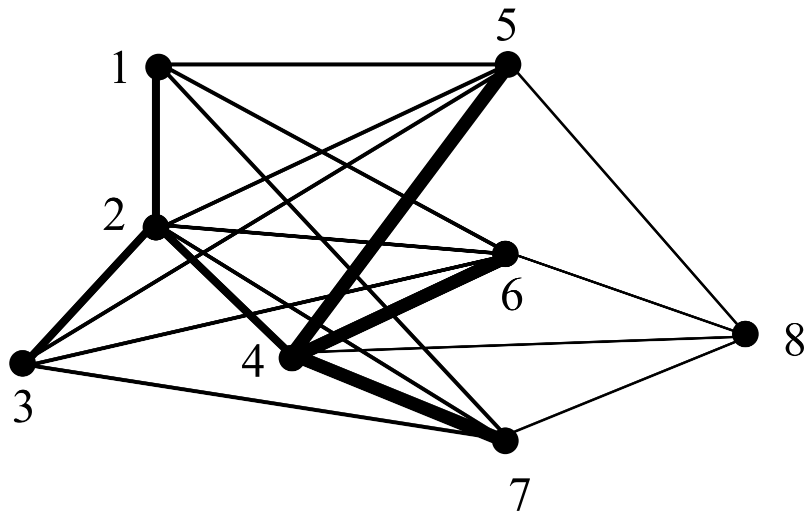

Figure 1: $K_{1,3}$-tree of order 8

Definition 2.3. *A graph F is called a H-*tree *if there exists a graph H such that F is isomorphic to $G\langle H\rangle$.*
Equivalently, a H-tree $G\langle H\rangle$ of order $\geq k+1$, (where $|H| = k$) can be reduced to H by sequentially removing the vertices of degree k from $G\langle H\rangle$.

For a vertex v of a graph G, a *neighbour* of v is a vertex adjacent to v in G. The *neighbourhood $N(v)$* of v is the set of all neighbours of v.

The following result is a simple characterization of H-trees involving their hereditary subgraphs and is simply the restatement of Definition 2.1.

Proposition 2.4. *Let H be any graph of order k. Then G is a H-tree of order $\geq k+1$ if and only if G contains a vertex v of degree k such that $N(v)$ induces H in G and $G - v$ is a H-tree.*

An immediate consequence of the above result is the following corollary.

Corollary 2.5. *For any graph H of order k and size m, let G be a H-tree of order $p \geq k$. Then*

1. $|E(G)| = m + k(p-k)$.
2. *G contains a subgraph isomorphic to $H + 2K_1$, provided $p \geq k+2$.*
3. *If H has t triangles, then the number of triangles in G is $t + m(p-k)$.*

3. Properties and Characterizations

Definition 3.1. *A graph F is called a P_k-*tree (or path-tree) *if there exists a path P_k of order k such that F is isomorphic to $G\langle P_k\rangle$.*

We define similarly, a C_k-tree (or cycle-tree). Generally speaking, every P_k (resp. C_k)-tree of order $\geq k+1$, can be reduced to P_k (resp. C_k) by sequentially removing the vertices of degree k from P_k (resp. C_k)-tree.

In [10], the following general open-problem is proposed for further research.

Open Problem 1. Characterize the class of star-trees $G\langle K_{1,n}\rangle$ for $n \geq 2$.

We now characterize path-trees $G\langle P_k\rangle$ for $k \geq 4$.

Theorem 3.2. *A graph G of order $p \geq k+1$, is a P_k-tree if and only if G is isomorphic to $P_k + (p-k)K_1$.*

Proof. Suppose that G is isomorphic to $P_k+(p-k)K_1$. Then G contains the vertices $v_1, v_2, \ldots, v_{p-k}$, each of degree k such that $N(v_i)$ induces P_k in G for $1 \leq i \leq p-k$. By repeated removal of each vertex v_i from G reduces to P_k. Hence, G is a P_k-tree.

We prove the converse by induction on p.
If $p = k+1$, then by the recursive definition, a P_k-tree G of order $k+1$, is isomorphic to $P_k + K_1$, which is obviously true.
Assume that the result is true for any positive integer $m < p$. Next, we consider a P_k-tree of order p. By Proposition 2.4 with $H = P_k$, G contains a vertex v of degree k such that $N(v)$ induces P_k in G and $G-v$ is again a P_k-tree of order $p-1$. By induction hypothesis, $G-v$ is isomorphic to $P_k + (p-k-1)K_1$. Consequently, $G-v$ is the join of two disjoint graphs : P_k and $I = (p-k-1)K_1$.

Suppose that v is adjacent to each vertex of P_k in G. Then the result follows immediately. Otherwise, v is adjacent to at least one vertex of I in G. Moreover, $deg\ (v) = k$ in G. There exist two disjoint nonempty sets : A and B such that $A \subseteq P_k$; $B \subseteq I$ with $A \cup B = N(v)$ and $|A| + |B| = k$.(Figure 2) We discuss four cases, depending on the cardinalities of A and B :
Case 1. $|A| = k-1$ and $|B| = 1$. Since $k \geq 4$, $\langle A\rangle$ contains at least one edge, say $e = xy$. Then for any vertex u of B, we have a triangle $uxyu$ in $N(v)$, which is not possible.
Case 2. $|A| = k-2$ and $|B| = 2$. Immediately, we have $|A| \geq 2$ (because $k \geq 4$).
There are two possibilities for discussion.
2.1. Suppose that A is independent. Certainly, there are two non-adjacent vertices x and y in A. Let us consider $B = \{a, b\}$. Immediately,

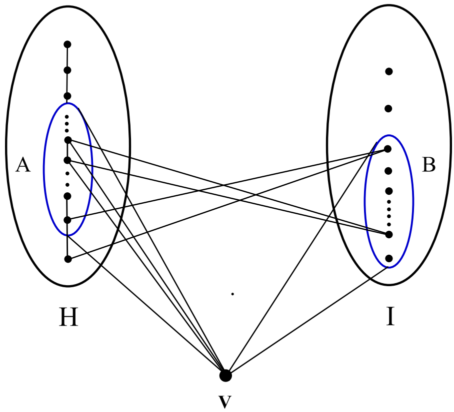

Figure 2:

$\langle\{x, y, a, b\}\rangle$ is isomorphic to C_4 and it appears in $\langle N(v)\rangle$. This is impossible.
2.2. Suppose that A is non-independent. Then $\langle A\rangle$ contains at least one edge. In this situation, Case 1 repeats.
Case 3. $|A| = 1$ and $|B| = k - 1$. It is easy to see that $\langle N(v)\rangle$ is a star $K_1 + \overline{K_{k-1}}$ and this is not possible.
Case 4. $|A| \geq 2$ and $|B| \geq 3$.
We discuss two possibilities, depending on A :
4.1. Suppose that A is non-independent. Then Case 1 repeats.
4.2. Suppose that A is independent. Then Case 2 repeats.

In each of the above cases, we see that $\langle N(v)\rangle$ is not isomorphic to P_k. This is a contradiction. □

In [7], it is shown that the notion of C_3-trees are equivalent to the family of 3-trees and it is also proved that this class of graphs are equivalent to the family of 3-connected, triangulated and K_5-free graphs of order ≥ 4. Further, it is noticed that the graphs in the class of C_4-trees have highly irregular structure. In fact, it is hard to find a characterization of C_4-trees. We first propose the following problem for further research.

Open Problem 2. Characterize the class of C_4-trees.

The following theorem is a characterization of C_k-trees for $k \geq 5$ and its proof is quite similar to that of Theorem 3.2, with the replacement of P_k by C_k.

Theorem 3.3. *A graph G of order $p \geq k + 1$, is a C_k-tree if and only if G is isomorphic to C_k + (p-k)K_1.*

The immediate consequence of theorems 3.2 and 3.3 is the following corollary.

Corollary 3.4. *1.* $\chi(G\langle P_k\rangle) = 3$ *for* $k \geq 4$.
2. $\chi(G\langle C_k\rangle) = \begin{cases} 3 & \text{if } k \geq 6 \text{ and is even.} \\ 4 & \text{if } k \geq 5 \text{ and is odd.} \end{cases}$

Proposition 3.5. *Let* $G\langle H\rangle$ *be a* H*-tree of order* $p \geq k+1$*, where* H *is either* P_k ; $k \geq 4$ *or* H *is* C_k : $k \geq 5$.
1. $G\langle H\rangle$ *is hamiltonian if and only if* $p \leq 2k$.
2. $G\langle H\rangle$ *is planar if and only if* $p \leq k+2$.

Proof. By theorems 3.2. and 3.3, $G\langle H\rangle$ is isomorphic to $H + (p-k)K_1$.
1. Assume that $G\langle H\rangle$ is hamiltonian and on contrary, $p \geq 2k+1$. Since $|V(H)| = k$, we have $|(p-k)K_1| = k+1$. Consider $S = V(H)$. Then $G - S$ is isomorphic to $(p-k)K_1$ and hence the number of components of $(G-S) \geq k+1$. This implies that $G\langle H\rangle$ is not hamiltonian. So, $p \leq 2k$. To prove the converse, it is sufficient to obtain a Hamilton-cycle in $G\langle H\rangle$, where $G\langle H\rangle$ is isomorphic to $H + tK_1$ for $1 \leq t \leq k$. Let $V(H) = \{u_1, u_2, \ldots, u_k\}$ and $V(tK_1) = \{v_1, v_2, \ldots, v_t\}$. Since $k \geq t$, we have $(k-t) = m \geq 0$. Immediately, a Hamilton cycle :
$u_1, u_2, \ldots, u_{m+1}, v_1, u_{m+2}, v_2, u_{m+3}, \ldots, v_{t-1}, u_k, v_t, u_1$ appears in $G\langle H\rangle$ (Figure 3). Hence, H-tree is hamiltonian.

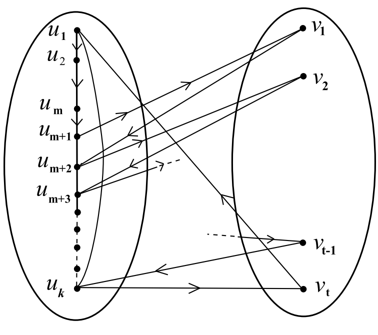

Figure 3: Hamilton-cycle

2. Assume that $G\langle H\rangle$ is planar and on contrary, $p \geq k+3$. Immediately, we observe that $(H + 3K_1) \subseteq G\langle H\rangle$. Since $K_{3,3}$ appears as an induced subgraph in $(H + 3K_1)$, it follows that $K_{3,3}$ appears as a forbidden subgraph in $G\langle H\rangle$ and hence by Kuratowski theorem, $G\langle H\rangle$ is not planar. This is a contradiction to our assumption. Hence, $p \leq k+2$.
It is easy to prove the converse. □

4. Dominations and b-coloring

For any graph G, $\gamma(G)$ denotes the *domination number* of G. A *Roman domination function (in short, RDF)* on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of this function is $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on G is the *Roman domination number of G* and is denoted by $\gamma_R(G)$.[3] The following result gives both the domination and Roman domination numbers of path-trees and cycle-trees and its proof is obvious.

Proposition 4.1. *Let $G\langle H\rangle$ be a H-tree of order $p \geq k+1$, where H is either P_k for $k \geq 4$ or C_k for $k \geq 5$. Then*

1. $\gamma(G\langle H\rangle) = \begin{cases} 1 & \text{if } p = k+1. \\ 2 & \text{otherwise.} \end{cases}$

2. $\gamma_R(G\langle H\rangle) = \begin{cases} 2 & \text{if } p = k+1. \\ 3 & \text{if } p = k+2. \\ 4 & \text{otherwise.} \end{cases}$

The *b-chromatic number b(G)* of a graph G is the largest integer k such that G admits a proper k-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring of G is a *b-coloring of G*[6] It is shown in [6] that for any path P_k and a cycle C_k for $k \geq 5$, $b(P_k) = b(C_k) = 3$.

Next, we determine the b-chromatic number of the path-trees and cycle-trees. For this, we establish the following lemma.

Lemma 4.2. *In any b-coloring of a graph $H + (p-k)K_1$, where H is any graph of order k and $p \geq k+1$, all the vertices of $(p-k)K_1$ receive the same color.*

Proof. Let $u_1, u_2, \ldots, u_k$, be the vertices of H and let $v_1, v_2, \ldots, v_{p-k}$ be the vertices of I, where $I = (p-k)K_1$. If $p = k+1$, then $|I| = 1$. The result obvious.
If $p \geq k+2$, then $|I| \geq 2$. If possible, then assume that in some b-coloring of $H + I$, the vertices of I receive $q \geq 2$ different colors, say $c_1, c_2, \ldots, c_q$. Since I is independent and each vertex of I is adjacent to all vertices of H, it follows that there is no color dominating vertex corresponding to the colors c_i $(1 \leq i \leq q)$ in $H + I$. This is not possible in any b-coloring of $H + I$, because each color class has at least one color dominating vertex. □

Theorem 4.3. *Let $G\langle H\rangle$ be a H-tree of order $p \geq k+1$, where H is either a path P_k for $k \geq 4$ or a cycle C_k for $k \geq 5$. Then*

1. $b(G\langle P_k\rangle) = \begin{cases} 3 & \textit{if } k = 4. \\ 4 & \textit{otherwise.} \end{cases}$

2. $b(G\langle C_k\rangle) = 4$.

Proof. By theorems 3.2, and 3.3, we have $G\langle H\rangle$ is isomorphic to $H+I$, where $I = (p-k)K_1$. We discuss two cases depending on k in **(1)** :
Case 1. Assume that $k = 4$. Since $b(P_k) = 2$ and from Lemma 4.2, all the vertices of I receive a single color, it follows that $b(G\langle P_k\rangle) \leq 3$.
To achieve the lower bound, color P_k properly by using the colors 1 and 2 and next, assign the color 3 to each vertex of I. Thus, we have $b(G\langle P_k\rangle) = 3$.
Case 2. Assume that $k \geq 5$. Since it is shown in [6] that $b(P_k) = 3$ and all the vertices of I receive a single color, it follows that $b(G\langle P_k\rangle) \leq 4$.
To achieve the lower bound, color P_k properly by all three colors 1, 2 and 3 and next, assign color 4 to each vertex of I. Thus, we have $b(G\langle P_k\rangle) = 4$.
For **(2)**, since $b(C_k) = 3$ and all the vertices of I receive a single color, it follows that $b(G\langle C_k\rangle) \leq 4$. To achieve the lower bound, color C_k properly by using all three colors 1, 2, 3, and next, assign the color 4 to each vertex of I. Thus, we have $b(G\langle C_k\rangle) = 4$. □

5. Line graphs and path (or cycle)-trees

In this section, we determine all the graphs, whose line graphs are either P_k-trees or C_k-trees for $k \geq 3$. We begin with the definition of line graph. The *line graph* $L(G)$ of a graph G, is the graph whose vertex set is the edge set of G and in which two vertices are adjacent, if the corresponding edges are adjacent in G.[5] Beineke [5, p.75] has shown that a graph is a line graph if and only if it has none of nine specified graphs as induced subgraphs, including $K_{1,3}$, $(K_1 \cup K_2) + 2K_1$ and $(C_5 + K_1)$. The problem of obtaining all the graphs, whose line graphs are P_k-trees for $1 \leq k \leq 2$, is already done in [8, 9] and therefore, we solve the problem for $k \geq 3$.

Proposition 5.1. *A P_k-tree of order $p \geq k+1$; $k \geq 3$, is the line graph of a graph G if and only if both the following conditions hold:*
1. $k = 3$ *; G is either $(K_2 + 2K_1)$ or a triangle with exactly one end-edge at some vertex.*
2. $k = 4$ *; G is a triangle with exactly two end-edges, one at some vertex.*

Proof. We first show that G is connected. If not, then $L(G)$ is disconnected and by Definition 2.1 with $H = P_k$, $L(G)$ is not a P_k-tree. This is a contradiction. Since $L(G)$ is a P_k-tree of order $p \geq k+1$ and $k \geq 3$, by Theorem 3.2, P_k-tree T is isomorphic to $P_k + (p-k)K_1$. Suppose $k \geq 5$. Then T contains a subgraph F isomorphic to $P_5 + K_1$. Since

$F \subseteq T$ and T is $L(G)$, immediately a forbidden subgraph isomorphic to $K_{1,3}$ appears in $L(G)$. This is impossible and it shows that $k \leq 4$. Thus, either $k = 3$ or $k = 4$.

Case 1. Assume that $k = 3$. Further, we observe that $p \leq 5$; since otherwise, $P_3 + 3K_1$ appears in T and $L(G)$ contains a forbidden subgraph $K_{1,3}$.

We discuss two possibilities depending on p.

1.1. If $p = 4$, then $L(G) = P_3 + K_1$ and hence G is isomorphic to triangle with exactly one end-edge at some vertex.

1.2. If $p = 5$, then $L(G) = P_3 + 2K_1$ and therefore, G is isomorphic to $K_2 + 2K_1$.

Case 2. Assume that $k = 4$. Moreover, we observe that $p = 5$; since otherwise, $P_4 + 2K_1$ appears in T and $L(G)$ contains a forbidden subgraph isomorphic to $(K_1 \cup K_2) + 2K_1$. Since $k = 4$ and $p = 5$, it follows that $L(G) = P_4 + K_1$ and hence G is isomorphic to a triangle with exactly two end-edges, one at some vertex.

It is easy to prove the converse. □

Finally, we determine all the graphs whose line graphs are C_k-trees for $k \geq 3$. However for $k = 3$, this problem is solved in [8] and now we solve this problem, for $k \geq 4$.

Proposition 5.2. *There are only two graphs whose line graphs are C_k-trees for $k \geq 4$. These graphs are $K_2 + 2K_1$ and K_4.*

Proof. Suppose that $L(G)$ is a C_k-tree of order $p \geq k+1$; $k \geq 4$. Clearly, G is connected. Assume that $k \geq 5$. Then $p \geq 6$ and immediately, $L(G)$ contains a subgraph F isomorphic to $C_k + K_1$. There are two possibilities, depending on k :

1. If $k = 5$, then $F = C_5 + K_1$ is a forbidden subgraph of $L(G)$.

2. If $k \geq 6$, then F contains a forbidden subgraph isomorphic to $K_{1,3}$.

In either case, we arrive at a contradiction. Hence, $k = 4$. Furthermore, we observe that $p \leq 6$; since otherwise, $L(G)$ contains a subgraph F isomorphic to $C_4 + 3K_1$. It is easy to check that a forbidden subgraph isomorphic to $K_{1,3}$ appears in F and hence in $L(G)$.

Next, we discuss two possibilities depending on p.

1. If $p = 5$, then $L(G) = C_4 + K_1$ and hence $G = K_2 + 2K_1$.

2. If $p = 6$, then $L(G) = C_4 + 2K_1$ and hence $G = K_4$. □

6. Relation between P_k-trees and split graphs

A nonempty subset S of $V(G)$ is an *independent set $I(G)$ in a graph G* if no two vertices of S are adjacent in G. A nonempty subset K of $V(G)$ is a *complete set $K(G)$ in G* if every two vertices of K are adjacent in G. The concept of a split graph appears in [4]. A *split graph* is defined to be a graph G, whose vertex set $V(G)$ can be partitioned into a complete set K and an independent set I such that $G = (K \cup I \cup (K, I))$, where (K, I) denotes a set of edges xy for $x \in K$ and $y \in I$. Notice that the partition $V(G) = K \cup I$ of a split graph G will not be unique always. Let us denote a split graph G with its bipartition (K, I) by $G(K, I)$. In [4, Theorem 6.3], it is proved that a graph G is a split graph if and only if G contains no induced subgraph isomorphic to $2K_2, C_4$ or C_5.

Now, we obtain the conditions under which P_k-trees to be the split graphs. We begin with the following definitions.

Definition 6.1. *A* double-star $D(m, n)$ *for* $m, n \geq 1$ *; is a tree, obtained from a complete graph* K_2*, by joining* m *isolated vertices to one end of* K_2 *and* n *isolated vertices to the other end of* K_2.

Definition 6.2. *For any triangle* K_3 *with vertices* a, b *and* c*, there are three special families of* K_2*-trees as follows :*

1. *A* m-graph *for* $m \geq 1$*, denoted by* $T(m)$*, is a* K_2*-tree, obtained from* K_3*, by joining* m *isolated vertices to both* a *and* b *of* K_3.

2. *A* (m, n)-graph *for* $m, n \geq 1$*, denoted by* $T(m, n)$*, is a* K_2*-tree, obtained from* $T(m)$*, by joining* n *isolated vertices to both* b *and* c *of* K_3 *in* $T(m)$.

3. *A* (m, n, k)-graph *for* $m, n, k \geq 1$*, denoted by* $T(m, n, k)$*, is a* K_2*-tree, obtained from* $T(m, n)$*, by joining* k *isolated vertices to both* a *and* c *of* K_3 *in* $T(m, n)$.

Proposition 6.3. *A* P_k*-tree of order* $p \geq k + 1$*, is a split graph if and only if the following statements hold:*
1. $k = 1$*. There are only two split graphs:*
a) $G(K_1, \bar{K}_{p-1})$ *is a star* $K_1 + \bar{K}_{p-1}$.
b) $G(K_2, \bar{K}_{p-2})$ *is a double-star* $D(m, n)$*, where* $(m + n + 2) = p$; $m, n \geq 1$.

2. $k = 2$*. There are only two split graphs:*
a) $G(K_2, \bar{K}_{p-2})$ *is a* K_2*-tree* $K_2 + \bar{K}_{p-2}$.
b) $G(K_3, \bar{K}_{p-3})$ *is one of the following three* K_2*-trees :* $T(n_1)$ *for* $n_1 + 3 = p$; $T(n_1, n_2)$ *for* $n_1 + n_2 + 3 = p$ *and* $T(n_1, n_2, n_3)$ *for* $n_1 + n_2 + n_3 + 3 = p$.

3. $k = 3$*. Either* $G(K_2, \bar{K}_2)$ *or* $G(K_3, K_1)$ *is a* P_3*-tree* $P_3 + K_1$.
4. $k = 4$*.* $G(K_3, \bar{K}_2)$ *is a* P_4*-tree* $P_4 + K_1$.

Proof. Suppose that a P_k-tree of order $p \geq k + 1$, is a split graph of the form : $G(K, I)$. Immediately, $k \leq 4$; since otherwise, $2K_2$ appears as a forbidden subgraph in P_k.
We discuss three cases, depending on k.
Case 1. Assume that $k = 1$. Then P_k is K_1. Clearly, a P_k-tree T is a nontrivial tree. In this case, the star $K_1 + \bar{K}_{p-1}$ and double- stars $D(m, n)$ with $(m + n = p - 2$; $m, n \geq 1)$, are the only split graphs of the

form: $G(K_1, \bar{K}_{p-1})$ and $G(K_2, \bar{K}_{p-2})$, respectively; since otherwise, $2K_2$ appears immediately as a forbidden subgraph in T.
Case 2. Assume that $k = 2$. Then P_k is K_2. Clearly, the notion of K_2-tree is equivalent to the notion of 2-tree.[7] By (3) of Remark 2.2 (with $k = 2$), a K_2-tree T is 2-connected, triangulated and K_4-free. Consequently, the complete sets K in T are the only K_2 and K_3.
Next, there are two possibilities to discuss on K.
2.1. If $K = K_2$, then T is isomorphic to $K_2 + \bar{K}_{p-2}$, is the split graph of the type: $G(K_2, \bar{K}_{p-2})$.
2.2. If $K = K_3$, then one of the following types of K_2-trees : $T(n_1)$, with $n_1 + 3 = p$; $T(n_1, n_2)$ with $(n_1 + n_2 + 3) = p$ and $T(n_1, n_2, n_3)$ with $(n_1 + n_2 + n_3 + 3) = p$, is a split graph of the form: $G(K_3, \bar{K}_{p-3})$.
Case 3. Assume k such that $(3 \leq k \leq 4)$. Since $k \geq 3$, P_k contains P_3 as an induced subgraph. By (2) of Corollary 2.5, a P_k-tree of order $p \geq k + 2$, contains a subgraph isomorphic to $P_k + \bar{K}_2$. Immediately, a forbidden subgraph C_4 appears in $P_3 + \bar{K}_2$ and hence, in $P_k + \bar{K}_2$. This is a contradiction and hence proves that $p = k + 1$. Now, we discuss two possibilities.
3.1. $k = 3$. Then both K_3 and K_4 are the complete sets in a P_3-tree of order 4. This shows that P_3-tree $P_3 + K_1$ is a split graph either of the type: $G(K_2, \bar{K}_2)$ or $G(K_3, K_1)$.
3.2. $k = 4$. Then K_3 is the only complete set in a P_4-tree of order 5. This shows that P_4-tree $P_4 + K_1$ is a split graph of the type $G(K_3, \bar{K}_2)$.

It is easy to prove the converse. □

Open Problem 3. Determine the conditions under which the C_k-trees for $k \geq 3$, are the split graphs.

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