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KENMOTSU MANIFOLDS WITH CONSERVATIVE CONFORMAL CONCIRCULAR AND CONHARMONIC CURVATURE TENSORS

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ABSTRACT

We study the geometry of Kenmotsu manifolds when the conformal, con circular and con harmonic curvature tensors are conservative.

Keywords

Contact, Kenmotsu, conformal and con circular & con harmonic

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1. Preliminaries

Definition 1.1: Let (M^n, g) $(n = 2m + 1)$ be a contact Riemannian manifold with contact form η, the associated vector form ξ, (1, 1) tensor field φ and the associated Riemannian metric g such that (Blair 1976)

$$\left.\begin{array}{ll} \varphi^2 X = -X + \eta(X)\xi, \quad \varphi(\xi) = 0, & \eta(\xi) = 1, \ \eta.\varphi = 0 \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\,\eta(Y) & g(X, \xi) = \eta(X) \end{array}\right\} \quad (1.1)$$

where X & Y are any vector fields on M.

Definition 1.2: An almost contact metric manifold with structure tensors (j, x, h, g) is called Kenmotsu manifold if

$$(\nabla_X \varphi)Y = -\eta(Y)\,\varphi X - g(X, \varphi Y)\,\xi \quad (1.2)$$

$$\nabla_X \xi = X - \eta(X)\xi \quad (1.3)$$

Further we have

$$S(X, \xi) = -(n - 1)\eta(X) \quad (1.4)$$

$$L\xi g = 2(g - \eta \otimes \eta) \quad (1.5)$$

$$(L\xi S)(Y, Z) = 2S(X, Y) + 2(n - 1)\eta(X)\eta(Y) \quad (1.6)$$

$$R(\xi, X)\xi = X - \eta(X)\xi \quad (1.7)$$

For a symmetric endomorphism Q of the tangent space at a point of M, we express the Ricci tensor S as

$$S(X, Y) = g(QX, Y) \quad (1.8)$$

2. Conformal curvature tensor satisfying div C= 0

The conformal curvature tensor is given by

$$C(X, Y)Z = R(X, Y)Z + \{1/(n-2)\}\{ S(X, Z)Y - S(Y, Z)X + g(X, Z)QY - g(Y, Z)QX\}$$

$$- \{r/(n-1)(n-2)\}\{g(X, Z)Y - g(Y, Z)X\} \qquad (2.1)$$

Differentiating (2.1) covariantly and contracting we obtain

$$(\text{div } C)(X, Y)Z = \{(n-3)/(n-2)\}\{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\}$$

$$-1/(n-1)\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\} \qquad (2.2)$$

Let us suppose that in a Kenmotsu Riemannian manifold

$$\text{Div } C = 0 \qquad (2.3)$$

Using (2.2) & (2.3) we have

$$(\nabla_X S)(Y, Z) - (\nabla_Y S(X, Z)$$

$$= \{(n-2)/(n-1)(n-3)\}\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\} \qquad (2.4)$$

Using (1.1), (1.3), (1.4), (1.5) & (1.6) we have

$$(\nabla\xi S)(Y, Z) = (L\xi S)(Y, Z) - S(\nabla_Y\xi, Z) - S(Y, \nabla_Z\xi) = 0 \qquad (2.5)$$

and $$(\nabla_Y S)(\xi, Z) = YS(\xi, Z) - S(\nabla_Y\xi, Z) - S(\xi, \nabla_Y Z)$$

$$= -S(Y, Z) - (n-1)g(Y, Z) \qquad (2.6)$$

Substitute (2.5) & (2.6) in (2.4) we have

$$S(Y, Z) + (n - 1)g(Y, Z)$$

$$= \{(n - 2)/(n - 1)(n - 3)\}\{g(Y, Z)dr(\xi) - g(x,Z)dr(\Psi)\} \quad (2.7)$$

Replace Z by φZ in (2.7)

$$S(Y, \varphi Z) = [\{(n-2)/(n-1)(n-3)\} - (n-1)]\, g(Y, \varphi Z)dr(\xi) \quad (2.8)$$

Again replace Z by φZ in (2.8) and simplifying we have

$$S(Y, Z) = [\{(n - 2)dr(\xi)/(n - 1)(n - 3)\} - (n - 1)]g(Y, Z)$$

$$+ [-\{(n - 2)dr(\xi)/(n - 1)(n-3)\}]\eta(Y)\eta(Z) \quad (2.9)$$

$$S(Y, Z) = Ag(Y, Z) + B\, \eta(Y)\eta(Z) \quad (2.10)$$

where $A = [\{(n - 2)dr(\xi)/(n - 1)(n - 3)\} - (n - 1)]$ and

$$B = -\{(n - 2)dr(\xi)/(n - 1)(n-3)\}$$

Thus we have the following result.

Theorem 2.1 : On a Kenmotsu manifold M, if the conformal curvature tensor C of type (1, 3) is conservative then M is h - Einstien, that is M satisfies equation (2.10), where the constants A and B satisfies the condition $A + B = -(n - 1)$, provided $n \neq 1, 3$.

Also from (1.8) & (2.10) we have

$$S(Y, Z) = g(QY, Z)$$

$$= Ag(Y, Z) + B\eta(Y)\eta(Z)$$

$$= g(AY, Z) + B\,\eta(Y)g(\xi, Z)$$

$$= g(AY + B\,\eta(Y)\xi,\ Z)$$

so that $QY = AY + B\,\eta(Y)\xi$ (2.11)

$$Q\xi = Ax + B\,\eta(x)x = Ax + Bx = (A + B)x$$

$$Q\xi = -(n - 1)\xi \tag{2.12}$$

By taking $X = Z = \xi$ and $Y = X$ in (2.1) we have

$$C(\xi, X)\xi = R(\xi, X)\xi + 1/(n - 2)\{S(\xi, \xi)X - S(X, \xi)\xi$$

$$+ g(\xi, \xi)QX - g(X, \xi)Q\xi\}$$

$$-\{ r/(n - 1)(n - 2)\}\{g(\xi, \xi)X - g(X, \xi)\xi\} \tag{2.13}$$

$$C(\xi, X)\xi = X - \eta(X)\xi + \{1/(n - 2)\}\{-(n - 1)X$$

$$+ (n - 1)\eta(X)\xi + AX + B\,\eta(X)\xi$$

$$+ \eta(X)(n - 1)\xi\} - \{r/(n - 1)(n - 2)\}\{X - \eta(X)\xi\} \tag{2.14}$$

$$C(\xi, X)\xi = (X - \eta(X)\xi)\ [\{(n-1)(- n - B) - r/ (n - 1)(n - 2)\}] \tag{2.15}$$

$\therefore\ C(\xi, X)\xi = 0 \quad$ where $r = -(n-1)(n+B)$

We have the following result.

Theorem 2.2 : If in a Kenmotsu manifold M^n $(n > 2)$ the relation Div C = 0 holds, then $C(\xi, X)\xi = 0$ holds for every X, if the scalar curvature $r = -(n-1)(n-B)$.

3. Concircular curvature tensor C satisfying div C = 0

The concircular curvature tensor C is given by

$$C(X, Y)Z = R(X, Y)Z - \{r/n(n-1)\}\{g(Y, Z)X - g(X, Z)Y\} \qquad (3.1)$$

Differentiating (3.1) covariantly and contracting we get

$$(\text{div } C)(X,Y)Z = \{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\}$$

$$- \{1/n(n-1)\}\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\}$$

Let us assume that $\quad \text{div } C = 0 \qquad (3.2)$

Using (3.1) & (3.2) we have

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$$

$$= \{1/n(n-1)\}\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\} \qquad (3.3)$$

Take $X = \xi$ in (3.3) & by virtue of (2.5) & (2.6) we have

$$S(Y, Z) + (n - 1)g(Y, Z)$$

$$= \{1/n(n - 1)\}\{g(Y, Z)dr(\xi) - g(\xi, Z)dr(Y)\} \quad (3.4)$$

Replace Z by φZ in (3.4)

$$S(Y, \varphi Z) = [\{dr(\xi)/n(n - 1)\} - (n - 1)]g(Y, \varphi Z) \quad (3.5)$$

Again replace Z by φZ in (3.5) and simplify we have

$$S(Y, Z) = [\{dr(\xi)/n(n - 1)\} - (n - 1)]\, g(Y, Z) + B\, \eta(Y)\eta(Z) \quad (3.6)$$

$$\therefore\ S(Y, Z) = A\, g(Y, Z) + B\, \eta(Y)\eta(Z) \quad (3.7)$$

where $A = \{dr(\xi)/n(n-1)\} - (n-1)$ and $B = -\{dr(\xi)/n(n-1)\}$

Thus we have the following result.

Theorem 3.1 : In a Kenmotsu manifold M, if the con circular curvature tensor C of type (1, 3) is conservative then M is η - Einstein, that is M satisfies equation (3.7), where the constants A and B satisfying the condition $A + B = -(n - 1)$, provided $n \neq 0, 1$.

Again, taking $X = Z = \xi$, $Y = X$ in (3.1) we have by virtue of

$$C(\xi, X)\xi = R(\xi, X)\xi - \{r/n(n - 1)\}[g(X, \xi)\xi - g(\xi, \xi)X] \quad (3.8)$$

$$= X - \eta(X)\xi - \{r/n(n - 1)\}[\eta(X)\xi - X]$$

$$= (X - \eta(X)\xi)[1 - r/n(n - 1)]$$

$\therefore \quad C(\xi, X)\xi = 0$ where $r = n(n-1)$

Theorem 3.2 : If in a Kenmotsu manifold M^n $(n>2)$ the relation Div $C = 0$ holds. Then $C(\xi, X)\xi = 0$ holds for every X, if the scalar curvature $r = n(n-1)$.

4. Conharmonic curvature L satisfying div L = 0

The conharmonic curvature tensor (L) is given by

$$L(X, Y)Z = R(X, Y)Z - \{1/(n-2)\}\{S(Y, Z)X$$

$$- S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY\} \qquad (4.1)$$

Differentiating (4.1) covariantly and contracting we obtain

$$(\text{Div } L)(X, Y)Z = \{(n-3)/(n-2)\}\{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\}$$

$$-\{1/(n-2)\}\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\} \qquad (4.2)$$

Let us suppose that in a Kenmotsu Riemannian manifold

$$\text{Div } L = 0 \qquad (4.3)$$

Using (4.2) & (4.3) we have

$$(\nabla_X S)(Y, Z) - (\nabla_Y S(X, Z)$$

$$= \{1/(n-3)\}\{g(Y, Z)dr(X) - g(X, Z)dr(Y)\} \qquad (4.4)$$

Using (1.1), (1.3), (1.4), (1.5) & (1.6) we have

$$(\nabla x S)(Y, Z) = (L\xi S)(Y, Z) - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) = 0 \quad (4.5)$$

and

$$(\nabla_Y S)(\xi, Z) = YS(\xi, Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z)$$

$$= - S(Y, Z) - (n - 1)g(Y, Z) \quad (4.6)$$

Substitute (4. 5) & (4.6) in (4.4) we have

$$S(Y, Z) + (n - 1)g(Y, Z)$$

$$= \{1 / (n - 3)\}\{g(Y, Z)dr(\xi) - g(\xi, Z)dr(Y)\} \quad (4.7)$$

Replace Z by φZ in (4.7)

$$S(Y, \varphi Z) = [\{dr(\xi) / (n - 3)\} - (n - 1)] g(Y, \varphi Z) \quad (4.8)$$

Again replace Z by φZ in (4.8) and simplify we have

$$S(Y, Z) = [\{dr(\xi) / (n - 3)\} - (n - 1)]g(Y, Z)$$

$$+ [-\{dr(\xi) / (n - 3)\}]\eta(Y)\eta(Z) \quad (4.9)$$

$$S(Y, Z) = Ag(Y, Z) + B\,\eta(Y)\eta(Z) \quad (4.10)$$

where $A = [\{dr(\xi) / (n - 3)\} - (n - 1)]$ and $B = -\{dr(\xi) / (n - 3)\}$

Thus we have the following result.

Theorem 4.1 : On a Kenmotsu manifold M, if the conformal curvature tensor L of type (1, 3) is conservative then M is h - Einstien, that is M satisfies equation (4.10), where the constants A and B satisfies the condition $A + B = -(n - 1)$, Provided $n \neq 1, 3$.

Again, by taking $X = Z = \xi$ and $Y = X$ in (4.1) we have

$$L(\xi, X)\xi = R(\xi, X)\xi - 1/(n-2)\{S(X, \xi)\xi$$

$$- S(\xi, \xi)X + g(X, \xi)Q\xi - g(\xi, \xi)QX\} \tag{4.11}$$

$$L(\xi, X)\xi = X - \eta(X)\xi - \{1/(n-2)\}\{-(n-1)\eta(X)\xi + (n-1)X - (n-1)\eta(X)\xi$$

$$- AX - B\eta(X)\xi\} \tag{4.12}$$

Using $A + B = -(n - 1)$

$$L(\xi, X)\xi = \{X - \eta(X)\xi\}\{-(n+B)/(n-2)\}$$

$$L(\xi, X)\xi \neq 0$$

We have the following result.

Theorem 4.2 : If in a Kenmotsu manifold M^n $(n>2)$ the relation Div $L = 0$ holds. Then

$L(\xi, X)\xi \neq 0$ holds for every x.

References:

1. U.C.De and Absas Ali Shaik, K-contact and Sasakian manifolds with conservative quasi-conformal curvature tensor, Bull. of Cal. Math. Soc. 89, 349-354 (1997)

2. Bagewadi C.S and V.S.Prasad, Kenmotsu manifolds with vanishing of divergent of quasi-conformal curvature tensor, Proceedings of International conference on Analysis, Geometry and Fluid mechanics and their applications, Prasaranga Kuvempu University (2003)

3. Blair D.E, Contact manifolds in Riemannian geometry, Lecturer Notes in Mathematics 509, Springer-Verlag (1976)