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A NOTE ON THE ESTIMATION OF THE AGE OF A GALTON - WATSON PROCESS

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Abstract

This note discusses the estimation of the age of a Galton-Watson branching process using an estimating equation. An asymptotic property of the derived estimator is established.

INTRODUCTION

Let a population behave like a Galton-Watson branching process { ; n ³ 0, X0 =1} with the offspring distribution {pk; k ³ 0}. Suppose that the generation size (Xn = k) is observed and the age, in generations, is to be estimated. Such a problem might be encountered in many situations. For example, one might be interested in the existence of a certain species in its present form or how long ago a mutation took place.

When the generation size is observed and the offspring distribution is completely known, a likelihood function is given by

$$L(n,k) = P(X_n = k \mid X_n > 0)$$

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$$= \frac{f_n^{\,k}(0)}{k![1-f_n(0)]}$$

where $f_n(s)$ is the n^{th} functional iterate of the offspring probability generating function (p.g.f.)

$$f(s) = \sum_{k=0}^{\infty} p_k s^k$$

$|s| < 1$ and $f_n^{(k)}$ denotes the kth derivative of $f_n(s)$ w.r.t. s.

Stigler (1970), the first to discuss the problem of estimating the age of a Galton-Watson process, obtained the maximum likelihood estimate (m.l.e.) of n when the offspring p.g.f. is a fractional linear generating function

$$f(s) = 1 - \frac{b}{1-c} + \frac{bs}{1-cs}, \; |s| < 1$$

which includes the p.g.f. of geometric distribution. Crump and Howe (1972) proposed non-parametric estimates of n in the supercritical case i..e., when the offspring mean m = f¢(1) > 1. Hwang and Hwang (1972) developed a numerical procedure to obtain the m.l.e. of n when the offspring distribution is Poisson. Ades et al. (1982) obtained a recurrence formula to compute P(Xn = k), k =1,2,3, . . . when the offpspring p.g.f. satisfies the functional equation (c+ds)f¢(s) = a + bf(s), where a,b,c, and d are constants. Also, they developed a procedure to obtain the m.l.e. of n numerically when the offspring distribution is negative binomial. Motivated by their method, Nanjundan (1985) and Nanjundan and Hanumantharayappa (1998) obtained numerically the m.l.e. of n when the offspring distribution is respectively binomial and Poisson.

It is worth noting that Stigler (1970) has established that the knowledge of additional generations does not improve the estimate.

ESTIMATING EQUATION

Let $\{X_n ; n \geq 0, X_0 = 1\}$ be a Galton-Watson branching process with the offspring mean $m = \sum_{k=1}^{\infty} k p_k$. Then, it can easily be verified that $\frac{X_n}{m^n} - 1 = 0$ is an unbiased estimating equation in the sense of Godambe (1976). In the super critical case, this unbiased estimating equation leads to the estimate $\hat{n} = \frac{\log X_n}{\log m}$

Note that the offspring distribution is to be completely known to compute the estimates of n mentioned above while it is enough if m is known to compute this estimate.

Since the estimate is based on the size of a single generation, it is too much to ask for the optimal properties. But it can be shown that $\hat{n}$ is strongly a - consistent.

An estimate $\hat{\theta}$ of a parameter θ is said to be α-consistent if, for $\alpha > 0$, $\frac{\hat{\theta} - \theta}{\theta^{\alpha}} \to 0$, as $\theta \to \infty$. This definition is due to Feldman and Fox (1968).

When $m > 1$ and $E(X_1 \log X_1) < \infty$, Stigum (1966) has shown that $\frac{X_n}{m^n} \xrightarrow{a.s.} W$, as $n \to \infty$, conditionally on $X_n > 0$, where W is a non-negative random variable. Here, $\hat{n} - n = \frac{\log X_n - n \log m}{\log m}$

$$= \frac{\log(X_n / m^n)}{\log m} \xrightarrow{a.s.} \frac{\log W}{\log m}, n \to \infty,$$

conditionally on $X_n > 0$, which in turn implies that

$$\frac{\hat{n}-n}{n^{\alpha}} \xrightarrow{a.s.} 0, n \to \infty.$$ And hence, is a strongly a -consistent estimate of n for any $\alpha > 0$

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