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Upper Vertex Triangle Free Detour Number of a Graph

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Abstract

For a graph G, the *x-triangle free detour set*, the *x-triangle free detour number*, the *minimal x-triangle free detour set*, the *upper x-triangle free detour number*, are defined and studied. Certain bounds are determined and the relation with the vertex triangle free detour number of a graph is found out. Some realization problems, properties related to the upper vertex detour number, the upper vertex detour monophonic number and the upper vertex geodetic number are also studied.

Keywords: Vertex triangle free detour set, Vertex triangle free detour number, Minimal vertex triangle free detour set, Upper vertex triangle free detour number.
Mathematics Subject Classification (2010): 05C12

1. Introduction

Let *graph* $G = (V, E)$ denote a finite undirected connected simple graph. For basic definitions and terminologies, we refer to Chartrand *et al.*[1] The concept of *triangle free detour distance* was introduced by Keerthi Asir and Athisayanathan.[3] A path P is called a *triangle free path* if no three vertices of P induce a triangle. For vertices u and v in a connected graph G, the *triangle free detour distance* $D_{\triangle f}(u, v)$ is the length of a longest $u-v$ triangle free path in G. A $u-v$ path of length $D_{\triangle f}(u, v)$ is called a *$u-v$ triangle free detour*.

The concept of *vertex detour number* of a graph was introduced and studied by Santhakumaran and Titus.[4] For any vertex x in a connected graph G, a set S of vertices of G is an *x-detour set* if each

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vertex v of G lies on an $x - y$ detour in G for some vertex y in S. The minimum cardinality of an x-detour set of G is defined as the *x-detour number* of G, denoted by $d_x(G)$ or simply d_x. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G. An x - detour set S_x is called a *minimal x-detour set* if no proper subset of S_x is an x-detour set. The *upper x-detour number*, denoted by $d_x^+(G)$, is defined as the maximum cardinality of a minimal x-detour set of G.

The concept of *vertex detour monophonic number* of a graph was introduced and studied by Titus et al. [8]. A *chord* of a path P is an edge joining two non - adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. A longest $u - v$ monophonic path is called an *$u - v$ detour monophonic path*. For any vertex x in a connected graph G, a set S of vertices of G is an *x-detour monophonic set* if each vertex v of G lies on an $x - y$ detour monophonic in G for some vertex y in S. The minimum cardinality of an x-detour monophonic set of G is defined as the *x-detour monophonic number* of G, denoted by $dm_x(G)$ or simply dm_x. An x-detour monophonic set of cardinality $dm_x(G)$ is called a dm_x-set of G. An x-detour monophonic set S_x is called a *minimal x-detour monophonic set* if no proper subset of S_x is an x-detour monophonic set. The *upper x-detour monophonic number*, denoted by $dm_x^+(G)$, is defined as the maximum cardinality of a minimal x-detour monophonic set of G.

The concept of *vertex geodetic number* of a graph was introduced and studied by Santhakumaran and Titus.[5] For any vertex x in a connected graph G, a set S of vertices of G is an *x-geodetic set* if each vertex v of G lies on an $x - y$ geodetic in G for some vertex y in S. The minimum cardinality of an x-geodetic set of G is defined as the *x-geodetic number* of G, denoted by $g_x(G)$ or simply g_x. An x-geodetic set of cardinality $g_x(G)$ is called a g_x-set of G. An x-geodetic set S_x is called a *minimal x-geodetic set* if no proper subset of S_x is an x-geodetic set. The *upper x-geodetic number*, denoted by $g_x^+(G)$, is defined as the maximum cardinality of a minimal x-geodetic set of G.

The concept of *triangle free detour number* was introduced and studied by Sethu Ramalingam *et al.*[6] A set $S \subseteq V$ is called a *triangle free detour set* of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The *triangle free detour number* $dn_{\triangle f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\triangle f}(G)$ is called a *triangle free detour basis* of G.

The concept of *vertex triangle free detour number* was introduced and studied by Sethu Ramalingam, Keerthi Asir and Athisayanathan [7]. For any vetex x in a connected graph G, a set $S \subseteq V$ is called a *x-triangle free detour set* of G if every vertex v in G lies on a $x - y$

triangle free detour in G for some vertex y in S. The *x-triangle free detour number* $dn_{\triangle f_x}(G)$ of G is the minimum order of its x-triangle free detour sets and any x-triangle free detour set of order $dn_{\triangle f_x}(G)$ is a *x-triangle free detour basis* of G. In this paper, we introduce *upper vertex triangle free detour number* in a connected graph G. Throughout this paper, G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1. [7] *For any vertex x in G, x does not belong to any $dn_{\triangle f_x}$-set of G.*

Theorem 1.2. *Let v be a vertex of a connected graph G. The following statements are equivalent:*
(i) v is a cut vertex of G.
(ii) There exist vertices u and w distinct from v such that v is on every $u - w$ path.
(iii) There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u - w$ path.[2]

Theorem 1.3. *If G is a connected graph with k end-blocks, then $dn_{\triangle f_x}(G) \geq k - 1$ for every vertex x in G.*

Theorem 1.4. *Let x be any vertex of a connected graph G.*
(i) Every end-vertex of G other than the vertex x (whether x is end-vertex or not) belongs to every x-triangle free detour set.
(ii) No cut vertex of G belongs to any x-triangle free detour set.[7]

Theorem 1.5. *Let G be a connected graph with cut vertices and let S_x be an x-triangle free detour set of G. Then every branch of G contains an element of $S_x \cup \{x\}$.[7]*

Theorem 1.6. *For every pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G with $d_x(G) = a$ and $dn_{\triangle f_x}(G) = b$.[7]*

Theorem 1.7. *For every pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G with $dn_{\triangle f_x}(G) = a$ and $dm_x(G) = b$.[7]*

Theorem 1.8. *For every pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G with $dn_{\triangle f_x}(G) = a$ and $g_x(G) = b$.[7]*

Theorem 1.9. *For any four positive integers a, b, c and d of with $2 \leq a \leq b \leq c \leq d$, there exists a connected graph G such that $d_x(G) = a$, $dn_{\triangle f_x}(G) = b$, $dm_x(G) = c$ and $g_x(G) = d$.[7]*

2. Upper Vertex Triangle Free Detour Number

Definition 2.1. *Let x be any vertex of a connected graph G. An x-triangle free detour set S_x is called a* minimal x-triangle free detour set *if no proper subset of* S_x *is an x-triangle free detour set. The* upper x-triangle free detour number, *denoted by* $dn^+_{\triangle f_x}(G)$*, is defined as the maximum cardinality of a minimal x-triangle free detour set of G.*

Remark 2.2. *For any vertex x in G, x does not belong to any minimal x-triangle free detour set of G.*

Proof. This follows from Theorem 1.1. □

Example 2.3. *For the graph G given in Figure 2.1, a minimal vertex triangle free detour sets and the upper vertex triangle free detour numbers are given in Table 2.1.*

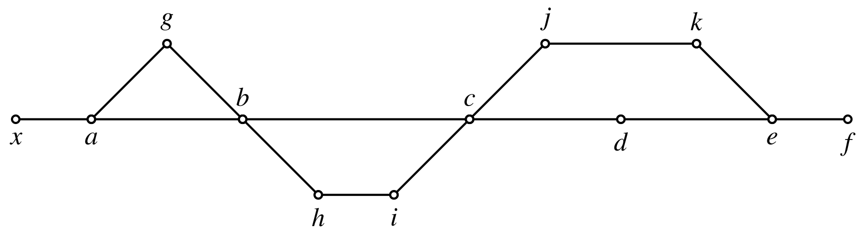

Figure 2.1 : G

For the graph G given in Figure 2.1, the sets $S_1 = \{d, f\}$, $S_2 = S_1 \cup \{g\}$, $S_3 = S_2 \cup \{h\}$ and $S_4 = S_3 \cup \{j\}$ are minimal x-detour set, minimal x-triangle free detour set, minimal x-detour monophonic set and minimal x-geodetic set respectively and hence $d^+_x(G) = 2$, $dn^+_{\triangle f_x}(G) = 3$, $dm^+_x(G) = 4$ and $g^+_x(G) = 5$. Thus the upper vertex detour number, upper vertex triangle free detour number, upper vertex detour monophonic number and upper vertex geodetic number of a graph G are distinct.

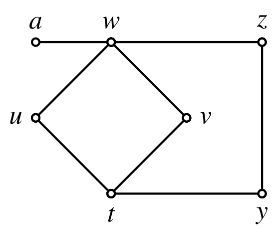

Figure 2.2 : G

Vertex t	Minimal $dn_{\triangle f_t}$-set	$dn^{+}_{\triangle f_t}(G)$
x	$\{g, d, f\}$	3
a	$\{x, g, d, f\}$	4
b	$\{x, g, d, f\}$	4
c	$\{x, g, d, f\}$	4
d	$\{x, g, f\}$	3
e	$\{x, g, d, f\}$	4
f	$\{x, g, d\}$	3
g	$\{x, d, f\}$	3
h	$\{x, g, d, f\}$	4
i	$\{x, g, d, f\}$	4
j	$\{x, g, d, f\}$	4
k	$\{x, g, d, f\}$	4

Table 2.1

Remark 2.4. *For any vertex x in a connected graph G, every minimum x-triangle free detour set is a minimal x-triangle free detour set, but the converse is not true. For the graph G given in Figure 2.2, $\{a, u, v\}$ is a minimal t-triangle free detour set but it is not a minimum t-triangle free detour set of G.*

Theorem 2.5. *Let x be any vertex of a connected graph G.*
(i) *Every end-vertex of G other than the vertex x(whether x is end-vertex or not) belong to every minimal x-triangle free detour set.*
(ii) *No cut vertex of G belongs to any minimal x-triangle free detour set.*

Proof. (i) Let x be any vertex of G. By Remark 2.2, x does not belong to any minimal x-triangle free detour set. So let $v \neq x$ be an end-vertex of G. Then v is the terminal vertex of an $x - v$ triangle free detour and v is not an internal vertex of any triangle free detour so that v belongs to every minimal x-triangle free detour set of G.

(ii) Let y be a cut vertex of G. Then by Theorem 1.2, there exists a partition of the set of vertices $V - \{y\}$ into subsets U and W such that for any vertex $u \in U$ and $w \in W$, the vertex y is on every $u - w$ path. Hence, if $x \in U$, then for any vertex w in W, y lies on every $x - w$ path so that y is an internal vertex of an $x - w$ triangle free detour. Let S_x be any minimal x-triangle free detour set of G. Suppose $S_x \cap W = \phi$. Let $w_1 \in W$. Since S_x is an x-triangle free detour set, there exists an element z in S_x such that w_1 lies in some $x - z$ triangle free detour $P : x = z_0, z_1, ..., w_1, ..., z_n = z$ in G. Then the $x - w_1$ subpath of P and $w_1 - z$ subpath of P both contain y so that P is not a path in G. Hence $S_x \cap W \neq \phi$. Let $w_2 \in S_x \cap W$. Then y is an internal vertex of an $x - w_2$ triangle free detour. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an $x - y$ triangle free detour is also lies on an

$x-w_2$ triangle free detour. Hence it follows that S is an x-triangle free detour set of G, which is a contradiction to S_x is a minimal x-triangle free detour set of G. Thus y does not belong to any minimal x-triangle free detour set. Similarly if $x \in W$, y does not belong to any minimal x-triangle free detour set. If $x = y$, then by Remark 2.2, y does not belong to any minimal x-triangle free detour set. □

Remark 2.6. *If x is an end-vertex of G, x does not belong to any minimal x-triangle free detour set by Remark 2.2.*

Theorem 2.7. *Let G be a connected graph with cut vertices and let S_x be a minimal x-triangle free detour set of G. Then every branch of G contains an element of $S_x \cup \{x\}$.*

Proof. Suppose that there is a branch B of G at a cut vertex v such that B contains no vertex of $S_x \cup \{x\}$. Then clearly, $x \in V-(S_x \cup V(B))$. let $u \in V(B)-\{v\}$. Since S_x is a minimal x - triangle free detour set, there is an element $y \in S_x$ such that u lies in some $x-y$ triangle free detour $P : x = u_0, u_1, ..., u, ..., u_n = y$ in G. By Theorem 1.2 the $x-u$ subpath of P and $u-y$ subpath of P both contain v, and it follows that P is not a path, contrary to assumption. □

Since every end-block B is a branch of G at some cut-vertex, it follows by Theorems 2.5 and 2.7 that every minimal x-triangle free detour set of G together with the vertex x contains at least one vertex from B that is not a cut-vertex. Thus the following corollaries are consequences of Theorem 2.7.

Corollary 2.8. *If G is a connected graph with k end-blocks, then $dn^+_{\triangle f_x}(G) \geq k-1$ for every vertex x in G. In particular, if x is a cut-vertex, then $dn^+_{\triangle f_x}(G) \geq k$.*

Theorem 2.9. *For any vertex x in G, $1 \leq dn_{\triangle f_x}(G) \leq dn^+_{\triangle f_x}(G) \leq n-1$.*

Proof. It is clear from the definition of x-triangle free detour set that $dn_{\triangle f_x}(G) \geq 1$. Since every minimum x-triangle free detour set is a minimal x-triangle free detour set, $dn_{\triangle f_x}(G) \leq \mathrm{dn}^+_{\triangle f_x}(G)$. Also since the vertex x does not belong to any minimal x-triangle free detour set, it follows that $dn^+_{\triangle f_x}(G) \leq n-1$. □

Remark 2.10. *The bounds for $dn_{\triangle f_x}(G)$ and $dn^+_{\triangle f_x}(G)$ in Theorem 2.9 are sharp. For the cycle $C_n (n \geq 4)$, $dn_{\triangle f_x}(G) = dn^+_{\triangle f_x}(G) = 1$ for any vertex x in C_n. Also for the complete graph K_n, $dn^+_{\triangle f_x}(G) = n$ - 1 for every vertex x in K_n. All the inequalities in Theorem 2.9 can be strict. For the graph G given in Figure 2.2, $dn_{\triangle f_w}(G) = 2$, $dn^+_{\triangle f_w}(G) = 3$ and $n = 7$. Thus $1 < dn_{\triangle f_x}(G) < dn^+_{\triangle f_x}(G) < n-1$.*

In the following theorem is an easy consequence of the definitions of the minimum vertex triangle free detour number and the upper vertex triangle free detour number of a graph.

Theorem 2.11. (*i*) *For any tree T with k end vertices, $dn^{+}_{\triangle f_x}(G) = k$ or $k-1$ according as x is a cut vertex or not.*
(*ii*) *For any vertex x in the cycle C_n of order $n \geq 4$, $dn^{+}_{\triangle f_x}(G) = 1$.*
(*iii*) *For any vertex x in the complete graph K_n, $dn^{+}_{\triangle f_x}(K_n) = n-1$.*
(*iv*) *For any vertex x in the complete bipartite graph $K_{n,m}$, $dn^{+}_{\triangle f_x}(K_{n,m}) = m$ or $dn^{+}_{\triangle f_x}(K_{n,m}) = m-1$ if $n = 1$ and $m \geq 2$.*

Theorem 2.12. *For every pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G with $d^{+}_{x}(G) = a$ and $dn^{+}_{\triangle f_x}(G) = b$.*

Proof. **Case 1.** For $1 \leq a = b$, any tree with a end vertices has the desired properties, by Theorem 2.5 and Theorem 2.11(i)

Case 2. For $1 \leq a < b$. Let $P_i : v_i(1 \leq i \leq b-a)$ be a $b-a$ copies of a path of order 1 and $P : x, u_1, u_2, u_3$ a path of order 4. Let G be the graph obtained by joining each $v_i(1 \leq i \leq b-a)$ in P_i and u_1 in P and u_2 in P. Adding a new vertices $w_1, w_2, ..., w_a$ and joining each $w_i(1 \leq i \leq a)$ to u_3. The resulting graph G of order $b+4$ is shown in Figure 2.3. Let $S_1 = \{x, w_1, w_2, ..., w_a\}$ be the set of all extreme vertices of G. It is easily verified that $S = S_1 - \{x\}$ is a x-detour set of G and so by Theorem 1.4, $d_x(G) = |S| = a$.

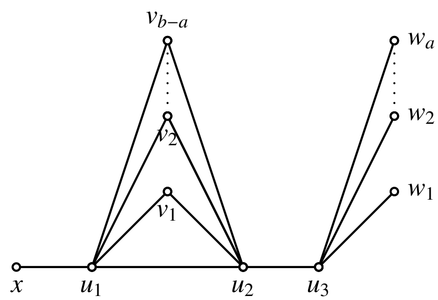

Figure 2.3 : G

Next, we show that $dn_{\triangle f_x}(G) = b$. By Theorem 2.5, every minimal x-triangle free detour set of G contains S. Clearly, S is not a minimal triangle free detour set of G. It is easily verified that each $v_i(1 \leq i \leq b-a)$ must belong to every minimal x-triangle free detour set of G.

Thus $T = S \cup \{v_1, v_2, ..., v_{b-a}\}$ is a minimal x-triangle free detour set of G, it follows from Theorem 2.5 that T is a maximum cardinality of a x-triangle free detour set of G and so $dn^+_{\triangle f_x}(G) = b$. □

Theorem 2.13. *For every pair a, b of integers with $1 \le a \le b$, there exists a connected graph G with $dn^+_{\triangle f_x}(G) = a$ and $dm^+_x(G) = b$.*

Proof. **Case 1.** For $1 \le a = b$, any tree with a end vertices has the desired properties, by Theorem 2.5 and Corollary 2.7.

Case 2. For $1 \le a < b$. Let $P_i : s_i, t_i (1 \le i \le b-a)$ be $b-a$ copies of a path of order 2 and $P : x, u_1, u_2, u_3$ a path of order 4. Let G be the graph obtained by joining each $s_i (1 \le i \le b-a)$ in P_i to u_1 in P and joining each $t_i (1 \le i \le b-a)$ in P_i to u_2 in P. Add new vertices $w_1, w_2, ..., w_a$ and join each $w_i (1 \le i \le a)$ to u_3. The resulting graph G of order $2b-a+4$ is shown in Figure 2.4. Let $S_1 = \{x, w_1, w_2, ..., w_a\}$ be the set of all extreme vertices of G. It is easily verified that $S = S_1 - \{x\}$ is a x-trianlge free detour set of G and so by Theorem 2.5, $dn_{\triangle f_x}(G) = |S| = a$.

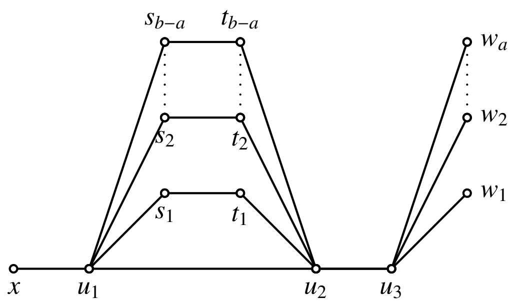

Figure 2.4 : G

Next, we show that $dm^+_x(G) = b$. By Theorem 1.5, every minimal x-detour monophonic set of G contains S. Clearly, S is not a minimal detour monophonic set of G. It is easily verified that each $s_i (1 \le i \le b-a)$ or each $t_i (1 \le i \le b-a)$ must belong to every minimal x-detour monophonic set of G. Thus $T = S \cup \{s_1, s_2, ..., s_{b-a}\}$ is a minimal x-detour monophonic set of G, it follows from Theorem 2.5 that T is a maximum cardinality of a minimal x-detour monophonic set of G and so $dm^+_x(G) = b$. □

Theorem 2.14. *For every pair a, b of integers with $1 \le a \le b$, there exists a connected graph G with $dn^+_{\triangle f_x}(G) = a$ and $g^+_x(G) = b$.*

Proof. This follows from Theorem 2.13. □

Theorem 2.15. *For every pair a, b of integers with $1 \leq a \leq b$, there is a connected graph G with $dn_{\triangle f_x}(G) = a$ and $dn^+_{\triangle f_x}(G) = b$ for some vertex x in G.*

Proof. For $a = b = 1$, $P_n (n \geq 2)$ has the desired properties. For $a = b$ with $b \geq 2$, let G be any tree of order $n \geq 3$ with b end-vertices. Then for any cut vertex x in G, $dn_{\triangle f_x}(G) = a = dn^+_{\triangle f_x}(G) = b$ by Theorem 2.11(i). Assume that $1 \leq a < b$. Let $F = K_2 \cup ((b-a+2)K_1) + \overline{K_2}$, where let $Z = V(K_2) = \{z_1, z_2\}$, $Y = V((b-a+2)K_1) = \{x, y_1, y_2, ..., y_{b-a+1}\}$ and $U = V(\overline{K_2}) = \{u_1, u_2\}$. Let G be the graph obtained from F by adding $a-1$ new vertices $w_1, w_2, ..., w_{a-1}$ and joining each w_i to x. The graph G is shown in Figure 2.5. Let $W = \{w_1, w_2, ..., w_{a-1}\}$ be the set of end vertices of G.

First we show that $dn_{\triangle f_x}(G) = a$. By Theorem 1.3, $dn_{\triangle f_x}(G) \geq a - 1 + 1 = a$. On the other hand, let $S = \{w_1, w_2, ..., w_{a-1}, z_1\}$. Then $D_{\triangle f}(x, z_1) = 4$ and each vertex of F lies on an $x - z_1$ triangle free detour. Hence S is an x-triangle free detour set of G and so $dn_{\triangle f_x}(G) \leq |S| = a$ Therefore, $dn_{\triangle f_x}(G) = a$. Also, we observe that a minimum x-triangle free detour set of G is formed by taking all the end vertices and exactly one vertex from Z.

Next we show that $dn^+_{\triangle f_x}(G) = b$. Let $M = \{w_1, w_2, ..., w_{a-1}, y_1, y_2, ..., y_{b-a+1}\}$. It is clear that M is an x-triangle free detour set of G. We claim that M is a minimal x-triangle free detour set of G. Assume, to the contrary, that M is not a minimal x-triangle free detour set. Then there is a proper subset T of M such that T is an x-triangle free detour set of G. Let $s \in M$ and $s \notin T$. By Theorem 1.4(i), clearly $s = y_i$, for some $i = 1, 2, ..., b - a + 1$. For convenience, let $s = y_1$. Since y_1 does not lie on any $x - y_j$ triangle free detour where $j = 2, 3, ..., b - a + 1$, it follows that T is not an x-triangle free detour set of G, which is a contradiction. Thus M is a minimal x-triangle free detour set of G and so $dn^+_{\triangle f_x}(G) \geq |M| = a - 1 + b - a + 1 = b$.

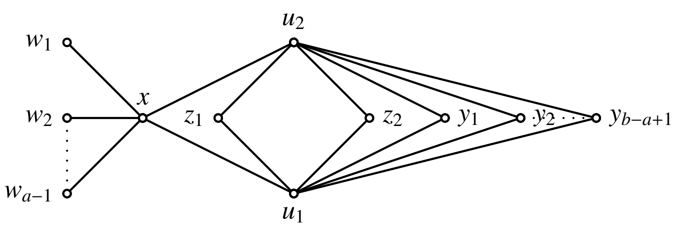

Figure 2.5 : G

Now we prove $dn^+_{\triangle f_x}(G) \leq b$. Suppose that $dn^+_{\triangle f_x}(G) > b$. Let N be a minimal x-triangle free detour set of G with $|N| > b$. Then there exists at least one vertex, say, $v \in N$ such that $v \notin M$. Thus $v \in \{u_1, u_2, z_1, z_2\}$.

Case 1. $v \in \{z_1, z_2\}$, say $v = z_1$. Clearly $W \cup \{z_1\}$ is an x-triangle free detour set of G and also it is a proper subset of N, which is a contradiction to N is a minimal x-triangle free detour set of G.

Case 2. $v \in \{u_1, u_2\}$, say $v = u_1$. Suppose $u_2 \notin N$. Then there is at least one y in Y such that $y \in N$. Cleary, $D_{\triangle f}(x, u_1) = 3$ and the only vertices of any $x - u_1$ triangle free detour are x, z_1, z_2, u_1 and u_2. Also x, u_2, z_1, z_2, u_1, y is an $x - y$ triangle free detour and hence $N - \{u_1\}$ is an x-triangle free detour set, which is a contradiction to N is a minimal x-triangle free detour set of G. Suppose $u_2 \in N$. It is clear that the only vertices of any $x - u_1$ or $x - u_2$ triangle free detour are x, u_1, u_2, z_1 and z_2. Since $u_1, u_2 \in N$, it follows that both $N - \{u_1\}$ and $N - \{u_2\}$ are x-triangle free detour sets, which is a contradiction to N a minimal x-triangle free detour set of G. Thus there is no minimal x-triangle free detour set N of G with $|N| > b$. Hence $dn^+_{\triangle f_x}(G) = b$. □

Remark 2.16. *The graph G of Figure 2.2 contains exactly three minimal x-triangle free detour sets, namely, $W \cup \{z_1\}$, $W \cup \{z_2\}$ and $W \cup (Y - \{x\})$. This example shows that there is no "Intermediate Value Theorem" for minimal x-triangle free detour sets, that is, if n is an integer such that $dn_{\triangle f_x}(G) < n < dn^+_{\triangle f_x}(G)$, then there exist a minimal x-triangle free detour set of cardinality n in G.*

Theorem 2.17. *For any three positive integers a, b and c with $a \geq 2$ and $a \leq c \leq b$, there exists a connected graph G with $dn_{\triangle f_x}(G) = a$, $dn^+_{\triangle f_x}(G) = b$ and a minimal x-triangle free detour set of cardinality c.*

Proof. Let $P : z_1, z_2, z_3, z_4$ and $Q : v_1, v_2, v_3, v_4$ be two paths. Let H be the graph obtained from P and Q by identifying the vertices z_2 in Q. Let G be the graph obtained from H by adding b new vertices $u_1, u_2, \ldots u_{a-2}, y_1, y_2, \ldots, y_{b-c+1}, x_1, x_2, ..., x_{c-a+1}$ and joining each $u_i (1 \leq i \leq a-2)$ with z_2; joining each $y_i (1 \leq i \leq b-c+1)$ with z_1 and z_4 and joining each $x_i (1 \leq i \leq c-a+1)$ with v_1 and v_4 in H. The graph G is shown in Figure 2.6.

Let $S = \{u_1, u_2, ..., u_{a-2}\}$ be the set of all extreme vertices of G and let $x = z_2$. Then by Theorem 1.4, every x-triangle free detour set of G contains S and also for any vertex $y \in V(G) - S$, $S \cup \{y\}$ is not an x-triangle free detour set of G. It is clear that $S_1 = S \cup \{z_4, v_4\}$ is a minimum x-triangle free detour set of G and so $dn_{\triangle f_x}(G) = |S_1| = a$.

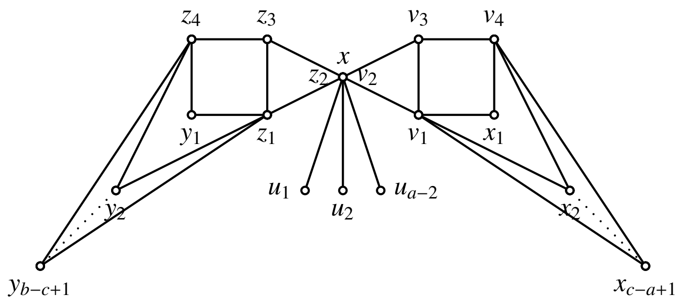

Figure 2.6 : G

Now we show that $dn^+_{\triangle f_x}(G) = b$. Let $M = S \cup \{y_1, y_2, ..., y_{b-c+1}, x_1, x_2, .., x_{c-a+1}\}$. It is clear that M is an x-triangle free detour set of G. We claim that M is a minimal x-triangle free detour set of G. Assume that M is not a minimal x-triangle free detour set of G. Then there exists a proper subset of M_1 of M such that M_1 is an x-triangle free detour set of G. Let $w \in M$ and $w \notin M_1$. By Theorem 2.5, either $w = y_i (1 \leq i \leq b - c + 1)$ or $w = x_j (1 \leq j \leq c - a + 1)$. If $w = y_i (1 \leq i \leq b - c + 1)$, then w does not lie on an $x - z$ triangle free detour path for any $x - z$ triangle free detour for any $z \in M_1$, which is a contradiction. Thus M is a minimal x-triangle free detour set of G and so $dn^+_{\triangle f_x}(G) \geq |M| = b$. Also, it is clear that every minimal x-triangle free detour set of G contains at most b elements and hence $dn^+_{\triangle f_x}(G) \leq b$. Hence $dn^+_{\triangle f_x}(G) = b$.

Finally we show that there is a minimal x-triangle free detour set of cardinality c. Let $T = S \cup \{z_4, x_1, x_2, ...x_{c-a+1}\}$. It is clear that T is an x-triangle free detour set of G. We claim that T is a minimal x-triangle free detour set. Assume that T is not a minimal x-triangle free detour set of G. Then there is a T_1 is an x-triangle free detour set of G. Let $t \in T$ and $t \notin T_1$. By Theorem 2.5 (i), clearly, $t = z_4$ or $t = x_j (1 \leq j \leq c - a + 1)$. If $t = z_4$, then $y_i (1 \leq j \leq c - a + 1)$ does not lie on any $x - y$ triangle free detour path for some $y \in T_1$, which is a contradiction. If $t = x_j (1 \leq j \leq c - a + 1)$, then x_j does not lie on any $x - y$ triangle free detour path for some $y \in T_1$, which is a contradiction. Thus T is a minimal x-triangle free detour set of G with cardinality c. □

Theorem 2.18. *For each positive integers a, b and $c \geq 3$ with $a < b$, there exists a connected graphs G such that $R_{\triangle f}(G) = a$, $D_{\triangle f}(G) = b$ and $dn^+_{\triangle f}(G) = c$ for some vertex x in G.*

Proof. We prove this theorem by considering three cases.

Case 1. $a = b = 1$. Let $G = K_{c+1}$. It is easily seen that $e_{\triangle f}(x) = 1$ for every vertex x in G and so $R_{\triangle f}(G) = 1 = D_{\triangle f}(G)$. Also, by Theorem 2.11(iii), $dn^+_{\triangle f}(G) = c$ for every vertex x in G.

Case 2. $1 = a < b$. Let $C_{b+2} : v_1, v_2, ..., v_{b+2}, v_1$ be a cycle of order $b + 2$. Let G be the graph obtained by adding $n - 1$ new vertices $u_1, u_2, u_3, ..., u_{c-1}$ to C_{b+2} and joining each of the vertices $u_1, u_2, u_3, ..., u_{c-1}$ to the vertex v_1 and also joining each vertex $v_i(3 \leq i \leq b + 1)$ to the vertex v_1. The graph G is shown in Figure 2.7. It is easily verified that $1 \leq e_{\triangle f}(x) \leq b$ for any vertex x in G and $e_{\triangle f}(v_1) = 1$, $e_{\triangle f}(v_2) = b$. Then $R_{\triangle f}(G) = 1$ and $D_{\triangle f}(G) = b$. Let $S = \{v_2, v_{b+2}, u_1, u_2, ..., u_{c-1}\}$ be the set of all extreme vertices of G and let $x = v_2$. Clearly S is the unique minimal triangle free detour set of G and so $dn^+_{\triangle f}(G) = |S| = c$.

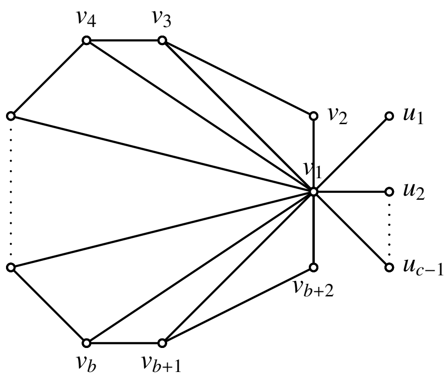

Figure 2.7: G

Case 3. $2 \leq a \leq b$. Let H be a graph obtained from a cycle C_{a+2} : $v_1, v_2, ..., v_{a+2}, v_1$ of order $a + 2$ and a path $P_{b-a+1} : u_0, u_1, u_2, ..., u_{b-a}$ of order $b - a + 1$ by identifying the vertex v_{a+1} in C_{a+2} and u_0 in P_{b-a+1}; also join each vertex $u_i(1 \leq i \leq b - a)$ in P_{b-a+1} with v_{a+2} in C_{a+2}. Now, let G be the graph obtained from H by adding $c - 1$ new vertices $w_1, w_2, ..., w_{c-1}$ and join each $w_i(1 \leq i \leq c - 1)$ with v_2 and v_{a+2} in H. The graph G is shown in Figure 2.8.

It is easily verified that $a \leq e_{\triangle f_x} \leq b$ for any vertex x in G. Also, $e_{\triangle f_x}(v_{a+2}) = a$ and $e_{\triangle f_x}(v_1) = b$. It follows that $R_{\triangle f}(G) = a$, $D_{\triangle f}(G) = b$. Now, let $x = u_{b-a}$ and let $S = \{v_1, w_1, w_2, ..., w_{c-1}\}$. Since every vertex of G lies on an $x - y$, where $y \in S$, triangle free detour path, S is an x-triangle free detour set of G. Then there exists a vertex z in S such that $z \notin S_1$. It is clear that z is either v_1 or $w_i(1 \leq i \leq c - 1)$. In all cases z does not lie on any $x - u$, where $u \in S_1$, triangle free detour path, it follows that S_1 is not an x-triangle free detour set of G. This shows that S is a minimal x-triangle free detour set of G and so $dn^+_{\triangle f_x}(G) \geq c$. Also, it is clear that any minimal x-triangle free detour set of G contains at most $c-1$ elements and hence $dn^+_{\triangle f_x}(G) \le c$. Thus $dn^+_{\triangle f_x}(G) = c$.

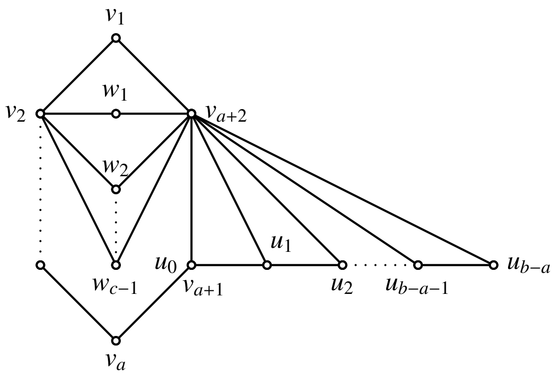

Figure 2.8: G

□

Theorem 2.19. *For any four positive integers a, b, c and d of with $2 \le a \le b \le c \le d$, there exists a connected graph G such that $d^+_x(G) = a$, $dn^+_{\triangle f_x}(G) = b$, $dm^+_x(G) = c$ and $g^+_x(G) = d$.*

Proof. Let $2 \le a \le b \le c \le d$. Let $P : x, a, b, c, d, e, f$ be a path of order 7 and adding $a-1$ new vertices $v_1, v_2, v_3, v_4, \ldots.., v_{a-1}$ to f.

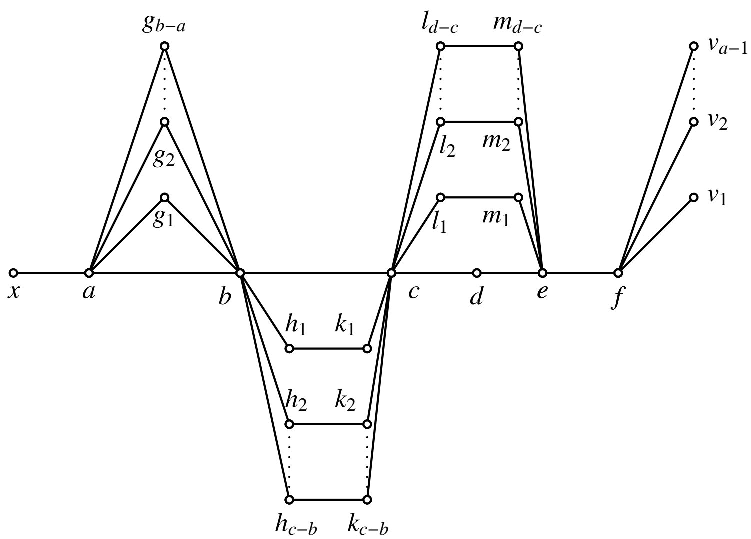

Figure 2.9: G

Let $P_i : g_i(1 \le i \le b-a)$ be a $b-a$ copies of K_1 and joining each $g_i(1 \le i \le b-a)$ in P_i to a and b in P. Let $P_j : h_j, k_j(1 \le j \le c-b)$ be a $c-b$ copies of a path of length 2 and joining each $h_j(1 \le j \le c-b)$

in P_j to b in P and joining each $k_j(1 \leq j \leq c-b)$ in P_j to c in P. Let $P_k : l_k, m_k(1 \leq k \leq d-c)$ be a $d-c$ copies of a path of order 2 and joining each $l_k(1 \leq k \leq d-c)$ in P_k to c in P and joining $m_k(1 \leq k \leq d-c)$ in P_k to e in P. The resulting graph G is shown in Figure 2.9

It is easily verify that $S_1 = \{d, v_1, v_2, ..., v_{a-1}\}$ is a minimal x-detour set, $S_2 = S_1 \cup \{g_1, g_1, g_2, ..., g_{b-a}\}$ is a minimal x-triangle free detour set, $S_3 = S_2 \cup \{h_1, h_2, h_3, ..., h_{c-b}\}$ is a minimal x-detour monophonic set and $S_4 = S_3 \cup \{l_1, l_2, l_2, ..., l_{d-c}\}$ is a minimal x-geodetic set. Thus $d_x^+(G) = a$, $dn_{\triangle f_x}^+(G) = b$, $dm_x^+(G) = c$ and $g_x^+(G) = d$.

□

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