H ARTICLES

MJS, Vol. 2, No. 1, May 2003 - Oct. 2003, pp.1-24

ISSN 0975-3311
https://doi.org/10.12725/mjs.3.1

STEADY PLANE MHD FLOWS THROUGH POROUS MEDIA IN THE MAGNETOGRAPH PLANE

C.S. Bagewadi and S. Bhagya***

Abstract

We obtain solutions for steady plane MHD flow through porous media when velocity and magnetic vectors are constantly and variably inclined and the magnitude of the magnetic vector is constant on each individual stream line in the magnetograph plane. It is shown that the path of magnetic and velocity vectors are circles congruency to each other. Also flow analysis is carried out by writing the expression of Legendre transformation in polar co-ordinates. It is shown that solutions obtained agree with the graphs.

Key words : Stream line, Magnetic line, Legendre transformation, Magnetograph plane.

1. Introduction

Flow of a viscous liquid in a porous medium is of great and increasing importance in the study of percolation through soils in hydrology, petroleum industry and in agricultural engineering. The flows in porous media generally

* Dean, Faculty of Science & Technology, Chairman-Department of Mathematics & Computer Science, Kuvempu University, Jnana Sahyadri, -577 451, Shimoga, Karnataka, India.

** Department of Mathematics, P.E.S. College, Bangalore-560 050, India.

involve extensively low Reynolds numbers and such flows are called Darcy flows. Such flows are valid under several limitations. It is shown that it can be possibly valid only in a certain seepage velocity domain outside which more general equations may be used to describe the flow. This happens because initial effects become important. Also the flow through porous boundaries is of great importance both in technological as well as bio-physical fields, examples of which are soil mechanics, transpiration cooling, food preservation, cosmetic industry, blood flow and artificial dialysis. In recent years the problem of fluid flow past porous media or in channels with mass transfer, heat transfer have gained more importance because of varied application. For example, I.V.fluid containers made of PVC are commonly used these days. Water from inside permeates out thus increasing the concentration of drug inside and sometimes becoming hazardous to life. Therefore, the study relating to suction or injection is very important. The early researchers considered the blood to be a Newtonian fluid but being a suspension of cells it behaves as a non-Newtonian fluid at low shear rates in small arteries.

O.P. Chandna and his co-workers [1,3,10,14,15] have published a series of papers on plane incompressible MHD flow of a viscous fluid. They have obtained solutions for these flows by transforming the basic equations from cartesion plane to velocity plane by using Legendre transformation. K.K. Singh & D.P. Singh [8] studied the solutions in variably inclined MHD plane flows in porous media. C.S. Bagewadi and Siddabasappa [6],[7],[9],[11],[12] extended the work of above authors and published a series of papers on MGD flow, EMFD flow and Rotating MHD flow. They have obtained solutions for these flows by transforming the basic equations from cartesion plane to velocity and magnetograph planes. These methods infact help to study the flows in a more general way by the use of Jacobian matrix. Yamamota [4],[13] examined the flow past porous bodies by applying the generalised law using the generalised momentum equations. Ram and Mishra [2] have studied the unsteady MHD flow of fluid through a porous medium in a circular pipe under action of a consant pressure gradient.

In the present paper, we study MHD flow of a viscous incompressible fluid of infinite electrical conductivity through porous media when (i) angle between V and H is constant (ii) the magnitude of magnetic vector is constant on each individual magnetic line in the magnetograph plane. In the 2nd section basic equations are written and are decomposed in the cartesion plane. The 3rd section deals with some preliminaries about magnetograph plane. In the 4th section equations written in cartesion plane for constantly inclined flows are recast into magnetograph plane and flow analysis is carried out. In the 5th section the equations for MHD flow when the magnitude of the magnetic lines are constant are recast in the magnetograph plane and flow analysis is carried. The results obtained in our paper are entirely different and infact extensions from the results obtained by the above authors. Hence our results are superior to the results obtained by the above authors. Also various graphs are plotted and it is concluded that these graphs agree with the theoritical results obtained.

2. BASIC EQUATIONS

The steady MHD flow of a viscous incompressible fluid of infinite electrical conductivity through porous media is governed by [8].

$$\text{div } V = 0 \tag{1}$$

$$\rho[(v.\text{grad})V] = -\text{grad}P + \eta\nabla^2 V + \mu JXH - (\mu/K)V \tag{2}$$

$$\text{Curl}\,(V \times H) = 0 \tag{3}$$

$$\text{div } H = 0 \tag{4}$$

where V is the Velocity vector, H is the Magnetic field vector, P is the Pressure function, r is the Constant density, η is the Constant co-efficient of Viscosity, μ is the Constant magnetic permeability, K is the Permeability of the medium and J is the Current density.

We consider the flow to be the two dimensional so that V and H lie in a plane defined by the rectangular co-ordinates (x,y) and all the flow variables are functions of x and y. Therefore the above system of equations is replaced by the following system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{..........} \quad (5)$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial P}{\partial x} = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \mu H_2\left(\frac{\partial H}{\partial x} - \frac{\partial H_1}{\partial y}\right) - \frac{\mu u}{K} \qquad \text{...........}(6)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial P}{\partial y} = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \mu H_2\left(\frac{\partial H}{\partial x} - \frac{\partial H_1}{\partial y}\right) - \frac{\mu v}{K} \qquad \text{...........}(7)$$

$$uH_2 - vH_1 = k \tag{8}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \qquad \text{..........}(9)$$

Now by introducing the functions

$$\omega = \frac{\partial V}{\partial x} - \frac{\partial u}{\partial y} \quad \text{(Vorticity)} \qquad \text{.........} (10)$$

$$j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \quad \text{(Current density)} \qquad \text{.........} (11)$$

$$h = P + (1/2)\,\rho\, q2 \quad \text{where } q^2 = u^2 + v^2 \qquad \text{.........} (12)$$

the above system of equations is replaced (5), (8), (9), (10), (11) and by the following system :

$$\eta \frac{\partial \omega}{\partial y} - \rho v \omega + \mu H_2 j = -\frac{\partial h}{\partial x} - \frac{\mu u}{K} \qquad \text{.........} (13)$$

(Linear Momentum)

$$\eta \frac{\partial \omega}{\partial x} - \rho u \omega + \mu H_1 j = \frac{\partial h}{\partial y} + \frac{\mu v}{K} \qquad \text{.........} (14)$$

of seven non linear partial differential equations in seven unknowns u, v, H_1, H_2, ω, j and h which are functions of x,y. The advantage of this system over to the first system is that it decreased from second order differential equations to the first order differential equations.

3. Study of Flows in The Magnetograph Plane

Let the function $H_1 = H_1(x,y)$ $H_2 = H_2(x,y)$ to be such that in the region of flow the Jacobian.

$$J(x,y) = \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} \; \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} \neq 0 \qquad |J| < \infty \qquad \text{........} (15)$$

We may consider x and y as functions of H_1 and H_2. by means of $x = x(H_1, H_2)$ and $y = y(H_1, H_2)$ we have the relations,

$$\frac{\partial H_1}{\partial x} = J \frac{\partial Y}{\partial H_2}, \quad \frac{\partial H_1}{\partial y} = -J \frac{\partial x}{\partial H_2} \qquad \text{........} (16)$$

$$\frac{\partial H_2}{\partial x} = -J\frac{\partial y}{\partial H_1}, \quad \frac{\partial H_2}{\partial y} = J\frac{\partial x}{\partial H_1} \qquad \text{............(17)}$$

Further more using (16),(17) we have

$$J(x,y) = \frac{\partial (H_1,H_2)}{\partial(x,y)} = \left[\frac{\partial (x,y)}{\partial (H_1,H_2)}\right]^{-1} = J(H_1,H_2) \qquad \text{......... (18)}$$

$$\frac{\partial f}{\partial x} = \frac{\partial (f,y)}{\partial (x,y)} = J\frac{\partial (f,y)}{\partial (H_1,H_2)} = \bar{J}\frac{\partial (f,y)}{\partial (H_1,H_2)} \qquad \text{.......... (19)}$$

$$\frac{\partial f}{\partial y} = -\frac{\partial (f,x)}{\partial (x,y)} = J\frac{\partial (x,f)}{\partial (H_1,H_2)} = \bar{J}\frac{\partial (x,f)}{\partial (H_1,H_2)} \qquad \text{......... (20)}$$

where $f = f(x, y)$ is any continuously differentiable function.

4. Constantly Inclined Plane Flows

We now consider constantly inclined plane flows and f denote the constant non zero angle between v and H. The vector and scalar products of V and H using the equation (8) yield.

$$uH_2 - vH_1 = qH \operatorname{Sin}\phi = k \qquad \text{........ (21)}$$

$$uH_1 + vH_2 = qH \operatorname{Cos}\phi = k \operatorname{Cot}\phi \qquad \text{........ (22)}$$

where $H = \sqrt{H_1^2 + H_2^2}$

Solving (21) and (22) for u and v in terms of H1 and H2 to get

$$u = \frac{k}{H^2}(H_1\cot\phi + H_2), \qquad v = \frac{k}{H^2}(H_2\cot\phi - H_1) \qquad \text{........... (23)}$$

Eliminate functions u and v from the system of equations (5), (8), (9), (10),(11), (13) and (14) by using equation (23) and obtain the following system of six partial differential equations.

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \qquad \text{.........(24)}$$

$$\eta\frac{\partial \omega}{\partial y} - \frac{\rho k}{H^2}(H_2\cot\phi - H_1)\,\omega + \mu H_2 j = -\frac{\partial h}{\partial x} - \frac{\mu u}{K} \qquad \text{......... (25)}$$

$$\eta \frac{\partial \omega}{\partial x} - \frac{\rho k}{H^2}(H_1 \cot\phi - H_2)\,\omega + \mu H_1 j = \frac{\partial h}{\partial y} + \frac{\mu v}{K} \qquad \text{.........} (26)$$

$$(H_1^2 - H_2^2 - 2H_1H_2\cot\phi)\left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial x}\right) + (H_2^2\cot\phi - H_1^2\cot\phi - 2H_1H_2)\left(\frac{\partial H_1}{\partial x} - \frac{\partial H_2}{\partial y}\right) + H^2\left(H_1\frac{\partial \cot\phi}{\partial x} + H_2\frac{\partial \cot\phi}{\partial y}\right) = 0 \qquad \text{......}(27)$$

$$k\frac{\partial}{\partial x}\left(\frac{H_2\cot\phi - H_1}{H^2}\right) - k\frac{\partial}{\partial y}\left(\frac{(H_1\cot\phi + H_2}{H^2}\right) = \omega \qquad \text{..........} (28)$$

$$\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = j \qquad \text{..........} (29)$$

For the six unknown functions $H_1, H_2, j\ \omega, h\ \phi$ of (x,y) once a solution of this system is determined the velocity vector field is obtained from (23) and then the pressure function is found by using the definition of h in (12). Using the transformation relations (16) to (20) in the equations (24) to (29) we find the following system of equations :

$$\frac{\partial x}{\partial H_1} + \frac{\partial y}{\partial H_2} = 0 \qquad \text{... ... ...} (30)$$

$$\eta\, J\frac{\partial(x,\omega)}{\partial(H_1,H_2)} - \frac{\rho k}{H^2}(H_2\cot\phi - H_1)\,\omega + \mu H_2 j = -J\frac{\partial(h,y)}{\partial(H_1,H_2)} \qquad \text{...........} (31)$$

$$\eta\, J\frac{\partial(\omega,y)}{\partial(H_1,H_2)} - \frac{\rho k}{H^2}(H_1\cot\phi + H_2)\,\omega + \mu H_1 j = J\frac{\partial(x,h)}{\partial(H_1,H_2)} \qquad \text{... ... ...} (32)$$

$$\frac{\partial x}{\partial H_1}\left[H_2(H_1^2 + H_2^2)\frac{\partial(\cot\phi)}{\partial H_2} - 2H_1H_2 + (H_1^2 + H_2^2)\cot\phi\right] + \frac{\partial x}{\partial H_2}\left[2H_1H_2\cot\phi + (H_1^2 + H_2^2) - H_2(H_1^2 + H_2^2)\frac{\partial \cot\phi}{\partial H_1}\right]$$

$$+\frac{\partial y}{\partial H_1}\left[2H_1H_2\cot\phi + (H_1^2 + H_2^2) - H_1(H_1^2 + H_2^2)\frac{\partial \cot\phi}{\partial H_2}\right] + \frac{\partial y}{\partial H_2}\left[H_1(H_1 + H_2)\frac{\partial \cot\phi}{\partial H_1} + 2H_1H_2 + (H_1^2 + H_2^2)\cot\phi\right] = 0 \qquad \text{...}(33)$$

$$J\left[\frac{\partial x}{\partial H_2} - \frac{\partial y}{\partial H_1}\right] = \omega \qquad \text{.........}(34)$$

$$k \cdot J\left(\frac{\partial (H_2 \cot\phi + H_1)/H_2, y)}{\partial(H_1, H_2)} - \frac{\partial (x, H_1 \cot\phi + H_2/H_2)}{\partial(H_1, H_2)}\right) = j \qquad \ldots\ldots\ldots (35)$$

These are six partial differential equations in six unknowns x (H_1, H_2), y (H_1, H_2) and four transformed functions w((H_1, H_2), h ((H_1, H_2), j (H_1, H_2) and $\phi(H_1, H_2)$. The equation (15) implies the existence of magnetic flux function Y(x,y) such that,

$$\frac{\partial\Psi}{\partial x} = -H_2, \qquad \frac{\partial\Psi}{\partial y} = H_1 \qquad \ldots\ldots\ldots (36)$$

Likewise, equation (30) implies the existence of a function L(H1, H2) called the Legendre transform function of the stream function $\Psi(x,y)$ so that,

$$\frac{\partial L}{\partial H_1} = -y, \qquad \frac{\partial L}{\partial H_2} = x \qquad \ldots\ldots\ldots (37)$$

and the functions $\Psi(x,y)$ and L(H1, H2) are related by

$$L(H_1, H_2) = H_2 x - H_1 y - \Psi(x,y) \qquad \ldots\ldots\ldots (38)$$

Introducing $L(H_1, H_2)$ into the system (30) to (35) with J given by (18), it follows that equation (30) is identically satisfied and this system may be replaced by :

$$\eta \; J \frac{\partial(\partial L/\partial H_2)}{\partial(H_1,H_2)} - \frac{\rho k}{H^2} (H_2 \cot\phi - H_1) \;\omega + \mu H_2 j = J \frac{\partial(h, \partial L/\partial H_1)}{\partial(H_1,H_2)} \qquad \ldots\ldots\ldots (39)$$

$$\eta \; J \frac{\partial(\omega, \partial L/\partial H_1)}{\partial(H_1,H_2)} - \frac{\rho k}{H^2} (H_1 \cot\phi + H_2) \;\omega + \mu H_1 j = -J \frac{\partial(\partial L/\partial H_2, h)}{\partial(H_1,H_2)} \qquad \ldots\ldots\ldots (40)$$

$$\frac{\partial^2 L}{\partial H_1}\left[H_2^2 - H_1^2 - 2H_1H_2(\cot\phi) + H_1(H_1^2 + H_2^2)\frac{\partial\cot\phi}{\partial H_1}\right] + \frac{\partial^2 L}{\partial H_2}\left[H_1^2 - H_2^2 + 2H_1H_2(\cot\phi) - H_2(H_1^2 + H_2^2)\frac{\partial\cot\phi}{\partial H_2}\right]$$

$$\frac{\partial^2 L}{\partial H_1 \partial H_2}\left[2(H_1^2 - H_2^2)(\cot\phi) - 4H_1H_2 - H_1(H_1^2 + H_2^2)\frac{\partial\cot\phi}{\partial H_1} + H_2(H_1^2 + H_2^2)\frac{\partial\cot\phi}{\partial H_2}\right] = 0 \qquad \ldots\ldots\ldots (41)$$

$$J\left[\frac{\partial^2 L}{\partial H_1} + \frac{\partial^2 L}{\partial H_2}\right] = \omega \qquad \ldots\ldots\ldots (42)$$

$$K\,J\left[\frac{\partial\left(\frac{\partial L}{\partial H_1}, \frac{H_2 \cot\phi + H_1}{H^2}\right)}{\partial(H_1,H_2)} + \frac{\partial\left(\frac{\partial L}{\partial H_1}, \frac{H_2 - H_1\cot\phi}{H^2}\right)}{\delta(H_1,H_2)}\right] = j \qquad \text{......... (43)}$$

$$J = \left[\frac{\partial^2 L}{\partial H_1} \times \frac{\partial^2 L}{\partial H_2} - \left(\frac{\partial^2 L}{\partial H_1 \partial H_2}\right)^2\right] \qquad \text{......... (44)}$$

For the six functions L (H1, H2),h(H1, H2),j(H1, H2), w(H1, H2) J(H1, H2) and f(H1, H2) we define as

$$T_1(H_1,H_2) = \frac{\partial\left(\frac{\partial L}{\partial H_2}, \omega\right)}{\partial(H_1,H_2)} = \frac{\partial\left(\frac{\partial L}{\partial H_2}, J\frac{\partial^2 L}{\partial H_1} + J\frac{\partial^2 L}{\partial H_2}\right)}{\partial(H_1,H_2)} \qquad \text{.......... (45)}$$

$$T_2(H_1,H_2) = \frac{\partial\left(\frac{\partial L}{\partial H_1}, \omega\right)}{\partial(H_1,H_2)} = \frac{\partial\left(\frac{\partial L}{\partial H_1}, J\frac{\partial^2 L}{\partial H_1} + J\frac{\partial^2 L}{\partial H_2}\right)}{\partial(H_1,H_2)} \qquad \text{.......... (46)}$$

and use the integrability condition

$$\left(J\frac{\partial^2 L}{\partial H_1 \partial H_2}\frac{\partial}{\partial H_2} - J\frac{\partial^2 L}{\partial H_2}\frac{\partial}{\partial H_1}\right)\left[J\partial\left(\frac{\frac{\partial L}{\partial H_1}, h}{\partial(H_1, H_2)}\right)\right] = \left(\frac{\partial^2 L}{\partial H_1}\frac{\partial}{\partial H_2} - J\frac{\partial^2 L}{\partial H_1 \partial H_2}\frac{\partial}{\partial H_1}\right)\left[J\partial\left(\frac{\frac{\partial L}{\partial H_2}, h}{\partial(H_1, H_2)}\right)\right]$$

i.e. $\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x}$ to eliminate h (H_1,H_2) from equation (39) and (40) to obtain

$$\eta\left[\frac{\partial\left(\frac{\partial L}{\partial H_2}, JT_1\right)}{\partial(H_1,H_2)} + \frac{\partial\left(\frac{\partial L}{\partial H_2}, JT_2\right)}{\partial(H_1,H_2)}\right] - \frac{\rho k}{H^2}\left[(H_2\cot\phi - H_1)\ T_1 + T_2(H_2 + H_1\cot\phi)\right]$$

$$\mu H_1\left[\frac{\partial\left(\frac{\partial L}{\partial H_2}, j\right)}{\partial(H_1,H_2)}\right]+\mu H_2\left[\frac{\partial\left(\frac{\partial L}{\partial H_2}, J\right)}{\partial(H_1,H_2)}\right] = 0 \qquad \text{.......... (47)}$$

The equations (41)and (47) constitute a system of two non linear partial differential equations in two unknowns L(H1,H2), ϕ(H1, H2) in the Hodograph plane. Once a solution L = L(H1 ,H2), $\phi=\phi$(H1 ,H2) of equation (41) and (47) is found for which J evaluated from (44) satisfies $o< \mid J \mid < \infty$, the solutions for the magnetic field components H1(x , y) and H2(x , y) are obtained by solving simultaneous equations,

$$\frac{\partial L}{\partial H_1} = -y \, , \qquad \frac{\partial L}{\partial H_2} = x \qquad \text{.............. (48)}$$

$$H = \sqrt{H_1^2 + H_2^2} \, , \quad \theta = \tan^{-1}(H_2/H_1) \, .$$

$$\text{or. } H_1 = H\cos\theta \, , \quad H_2 = H\sin\theta \qquad \text{........ (49)}$$

$$\left.\begin{aligned} \frac{\partial}{\partial H_1} &= \mathrm{Cos}\theta \frac{\partial}{\partial H} - \frac{\mathrm{Sin}\theta}{H}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial H_2} &= \mathrm{Sin}\theta \frac{\partial}{\partial H} - \frac{\mathrm{Cos}\theta}{H}\frac{\partial}{\partial \theta} \end{aligned}\right| \qquad \text{........... (50)}$$

Defining L* (H, θ), f* (H, θ), w*(H, θ), j*(H, θ), J*(H, θ) to be respectively the Legendre transforms variable angle, vorticity, current density, Jacobean function in (H, θ) co-ordinates and using (49) and (50), we have

$$\frac{\partial(F,G)}{\partial(H_1,H_2)} = \frac{\partial(F^*,G^*)}{\partial(H,\theta)} \times \frac{\partial(H,\theta)}{\partial(H_1,H_2)} = \frac{1}{H}\frac{\partial(F^*,G^*)}{\partial(H,\theta)} \qquad \text{.......... (51)}$$

where F(H1,H2) = F*(H, θ) and $G(H_1,H_2)$ = G*(H, θ) are continuously differentiable functions. We obtain that L* (H, q) and ϕ *(H, θ) satisfy

$$\frac{\partial^2 L^*}{\partial H^2}\left[\left(1+\frac{\partial \cot\phi^*}{\partial\theta}\right)K\,j^* + K\cot\phi^*\frac{\partial j^*}{\partial\theta}\right] + \left(\frac{1}{H_2}\frac{\partial^2 L^*}{\partial\theta_2}+\frac{1}{H}\frac{\partial L^*}{\partial H}\right)\left(HK\frac{\partial j^*}{\partial H} - K^*\right)$$

$$+\left(\frac{1}{H}\frac{\partial^2 L^*}{\partial\theta\partial H}\frac{1}{H_2}\frac{\partial L^*}{\partial\theta}\right)\left(K^*\,(2\mathrm{Cot}\phi - H\frac{\partial\cot\phi^*}{\partial H})\right) - \cot\phi^*\,Hk\frac{\partial j^*}{\partial H} - k\frac{\partial j^*}{\partial\theta} = 0$$

............(52)

$$\frac{\partial\left(\mathrm{Sin}\theta\frac{\partial L^*}{\partial H}+\frac{\mathrm{Cos}\theta}{H}\frac{\partial L^*}{\partial\theta}J^*T_1^*\right)}{\partial(H,\theta)} + \frac{\partial\left(\mathrm{Cos}\theta\frac{\partial L^*}{\partial H}-\frac{\mathrm{Sin}\theta}{H}\frac{\partial L^*}{\partial\theta}J^*T_2^*\right)}{\partial(H,\theta)}$$

$$+\mu H\mathrm{Sin}\,\theta\,\frac{\partial\left(\mathrm{Sin}\theta\frac{\partial L^*}{\partial H}+\frac{\mathrm{Cos}\theta}{H}\frac{\partial L^*}{\partial\theta}J^*\right)}{\partial(H,\theta)} + \mu H\mathrm{Cos}\theta\,\frac{\partial\left(\mathrm{Cos}\theta\frac{\partial L^*}{\partial H}-\frac{\mathrm{Sin}\theta}{H}\frac{\partial L^*}{\partial\theta}J^*\right)}{\partial(H,\theta)}$$

$$-\rho\,k^*\,T_1^*(\mathrm{Sin}\theta\ \mathrm{Cot}\phi^* - \mathrm{Cos}\,\theta) + T_2^*\,(\mathrm{Sin}\theta + \mathrm{Cos}\theta\cot\phi^*) = 0 \qquad \text{.......... (53)}$$

Where

$$J^*(H,\theta) = \frac{H^4}{H^2\frac{\partial^2 L^*}{\partial H^2}\left(H\frac{\partial L^*}{\partial H}+\frac{\partial^2 L^*}{\partial\theta^2}\right) - \left(\frac{\partial L^*}{\partial\theta} - H\frac{\partial^2 L}{\partial H\partial\theta}\right)^2} \qquad \text{......... (54)}$$

$$j^*(H,\theta) = J^*\left(\frac{\partial^2 L}{\partial H^2}+\frac{1}{H^2}\frac{\partial^2 L^*}{\partial\theta^2}+\frac{1}{H}\frac{\partial L^*}{\partial H}\right) \qquad \text{......... (55)}$$

$$T_1^*(H,\theta) = \frac{1}{H}\,\frac{\partial\left(\sin\theta\frac{\partial L^*}{\partial H}+\frac{\cos\theta}{H}\frac{\partial L^*}{\partial\theta},\ \omega^*\right)}{\partial(H,\theta)} = T_1(H_1,H_2) \qquad \text{......... (56)}$$

$$T_2^*(H,\theta) = \frac{1}{H}\,\frac{\partial\left(\cos\theta\,\frac{\partial L^*}{\partial H} + \frac{\sin\theta}{H}\,\frac{\partial L^*}{\partial \theta},\ \omega^*\right)}{\partial(H,\theta)} = T_2(H_1,H_2) \qquad \text{.........} (57)$$

and

$$\omega^*(H,\theta) = \frac{J^*}{H^2}\left[k^*\frac{\partial\cot\phi^*}{\partial\theta}\frac{1}{H^2}\frac{\partial L^*}{\partial\theta} - \frac{1}{H}\frac{\partial^2 L^*}{\partial\theta\partial H} + \frac{\partial\cot\phi^*}{\partial H}\frac{\partial L^*}{\partial H} + \frac{1}{H}\frac{\partial^2 L}{\partial\theta^2}\right.$$

$$+\cot\phi^*\left(\frac{\partial^2 L}{\partial H^2}\frac{1}{H} - \frac{\partial L^*}{\partial H}\frac{1}{H^2}\frac{\partial^2 L^*}{\partial\theta^2}\right) + \frac{2}{H}\left(\frac{1}{H}\frac{\partial L^*}{\partial\theta} - \frac{\partial^2 L}{\partial\theta\partial H}\right) - K^*\left(\frac{\cot\phi^*}{H}\frac{\partial j^*}{\partial\theta}\frac{\partial^2 L}{\partial\theta\partial H} - \frac{1}{H}\frac{\partial L^*}{\delta\theta}\right)$$

$$\left.-\cot\phi^*\frac{\partial j^*}{\partial H}\left(\frac{\partial L^*}{\partial H} + \frac{1}{H}\frac{\partial^2 L^*}{\partial\theta^2}\right) + \frac{\partial j^*}{\partial\theta}\frac{\partial^2 L^*}{\partial H^2} + \frac{\partial j^*}{\partial H}\left(\frac{1}{H}\frac{\partial L^*}{\partial\theta} - \frac{\partial^2 L^*}{\partial\theta\partial H}\right)\right] \qquad \text{.........} (58)$$

Once a solution $L^* = L^*(H,\theta)$, $\phi^* = \phi^*(H,\phi)$ of the system of equations (51) and (52) is determined, we employ

$$x = \sin\theta\,\frac{\partial L^*}{\partial H} + \frac{\cos\theta}{H}\,\frac{\partial L^*}{\partial\theta}$$

$$y = \frac{\sin\theta}{H}\,\frac{\partial L^*}{\partial\theta} - \cos\theta\,\frac{\partial L^*}{\partial H} \qquad \text{............} (59)$$

and (48) to obtain $H_1 = H_1(x,y)$, $H_2 = H_2(x,y)$ in physical plane. The remaining flow variables are then obtained in the physical plane by using the flow equations in the physical plane

Solution

1. Vertex flow : In this flow, we wish to determine the solution of a flow problem when the Legendre transform function is of the form $L^*(H,\theta) = F(H)$ in (H,θ) co-ordinates.

$$L(H_1,H_2) = F\sqrt{H_1^2 + H_2^2}$$ in (H_1,H_2) co-ordinates in the hodograph plane.

Let us assume $L^*(H, \theta) = F(H)$ ....(60)

to be the Legendre tranform function for the system of equations (52) and (53) such that F^1 $(H)^1$ 0, F^{11} $(H)^1$ 0. From equation (60) and (51) we find that $\phi^*(H, \theta)$ satisfies

$$\frac{\partial \cot\phi^*}{\partial\theta} = \frac{1}{K^*}\left[K\left(\frac{F^1}{HF^{11}} - 1\right) + \left(\frac{F^1}{H(F^{11})^2} - \frac{1}{F^{11}} + \frac{F^{111}F1}{(F^{11})^3}\right)\right] \qquad \text{......... (61)}$$

$$\text{where } j^* = \frac{H}{F^1} + \frac{1}{F^{11}} \qquad \text{.......... (62)}$$

As calculated from (55) integration of (61) yields $\cot\phi^* = G_1(H)\theta + G_2(H)$ ....(63)

Where

$$G_1(H) = \frac{1}{K^*}\left[K\left(\frac{F^1}{HF^{11}} - 1\right) + \left(\frac{F^1}{H(F^{11})^2} - \frac{1}{F^{11}} + \frac{F^{111}F1}{(F^{11})^3}\right)\right] \qquad \text{......... (64)}$$

and G2(H) is an arbitary constant of H. By using (60) and (63) in (54) to (58), we find $\omega^* = A(H)\theta + B(H)$ .... (65)

$$T_1^* = \frac{F^{11}}{H} A(H)\sin\theta - \left(\frac{F^1}{H}\cos\theta\right)\left(A^1(H)\theta + B^1(H)\right) \qquad \text{............(66)}$$

$$T_2^* = \frac{F^{11}}{H} A(H)\cos\theta - \left(\frac{F^1}{H}\sin\theta\right)\left(A^1(H)\theta + B^1(H)\right) \qquad \text{............ (67)}$$

$$J^* = \frac{H}{F^{11}F^1} \qquad \text{........... (68)}$$

Where

$$A(H) = \frac{KG_1}{HF^{11}} + KG_1\left(\frac{1}{HF^1} - \frac{1}{H^2F^{11}}\right) + G_1\left(\frac{1}{F^{11}F^1} + \frac{1}{H(F^{11})^2}\right) + G_1\left(\frac{1}{HF^{11}F^1} - \frac{1}{H^2(F^{11})^2} - \frac{F^{111}}{H(F^{11})^3}\right) \qquad \text{............ (69)}$$

$$B(H) = \frac{KG_2}{HF^{11}} + KG_2\left(\frac{1}{HF^1} - \frac{1}{H^2F^{11}}\right) + G_2\left(\frac{1}{F^{11}F^1} + \frac{1}{H^2(F^{11})^2}\right) + G_2\left(\frac{1}{HF^{11}F^1} - \frac{1}{H^2(F^{11})^2} - \frac{F^{111}}{H(F^{11})^3}\right) \qquad \text{............ (70)}$$

Thus $L^*(H, \theta)$ and $\cot\phi^*$ given by (60) and (63) respectively, satisfy equation (52) but in order to be a solution of equations (52) and (53) the unknown functions F(H) and G2(H) must satisfy

$$\frac{\rho}{H} K^*\left(\cot\phi^* \frac{\partial\omega^*}{\partial\theta} F^{11} + \frac{\partial\omega^*}{\partial H} F^1\right) - \eta\left(\frac{\partial(\sin\theta F^1, J^*T1^*)}{\partial(H, \theta)} + \frac{\partial(\cos\theta F^1, J^*T2^*)}{\partial(H, \theta)} - \frac{\omega^*}{K}\right) = 0 \qquad \text{............ (71)}$$

with j^*, $\cot\phi^*$, ω^*, T_1^*, T_2^* and J^* as in (62) to (68) we find that $F(H) = M_1H^2 + M_2$, $G_2(H) = M_3$ is a solution set of (71) such that $F^1(H) \neq 0$, $F^{11}(H) \neq 0$ where $M_1 \neq 0$, M_2 and M_3 are arbitrary constants

Hence $L^*(H,\theta) = M_1H^2 + M_2$ .... (73)

$\phi^*(H,\theta) = \cot^{-1}M_3$ .... (74)

is a solution set of the system of partial differential equations (52) and (53) using the Legendre transform function from (73) and (74) in equation (59) and (65) to (68) expressions for the magnetic component, the vorticity and the current density are obtained as

$$H_1 = \frac{-y}{2M_1}, \quad H_2 = \frac{x}{2M_1}, \quad \omega = 0, \quad j = \frac{1}{M_1} \qquad \text{......... (75)}$$

The velocity components by using (73), (74) and (75) in (20) are given by

$$u = \frac{K}{H^2}\left(\frac{-y}{2M_1}\cot\phi + \frac{x}{2M_1}\right) = \frac{k}{H^2}\left(\frac{(-y\cot\phi + x)}{2M_1}\right) = \frac{k}{H^2}\left(\frac{(-yM_3 + x)}{2M_1}\right) \qquad \text{...... (76)}$$

$$v = \frac{K}{H^2}\left(\frac{(xM_3 + y)}{2M_1}\right) \qquad \text{..........(77)}$$

Finally employing (75) and (76) in (22) and (23) ,and integrating we find the function h(x,y) and substituting h(x,y) in (10) we find the pressure function as

$$P = \frac{\mu}{4M_1}(x_2+y_2) - \frac{nM_1}{M_2} k \left[\ln (x_2+y_2) + 2M_3\tan^{-1}(x/y)\right] - \frac{2\rho\delta^2}{M_1}\,\frac{M1\,(1+M_3)}{x_2+y_2} + M_4$$

where M4 is an arbitray constant summing up

A variably inclined steady MHD flow through porous media using Hodograph transformation problem with the families of stream lines and magnetic lines given by $x^2+y^2=$ constant and $M_3\,(\ln(x^2+y^2)) - \tan^{-1}(x/y) =$ constant

5. Flow With Magnetic Magnitude Constant on Each Individual Stream Line

Using (24), equation (27) yields

$$2H_1H_2\frac{\partial H_1}{\partial x} + (H_2^2 - H_1^2)\left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial x}\right) - 2H_1H_2\frac{\partial H_2}{\partial x} - 2\cot\phi\, H_1^2\frac{\partial H_1}{\partial x} + H_1H_2\left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial x}\right) + H_2\left(\frac{\partial H_2}{\partial y}\right) =$$

..........(78)

If the magnetic magnitude is constant on each individual stream line then we have H.∇H2 = 0

$$H_1^2\frac{\partial H_1}{\partial x} + (H_1H_2)\left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial x}\right) + H_2^2\frac{\partial H_2}{\partial y} = 0 \qquad \text{..........(79)}$$

Using (79) in (78), the continuity condition becomes

$$2H_1H_2\frac{\partial H_1}{\partial x} + (H_2^2 - H_1^2)\left(\frac{\partial H_1}{\partial y} + \frac{\partial H_2}{\partial x}\right)^2 - 2H_1H_2\frac{\partial H_2}{\partial x} = 0 \qquad \text{..........(80)}$$

So, if the magnetic magnitude is constant along each individual stream line of an incompressible fluid, then H_1, H_2 must satisfy the equation (9), (79) and (80), we discuss thes following cases :

Case 5.1 : Let $J \neq 0$: In the flow region under consideration let the flow variables $H_1(x,y)$, $H_2(x,y)$ be such that the transformation Jacobian

$$J = \frac{\partial(H_1, H_2)}{\partial(x,y)} \neq 0. \text{ I.e. } 0 < |J| < \infty$$

then from (49) and (50) we get

$$(H\cos\theta, H\sin\theta) \frac{\partial^2(H\cos\theta, H\sin\theta)}{\partial H_1^2} - (H^2\cos^2\theta - H^2\sin^2\theta) - \frac{\partial^2(H\cos\theta, H\sin\theta)}{\partial H_1 \delta H_2} - (H\cos\theta, H\sin\theta) \frac{\partial^2(H\cos\theta}{\partial H}$$

$$H \frac{\partial^2 L^*}{\partial H \partial\theta} - \frac{\partial L^*}{\partial\theta} = 0 \qquad \text{..........(81)}$$

$$H^2 \frac{\partial^2 L^*}{\partial H^2} - H \frac{\partial L^*}{\partial H} - \frac{\partial^2 L^*}{\partial\theta^2} = 0 \qquad \text{..........(82)}$$

General solution of (81) is given by,

$$L^*(H,\theta) = H(\alpha(\theta)) + \beta(H) \qquad \text{.......... (83)}$$

where $\alpha(\theta)$ and $\beta(H)$ are arbitrary functions of their arguments. Using (83) in (82) we get

$$(H\beta^{11} - \beta^1) - (\alpha^{11} + \alpha) = 0 \text{ and } \quad H(\beta^{11} - \beta^1) = (\alpha^{11} + \alpha) = m_1 \text{ (say)}$$

where m_1 is a constant and primes denote the differentiation with respect to the arguments

Now $H\beta^{11} - \beta^1 = m^1$ gives $\beta(H) = m/2H^2 - m_1H + m_2$ .... (84)

where m_1 and m_2 are constants and

$\alpha^{11} + \alpha = m_1$ gives $\alpha(\theta) = A\cos\theta + B\sin\theta + m_1$ .... (85)

where A and B are arbitrary constant. Substituting (84) and (85) in (83) gives

$L^*(H,\theta)=H(A\cos\theta+B\sin\theta+m_1)+1/2mH^2-m_1H+m_2$

The above with the help of (49) and (50) can be written as

$$L(H_1,H_2)=AH_1+BH_2+(1/2m)(H_1^2+H_2^2)+m_2 \qquad \text{.........(86)}$$

Then $x= B+mH_2$, $y=-(A+mH_1)$, $H_2=(1/m)(x-B)$, $H_1=-(y+A)/m$ .... (87)

$$j = \frac{\partial((1/m)(x-B))}{\partial x} - \frac{\partial(-(y+A)/m)}{\partial y}$$

$$j = 2/m \qquad \text{......... (88)}$$

$$u = km \frac{(x-B) - (y+A)\cot\phi}{(x-B)^2 + (y+A)^2} \quad \text{and}$$

$$v = km \frac{(x-B)\cot\phi + (y+A)}{(x-B)^2 + (y+A)^2}$$

$$h = h_0 + \frac{\rho k}{m^2}[(x-B)^2+(y+A)^2] + \frac{\eta}{km}(Ax-By+xy) \qquad \text{... ... ... (89)}$$

If $K \rightarrow \infty$, i.e. porous media is removed, we get all the results of Sattar and Chandna for velocity vector. The pressure is given by

$$P = P_0 + \frac{\rho k}{m^2}(1/2(x^2+y^2)) + (Ay-Bx) + \frac{\eta}{km}(Ax-By+xy) \qquad \text{... ... ... (90)}$$

and stream lines are given by $(x-B)^2+(y+A)^2 = \text{constant}$ ....(91)
which represent concentric circles whose centre at (B, -A).

Case 5.2:. Let J = 0 : Let the flow variables H1(x,y), H2(x,y) be such that in the region of flow the Jacobian

$$J = \frac{\partial(H_1,H_2)}{\partial(x,y)} = 0$$

In such a case either H^1 is a function of H_2 or H_2 is a function of H_1. We consider H_2 as a function of H_1.

$$H_2 = \phi(H_1) \qquad \text{.......... (92)}$$

where ϕ is an arbitary constant of H_1. Using (92) in equation (30), we obtain

$$\frac{\partial H_1}{\partial x} + \phi^1(H_1)\frac{\partial H_1}{\partial y} = 0 \qquad \text{......... (93)}$$

Eliminating $(\partial H_1/\partial x)$ from (93) and (80) we get

$$(H_1 + \phi\phi^1) \quad (H_1\phi^1 - \phi)\, H_1\, (\partial H_1/\partial y) = 0$$

From the above we have the following :

case 5.2.1 : $\frac{\partial H_1}{\partial y} = 0, \quad H_1\phi^1 - \phi \neq 0$

case 5.2.2 : $H_1\phi^1 - \phi = 0$

case 5.2.3 : $H_1 + \phi\phi^1 = 0$

case 5.2.4 : $\frac{\partial H_1}{\partial y} = 0,$ implies $\frac{\partial H_1}{\partial x} = 0$

i.e. $H_1 = k_1$ and $H_2 = k_2$ hence the solutions are $\omega=0$, $j=0$, H_1=constant, H_2=constant,

$$h = h_0 - (\eta/km)(k_1x+k_2y)$$

$$P = P_0 - (\eta/km)(k_1x+k_2y)$$

$$P_0 = \text{constant} = h_0 = (\rho/2)(k_1^2 + k_2^2)$$

The equation of stream line is given by

$K_2x - K_1y$ = constant and represents a parallel straight line.

Case 5.2.2 : $H_1\phi^1 - \phi = 0$ implies $\phi = k_3H_1$ i.e. $\phi_1 = k3$.

$$\text{i.e.} \quad \frac{\partial H_1}{\partial x} + k_3 \frac{\partial H_1}{\partial y} = 0$$

General solution of the equation is given by $H_1 = f(k_3x - y)$ and $H_2 = k_3f(k_3x-y)$ which implies $f1(k_3x-y) = 0$

Case 5.2.3 : Put restriction on ϕ such that $H_1 + \phi^2 =$ constant

hence $H_1 = f(k_3x - y)$ and $H_2 = k_3f(k_3x-y)$, $\omega = 0, j = 0$

$$u = \frac{k(\cot\phi - k_3)}{(1+k_3)f}, \quad v = \frac{k(1 + \cot\phi k_3)}{(1+k_3)f}, \quad h = h_0 - (\eta/k)(xf+k_3yf)$$

$$p = p_0 - (\rho/2)(1+k_3^2)f^2 - (\eta/k)f(x+k_3y)$$

and stream line is given by $k_3x-y =$ constant and are parallel straight lines.

Conclusion

It is shown that when the velocity and magnetic vectors of MHD flow are constantly variably inclined then the streamlines are concentric circles and the magnetic lines are spirals. These are shown by means of graphs (figures 3 & 4). Also graphs are plotted for velocity components u,v against H1 (figures 1a,1b & 2). In figure 1a, the curves almost coincide at angles 45° & 75°. In figure 1b, the curves almost coincides at angles 70° & 67°, 68° & 65° respectively. But however for angle 66° the curve steeps above. This is because velocity and magnetic vectors are variably inclined.

References

1. Chandna. O. P. Toews.H. and Nath.V.I.(1975) Can. J. Phys. 53.
2. Ram.G. and Mishra.R.S. (1977) Ind. Jour.Pure and applied maths 8, 161
3. Sattar M.A. and Chandna O.P. (1988) Jour.maths.phy.sc. 22,3
4. Yamamoto K. (1971) Jour.phy.soc. Japan 31, 572

5. L.D.Landau and E.M.Lifshitz, Fluid mechanics, Pergamon press, Oxford. Vol.6 (1959).

6. C.S.Bagewadi and Siddabasappa (1991) Ganitha, Lucknow, Vol.42, 13-21.

7. C.S.Bagewadi and Siddabasappa (1993) Bulletin of Cal. Math. Soc. vol 85, 513-520.

8. K.K.Singh and D.P.Singh (1993) Bull. Cal. Math. Soc 85, 255-262

9. C.S.Bagewadi and Siddabasappa (1991), Journal of karnataka University, Sci. Dharwad, Vol.35, 131-146.

10. O.P.Chandna and Nguyen P.V. (1992) Int. J .Engg. .Sci. 30,69.

11. Siddabasappa & C.S.Bagewadi (1992) Kar.Uni.Science Journal, Dharwad, Vol.36, 103-113.

12. C.S.Bagewadi & Siddabasappa (1995) Bulletin of Calcutta, Mathematical Society, Calcutta, Vol .87, 411-485.

13. Yamamoto. K. (1973) Jour.Phys. Soc. Japan 34, 814.

14. O.P.Chandna, R.M. Barron and K.T.Chew (1982), J.Engg. Math., Vol.16, 223-243.

15. R.M.Barron and O.P.Chandna (1981), J.Engg. Math., Vol.5, No.3, 211-220.

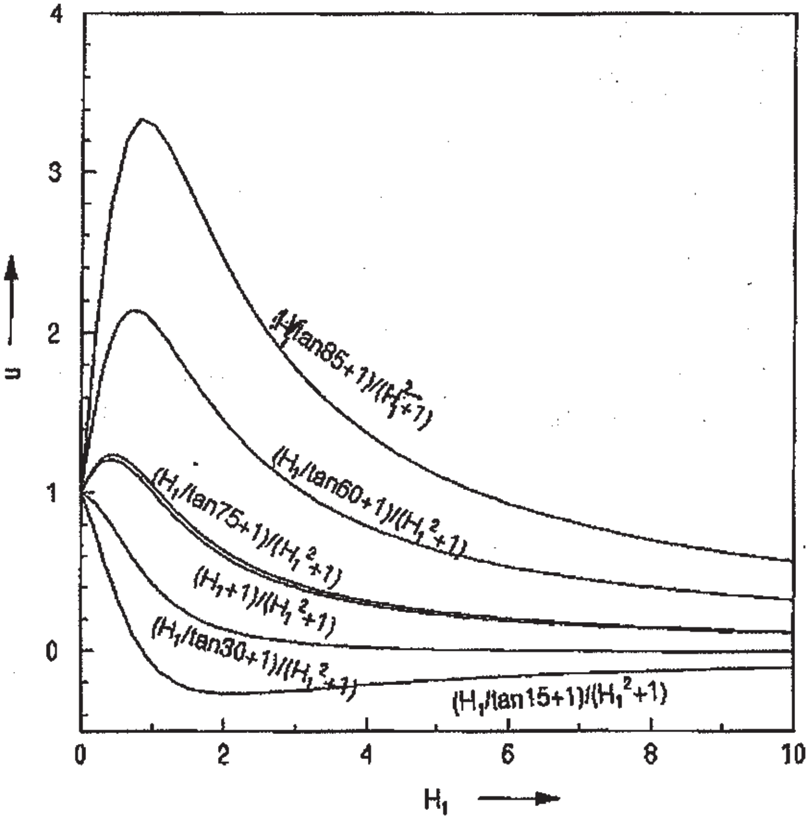

Fig. 1a Variation of u with respect to H_1

$$k = 1, \; H_2 = 1$$

$$u = \frac{k}{H^2} (H_1 \cot \phi + H_2)$$

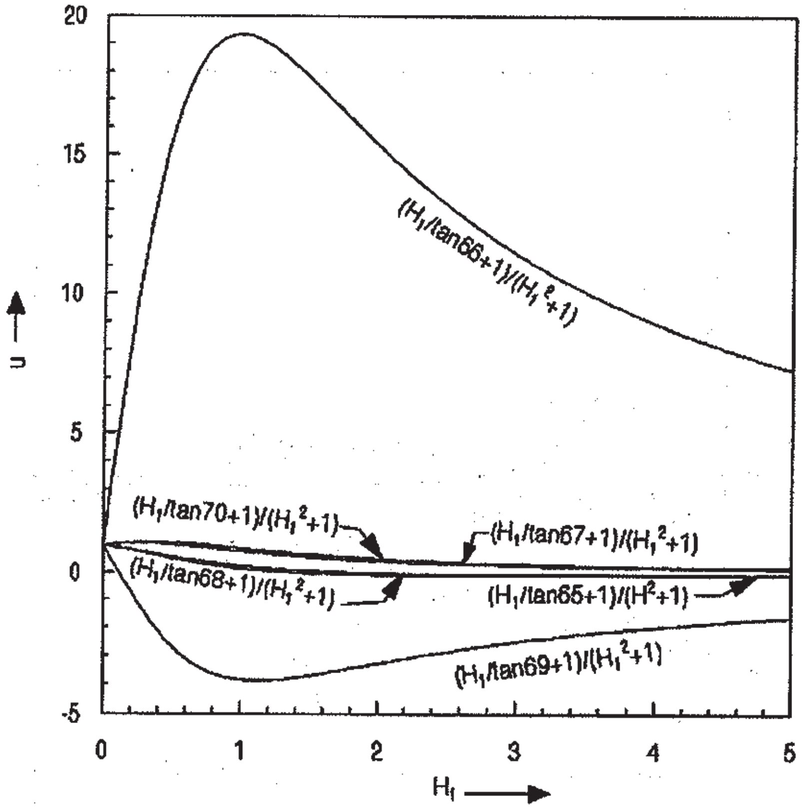

Fig. 1b Variation of u with respect to H_1

$k = 1, H_2 = 1$

$$u = \frac{k}{H^2} (H_1 \cot \phi + H_2)$$

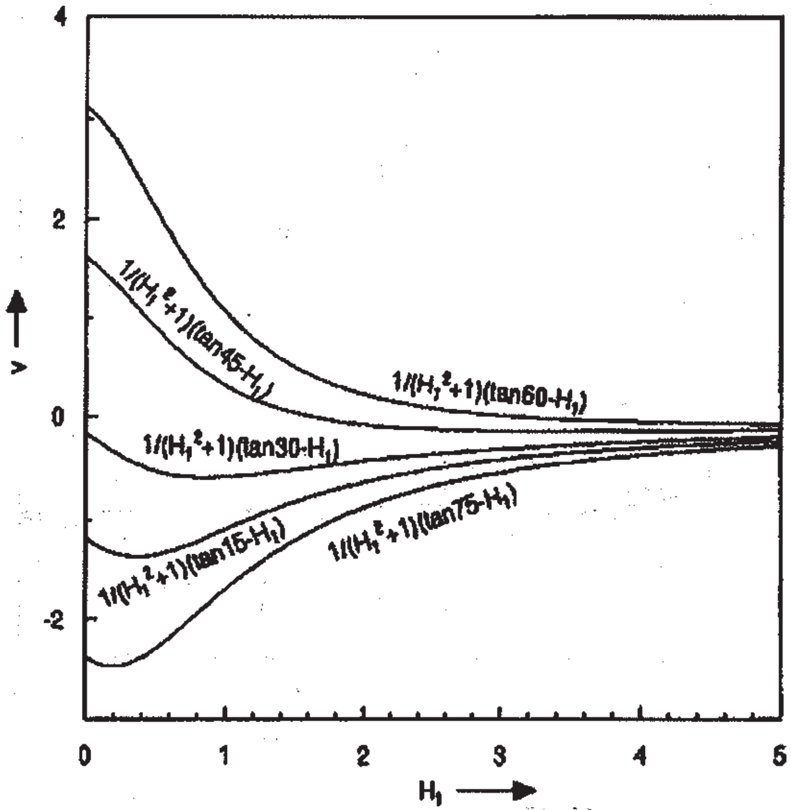

Fig. 2 Variation of v with respect to H_1

$k = 1, \; H_2 = 1$

$$v = \frac{k}{H^2}\left(H_2 \cot \phi - H_1\right)$$

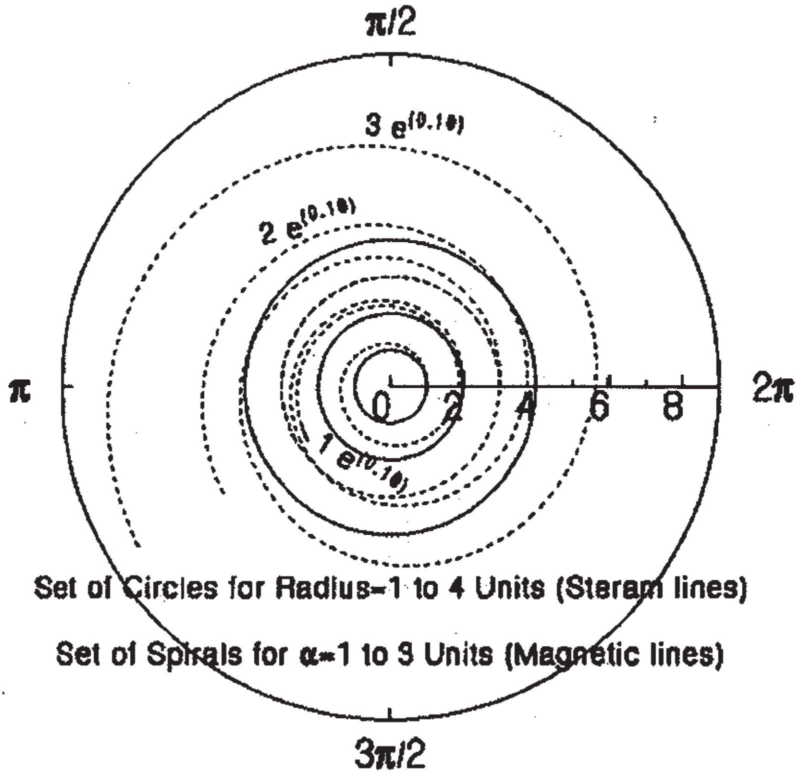

Fig. 3 Variation of Stream lines and Magnetic lines

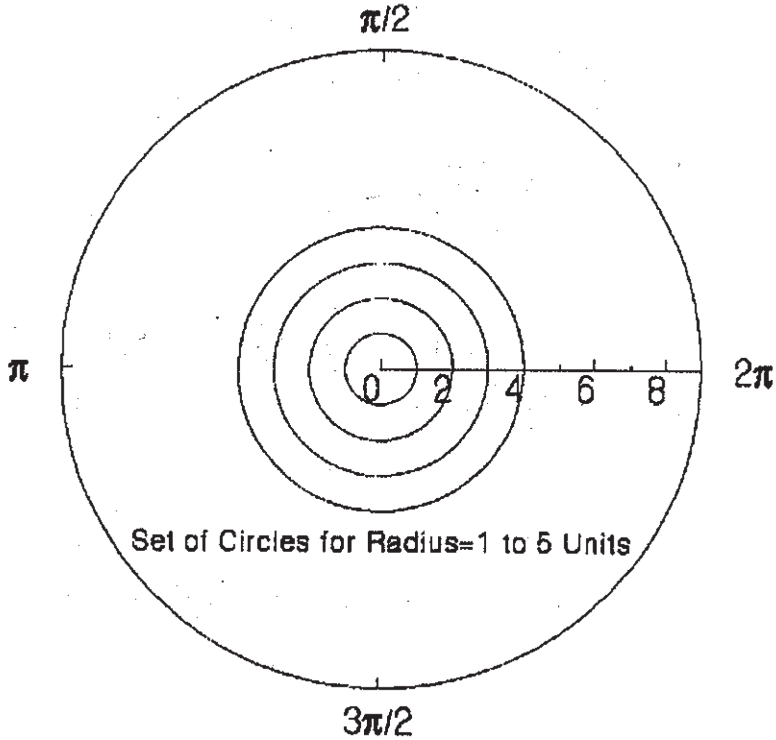

Fig. 4 Variation of $u^2+v^2=\sqrt{(x^2+y^2)}$