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TWO-PIECE CUBIC SPLINE FUNCTIONS

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Abstract

P.M. Prenter defines a cubic spline function in an interval [a,b] as a piecewise cubic polynomial which is twice continuously differentiable in the entire interval [a,b]. The smooth cubic spline functions fitting the given data are the most popular spline functions and when used for interpolation, they do not have the oscillatory behaviour which characterizes high-degree polynomials. The natural spline has been shown to be the unique function possessing the minimum curvature property of all functions interpolating the data and having square integrable second derivative. In this sense, the natural cubic spline is the smoothest function which interpolates the data. Here Two-piece Natural Cubic Spline functions have been defined. An approximation with no indication of its accuracy is utterly valueless. Where an approximation is intended for the general use, one must, of course, go for the trouble of estimating the error as precisely as possible. In this section, an attempt has been made to derive closed form expressions for the error-functions in the case of Two-piece Spline functions.

TWO-PIECE CUBIC SPLINE

Let $f(x)$ be a function defined as thrice differentiable in the interval $[x_{i-1}, x_{i+1}]$ and let the cubic spline interpolant of $f(x)$ be $s_j(x)$, $j = i-1, i$ in the subintervals $[x_j, x_{j+1}]$ and defined [2] as follows.

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$s(x) = s_j(x) = a_j (x - x_j)^3 + b_j (x - x_j)^2 + c_j (x - x_j) + d_j, \quad j = i - 1, i.$

Since $s_i - 1(x)$ matches with $f(x)$ at the nodes $x = x_{i-1}$ and $x = x_i$,

$s_{i-1}(x_{i-1}) = f(x_{i-1}) \Rightarrow d_{i-1} = f(x_{i-1}) = f_{i-1}$, say ... (1)

$s(x) = s_j(x) = a_j (x - x_j)^3 + b_j (x - x_j)^2 + c_j (x - x_j) + d_j, \quad j = i - 1, i.$

and

$s_{i-1}(x_i) = f(x_i) \Rightarrow a_{i-1}h^3 + b_{i-1}h^2 + c_{i-1}h + d_{i-1} = f_i$ ... (2)

where $x_i - x_{i-1} = h = x_{i+1} - x_i$.

Since si(x) matches with f (x) at the nodes $x = x_i$ and $x = x_{i+1}$,

$s_i(x) = f(x_i) \Rightarrow d_i = f_i$ ... (3)

and

$s_i(x_{i+1}) = f(x_{i+1}) \Rightarrow a_ih^3 + b_ih^2 + c_ih + d_i = f_{i+1}$ ... (4)

Since the first and second derivatives of the spline functions match at the interior point namely x_i, we have

$s'_{i-1}(x_i) = s'_i(x_i) \Rightarrow 3a_{i-1}h^2 + 2b_{i-1}h + c_{i-1} = c_i$ ... (5)

and

$s''_{i-1}(x_i) = s''_i(x_i) \Rightarrow 6a_{i-1}h + 2b_{i-1} = 2b_i$ ... (6)

In the case of natural splines, the second derivatives at the end-nodes are zero.

i.e., $s''_{i-1}(x_{i-1}) = 0 \Rightarrow 2b_{i-1} = 0$ ... (7)

and $s''_i(x_{i+1}) = 0 \Rightarrow 6a_ih + 2b_i = 0$ ... (8)

Solving equations (1) to (8), the unknowns in $s_{i-1}(x)$ and $s_i(x)$ are got as given below

$$a_{i-1} = \frac{1}{4h^3} [f_{i+1} - 2f_i + f_{i-1}]$$

$$b_{i-1} = 0$$

$$c_{i-1} = \frac{1}{4h} [- f_{i+1} + 6f_i - 5f_{i-1}]$$

$$d_{i-1} = f_{i-1}$$

$$a_i = -\frac{1}{4h^3}\left[f_{i+1} - 2f_i + f_{i-1} \right]$$

$$b_i = \frac{3}{4h^2}\left[f_{i+1} - 2f_i + f_{i-1} \right]$$

$$c_i = \frac{1}{2h}\left[f_{i+1} - f_{i-1} \right]$$

and $d_i = f_i$

$$\text{Hence, } s_{i-1}(x) = \frac{1}{4h^3}(f_{i+1} - 2f_i + f_{i-1})(x - x_{i-1})^3$$

$$+ \frac{1}{4h}(- f_{i+1} + 6f_i - 5f_{i-1})(x - x_{i-1}) + f_{i-1} \qquad \ldots (9)$$

$$\text{and } s_i(x) = \frac{1}{4h^3}(f_{i+1} - 2f_i + f_{i-1})(x - x_i)^3$$

$$+ \frac{3}{4h^2}(f_{i+1} - 2f_i + f_{i-1})(x - x_i)^2$$

$$+ \frac{1}{2h}(f_{i+1} - f_{i-1})(x - x_i) + f_i$$

Error Analysis based on Taylor Series Method

While employing Taylor Series Method to analyse the error involved in approximating a function by a cubic spline polynomial, following lemmas are made use of.

Lemma 1

If $f \varepsilon C^1[a,b]$ and c_1 and c_2 are of the same sign, then

$c_1 f(x_1) + c_2 f(x_2) = (c_1 + c_2) f(\xi)$ where $\xi \varepsilon (x_1, x_2)$ and $(x_1, x_2) \varepsilon [a,b]$. [4]

Lemma 2

If $f \varepsilon C^1[a,b]$ and c_1 and c_2 are of the same sign, then

$c_1 f(x_1) - c_2 f(x_2) = (c_1 - c_2) f(x_1) + c_2 (x_1 - x_2) f'(\xi)$ where $\xi \varepsilon (x_1, x_2)$ and $(x_1, x_2) \varepsilon [a,b]$. [1]

Now, for the Two-piece Cubic Spline interpolation,

let $E(x) = E_j(x) = f(x) - s_j(x)$, where $j = i - 1$

and i respectively, denote the error [3, 5, 6, 7, 8] for $x_j < x < x_{j+1}$.

Consider $E_{i-1}(x) = f(x) - s_{i-1}(x)$, $x_{i-1} < x < x_i$.

Upon setting $x = x_{i-1} + \theta h$, $0 < \theta < 1$, the associated relation will be

$$\bar{E}_{i-1}(\theta) = \bar{f}(\theta) - \bar{s}_{i-1}(\theta)$$

$$= f_{i-1} + \theta h f'_{i-1} + \frac{\theta^2 h^2}{2} f''_{i-1} + \frac{\theta^3 h^3}{3!} f''(\xi)$$

$$- \left[\frac{\theta^3}{4} (f_{i+1} - 2f_i + f_{i-1}) + \frac{\theta}{4} (-f_{i+1} + 6f_i - 5f_{i-1}) + f_{i-1} \right],$$

where the unknown $\xi \varepsilon (x_{i-1}, x_i)$.

[$\bar{E}$, $\bar{f}$ notations indicate that these functions of θ may be different from the functions E, f of x, respectively. Of course, $\bar{E}(\theta) = E(x)$ and $\bar{f}(\theta) = f(x)$.]

$$\bar{E}_{i-1}(\theta) = f_{i-1} + \theta h f'_{i-1} + \frac{\theta^2 h^2}{2} f''_{i-1} + \frac{\theta^3 h^3}{3!} f'''(\xi)$$

$$-\frac{\theta^3}{4}\{[f_{i-1} + 2hf'_{i-1} + 2h^2 f''_{i-1} + \frac{4}{3} h^3 f'''(\xi_1)]$$

$$-2[f_{i-1} + hf'_{i-1} + \frac{h^2}{2} f''_{i-1} + \frac{h^3}{6} f'''(\xi_2)] + f_{i-1}\}$$

$$-\frac{\theta^3}{4}\{[f_{i-1} + 2hf'_{i-1} + 2h^2 f''_{i-1} + \frac{4}{3} h^3 f'''(\xi_1)]$$

$$+6[f_{i-1} + hf'_{i-1} + \frac{h^2}{2} f''_{i-1} + \frac{h^3}{6} f'''(\xi_2)] - 5f_{i-1}\} - f_{i-1}.$$

where $\xi_1 \in (x_{i-1}, x_{i+1})$ and $\xi_2 \in (x_{i-1}, x_i)$.

Upon simplification,

$$\bar{E}_{i-1}(\theta) = -\frac{\theta}{2}(1-\theta)^2 \frac{h^2}{2} f''_{i-1} + \frac{h^3}{6}[\theta^3 f'''(\xi)$$

$$-\frac{\theta}{2}(3-\theta^2)\ f'''(\xi_2) + 2\theta(1-\theta^2)\ f'''(\xi_1)]$$

$$= -\frac{\theta}{2}(1-\theta)^2 \frac{h^2}{2} f''_{i-1} + \frac{h^3}{6}[\theta(2-\theta^2) f'''(\xi_3)$$

$$-\frac{\theta}{2}(3-\theta^2) f'''(\xi_2)]$$

where $\xi_3 \in (\xi, \xi_1) \in (x_{i-1}, x_{i+1})$ using lemma 1.

$$= -\frac{\theta}{2}(1-\theta)^2 \frac{h^2}{2} f''_{i-1} + \theta(1-\theta^2)\frac{h^3}{12}\ f'''(\xi_3)$$

$$+\frac{\theta}{2}(3-\theta^2)\frac{h^3}{6}[f'''(\xi_3) - f'''(\xi_2)]$$ using lemma 2.

$$= -\frac{\theta}{2}(1-\theta)^2 \frac{h^2}{2} f''_{i-1} + \theta(1-\theta^2)\frac{h^3}{12} f'''(\xi_3)$$

$$+ \theta(3-\theta^3)\frac{h^3}{12}(\xi_3-\xi_2) f^{iv}(\xi_4)]$$

where $\xi_4 \; \varepsilon \; (\xi_2, \xi_3) \; \varepsilon \; (x_{i-1}, x_{i+1})$.

$$\text{Hence, } \quad \bar{E}_{i-1}(\theta) = -\theta(1-\theta)^2 \frac{h^2}{4} f''_{i-1} + \theta(1-\theta^2)\frac{h^3}{12} f'''(\xi_3)$$

$$+ \theta(3-\theta^2)\,\theta^* \frac{h^4}{12} f^{iv}(\xi_4)$$

where $\theta^* h = \xi_3 - \xi_2$ and $\theta^* \; \varepsilon \; [0, 2]$ as $\xi_2, \xi_3 \; \varepsilon \; (x_{i-1}, x_{i+1})$.

Let Max $|f'''(\xi_3)| = M_3$ and Max $|f^{iv}(\xi_4)| = M_4$, $x \; \varepsilon \; (x_{i-1}, x_{i+1})$.

Maximum of $|\theta(1-\theta)^2|$ is $\frac{4}{27}$, attained when $\theta = \frac{1}{3}$.

Maximum of $|\theta(1-\theta)^2|$ is $\frac{2}{3\sqrt{3}}$, when $\theta = \frac{1}{\sqrt{3}}$

Maximum of $|\theta(3-\theta)^2|$ is 2, when $\theta = 1$.

Maximum of $\theta^* = 2$.

It follows that,

$$|E_{i-1}(x)| = |\bar{E}_{i-1}(\theta)| \leq \frac{1}{27} h^2 |f''_{i-1}| + \frac{1}{18\sqrt{3}} h^3 M_3 + \frac{1}{3} h^4 M_4 \quad \ldots (11)$$

Now consider,

$E_i(x) = f(x) - s_i(x)$, $x_i < x < x_{i+1}$

Upon setting $x = x_i + \theta h$, $0 < \theta < 1$,

$E_i(\theta) = \overline{f}(\theta) - \overline{s_i}(\theta)$

$$= f_i + \theta h f'_i + \frac{\theta^2 h^2}{2} f''_i + \frac{\theta^3 h^3}{6} f'''(\xi_5)$$

$$- \{ - \frac{\theta^3}{4}(f_{i+1} - 2f_i + f_{i-1}) + \frac{3\theta^2}{4}(f_{i+1} - 2f_i + f_{i-1})$$

$$+ \frac{\theta}{2}(f_{i+1} - f_{i-1}) + f_i\} \quad \text{where } \xi_5 \in (x_i, x_{i+1}).$$

$$= f_i + \theta h f'_i + \frac{\theta^2 h^2}{2} f''_i + \frac{\theta^3 h^3}{6} f'''(\xi_5)$$

$$+ \frac{\theta^3}{4}[f_i + hf'_i + \frac{h^2}{2} f''_i + \frac{h^3}{6} f'''(\xi_6) - 2f_i$$

$$+ f_i - hf'_i + \frac{h^2}{2} f''_i - \frac{h^3}{6} f'''(\xi_7)]$$

$$- \frac{3\theta^2}{4}[f_i + hf'_i + \frac{h^2}{2} f''_i + \frac{h^3}{6} f'''(\xi_6) - 2f_i$$

$$+ f_i - hf'_i + \frac{h^2}{2} f''_i - \frac{h^3}{6} f'''(\xi_7)]$$

$$- \frac{\theta}{2}\{f_i + hf'_i + \frac{h^2}{2} f''_i + \frac{h^3}{6} f'''(\xi_6)$$

$$- [f_i - hf'_i + \frac{h2}{2} f''_i - \frac{h3}{6} f'''(? 7)] \}$$

where $\xi_6 \ \varepsilon \ (x_i , x_{i+1})$ and $\xi_7 \ \varepsilon \ (x_{i-1} , x_i)$.

$$\overline{E}_i(\theta) = -\frac{\theta^2}{2}(1-\theta)\frac{h^2}{2} f''_i + \frac{\theta^3 h^3}{6} f'''(\xi_5)$$

$$-\frac{\theta}{4}(2+3\theta-\theta^2)\frac{h^3}{6} f'''(\xi_6) - \frac{\theta}{4}(2-3\theta+\theta^2)\frac{h^3}{6} f'''(\xi_7)$$

$$\overline{E}_i(\theta) = -\frac{\theta^2}{2}(1-\theta)\frac{h^2}{2} f''_i + \frac{\theta^3 h^3}{6} f'''(\xi_5) - \frac{\theta h^3}{6} f'''(\xi_8)$$

where $\xi_8 \ \varepsilon \ (\xi_6 , \xi_7) \ \varepsilon \ (x_{i-1} , x_{i+1})$ using lemma 1.

$$= -\frac{\theta^2}{2}(1-\theta)\frac{h^2}{2} f''_i - \frac{\theta}{6}(1-\theta^2) h^3 f'''(\xi_5)$$

$$+ \frac{\theta h^3}{6}(\xi_5 - \xi_8) f^{iv}(\xi_9)$$

where $\xi_5 - \xi_8 = \theta^* h$.

Let Max $|f'''(\xi_5)| = M_3$ and Max $|f^{iv}(\xi_9)| = M_4$
It follows that,

$$| E_i(x) | = | \overline{E}_i(\theta) | \leq \frac{1}{27} h^2 | f''_i | + \frac{1}{9\sqrt{3}} h^3 M_3 + \frac{1}{3} h^4 M_4 \quad \ldots (12)$$

where $x \ \varepsilon \ [x_i , x_{i+1}]$.

A perusal of the inequality (12) paired with (11) reveals that

$$| E(x) | \leq \frac{1}{27} h^2 L_2 + \frac{1}{9\sqrt{3}} h^3 M_3 + \frac{1}{3} h^4 M_4 , \ x \ \varepsilon \ [x_{i-1} , x_{i+1}] \quad \ldots (13)$$

where $L_2 = \text{Max}\,(|f''_i|, |f''_{i-1}|)$.

However, it is important to observe that, had we changed the origin to (xi + 1 + xi - 1 , 0) and reversed the x-axis, then h, M_3 and M_4 would have remained the same and (12) would read (11) with f''_{i-1} replaced by f''_{i+1} in the first term. Thus,

$$|E_i(x)| \leq \frac{1}{27} h^2 |f''_{i+1}| + \frac{1}{18\sqrt{3}} h^3 M_3 + \frac{1}{3} h^4 M_4 \quad \ldots (14)$$

for $x \in [x_i, x_{i+1}]$. Inequality (14) appears to be better than inequality (13) since the numerical coefficient is halved in the second term. In conclusion, the following inequality is recommended.

$$\boxed{\begin{array}{l} |E(x)| \leq \dfrac{1}{27} h^2 L_2 + \dfrac{1}{18\sqrt{3}} h^3 M_3 + \dfrac{1}{3} h^4 M_4, \qquad \ldots (15) \\ \text{where } x \in [x_{i-1}, x_{i+1}] \text{ and } L_2 = \text{Max}\,(|f''_{i-1}|, |f''_{i+1}|). \end{array}}$$

Here, we overlook (13).

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