MJS, Vol. 4, No. 2, July 2005 - Dec. 2005, pp. 22-31

ISSN 0975-3311
https://doi.org/10.12725/mjs.7.4

NEIGHBORHOODS AND PARTIAL SUMS OF MEROMORPHIC UNIVALENT FUNCTIONS

G. Murugusundaramoorthy and S.V.S. Velayudam***

ABSTRACT

In this paper we determine neighborhood results and partial sums for certain class of meromorphic univalent functions with positive coefficients defined by Ruscheweyh derivatives.

2000 Mathematics Subject Classification: 30C45.

Keywords and Phrases: *Meromorphic, neighborhood, convolution, partial sums.*

1. Introduction

Let M denotes the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

* Department of Mathematics, Vellore Institute of Technology, Deemed University, Vellore - 632014, India. email: gmsmoorthy@yahoo.com.

** Department of Mathematics, M.N.M. Jain Engineering College, Thorappakam, Chennai - 600 096, India.

which are regular and univalent in the punctured disc $E = \{z : 0 < |z| < 1\}$ A function $f(z)$ belonging to M is said to be meromorphically starlike of order α if it satisfies

$$\text{Re}\left\{\frac{-zf'(z)}{f(z)}\right\} > \alpha$$

for some α $(0 \leq \alpha < 1)$ and all $z \in U = \{z : |z| < 1\}$. We denote $M(\alpha)$ the class of all meromorphically starlike functions of order α. The class $M(\alpha)$ and related classes have been extensively studied in [2, 5, 6, 8].

Let $M(\alpha, \lambda, A, B)$ denote the class of functions f in M satisfying the condition

$$\left|\frac{z^2(D^\lambda f(z)'+1}{Bz^2(D^\lambda f(z))'+A}\right| < \alpha \tag{1.2}$$

for some $\alpha > 0$, $-1 \leq B < A \leq 1$ and $|B\alpha| \leq 1$ and for all $z \in U$, where $D^\lambda : M \to M$ is the operator defined by

$$D^\lambda f(z) = \frac{1}{z(1-z)^{\lambda+1}} * f(z), \ (\lambda > -1). \tag{1.3}$$

From the identity

$$\frac{1}{z(1-z)^{\lambda+2}} = \frac{1}{z(1-z)^{\lambda+1}} * \left(\frac{\lambda+2}{z(1-z)(\lambda+1)} + \frac{2z-1}{z(1-z)^2(\lambda+1)}\right).$$

We get

$$z(D^\lambda f(z))' = (\lambda+1)D^{\lambda+1}f(z) - (\lambda+2)D^\lambda f(z), \ \lambda > -1 \tag{1.4}$$

For $\lambda = n \in N_0 = \{0, 1, 2......\}$ we note that the relation (1.3) may be expressed as

$$D^n f(z) = \frac{1}{z}\left(\frac{z^{n+1}f(z)}{n!}\right)^{(n)}$$

$$D^{\lambda}f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}B_k(\lambda)a_k z^k,$$

where

$$B_k(\lambda)=\frac{(\lambda+1)(\lambda+2)(\lambda+3)\dots(\lambda+k-1)}{(k+1)!}$$

A function

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}|a_k|z^k\in M \tag{1.5}$$

is said to be in the class $M^*(\alpha,\lambda,A,B)$ if it satisfies the condition (1.3) with $-1\le B\le 0$.

The main object of this paper is to present a systematic investigation of various properties of the general class $M^*(\alpha,\lambda,A,B)$ and to determine neighborhood results and partial sums for the general class of meromorphic univalent functions with positive coefficients.

2. Properties of the class $M^*(\alpha,\lambda,A,B)$

Theorem 1. Let $f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}|a_k|z^k$ *be analytic and univalent in* U. *Then* $f(z)\in$ $M^*(\alpha,\lambda,A,B)$ *if and only if*

$$\sum_{k=1}^{\infty}\frac{k(1-B\alpha)}{\alpha(A-B)}B_k(\lambda)|a_k|\le 1. \tag{2.1}$$

The result is sharp for the function f(z) given by

$$f(z)=\frac{1}{z}+\frac{\alpha(A-B)}{kB_k(\lambda)(1-B\alpha)}z^k. \tag{2.2}$$

Proof. Let $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} | a_k | z^k \in M^*(\alpha, \lambda, A, B)$. Then

$$\left| \frac{z^2 (D^\lambda f(z))' + 1}{Bz^2 (D^{\lambda f(z)})' + A} \right| < \alpha$$

Substituting for $(D^\lambda f(z))'$, we get

$$\left| \frac{z^2 \left(\frac{-1}{z^2} + \sum_{k=1}^{\infty} kB_k(\lambda) a_k z^{k-1} \right) + 1}{Bz^2 \left(\frac{-1}{z^2} + \sum_{k=1}^{\infty} kB_k(\lambda) a_k z^{k-1} \right) + A} \right| < \alpha$$

That is,

$$\left| \frac{\sum_{k=1}^{\infty} kB_k(\lambda) a_k z^{k+1}}{(A-B) + \sum_{k=1}^{\infty} BkB_k(\lambda) z^{k+1}} \right| < \alpha \tag{2.3}$$

Since $|\mathrm{Re}(z)| \le |z|$ for any z, choosing values of z to be real, (2.3) yields

$$\sum_{k=1}^{\infty} k(1 - B\alpha) B_k(\lambda) | a_k | \le \alpha (A - B)$$

On other hand if we let $z \in \partial E$, then we find that

$$\left| \frac{z^2 (D^\lambda f(z))' + 1}{Bz^2 (D^\lambda f(z))' + A} \right| \le \frac{\sum_{k=1}^{\infty} kB_k(\lambda) a_k}{(A-B) + \sum_{k=1}^{\infty} BkB_k(\lambda)} < \alpha$$

Finally, by observing that the function f(z) given by (2.2) is indeed an extremal function for the assertion (2.1), which completes the proof of Theorem 1.

The following theorem is an easy consequence of Theorem 1.

Theorem 2. *Let each of function $f_i(z)$ defined by*

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} |a_{k,i}| z^k \quad (i = 1, 2, 3...) \tag{2.4}$$

be in the class $M^*(\alpha, \lambda, A, B)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{\infty} \mu_i f_i(z) \quad (\mu_i \geq 0 \text{ and } \sum_{i=1}^{\infty} \mu_i = 1) \tag{2.5}$$

is also in the class $M^*(\alpha, \lambda, A, B)$.

Theorem 3. *Let $f(z) \in M^*(\alpha, \lambda, A, B)$ then we have $f(z)$ is meromorphically starlike of order δ in $|z| < r$, that is*

$$\operatorname{Re}\left\{\frac{-zf'(z)}{f(z)}\right\} > \delta, \ (|z| < r_1, \text{when } 0 \leq \delta < 1)$$

$$r_1 = \inf_{k \geq 1} \left\{B_k(\lambda) \frac{k(1-\delta)(1-\alpha B)}{\alpha(A-B)(k+\delta)}\right\}^{\frac{1}{k+1}}$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. From Theorem 1, we have

$$\sum_{k=1}^{\infty} \frac{k+\delta}{(1-\delta)} |a_k| z^{k+1} < \sum_{k=1}^{\infty} B_k(\lambda) \frac{k(1-\alpha B)}{(A-B)\alpha} |a_k| \leq 1 \ (|z| < r_1)$$

Therefore for $|z| < r_1$,

$$\left|\frac{\frac{zf'(z)}{f(z)} + 1}{\frac{zf'(z)}{f(z)} - (1-2\delta)}\right| = \left|\frac{\sum_{k=1}^{\infty} (k+1) a_k z^k}{2(1-\delta) - \sum_{k=1}^{\infty} (k-(1-2\delta)) a_k z^k}\right|$$

$$= \frac{\sum_{k=1}^{\infty} (k+1) B_k(\lambda) |a_k| |z|^{k+1}}{2(1-\delta) - \sum_{k=1}^{\infty} (k-1+2\delta) |a_k| |z|^{k+1}} < 1$$

which implies that (2.6) is true.

3. Neighborhoods

The concept of neighborhood of analytic functions was first introduced by Goodman [4] and then generalized by Ruscheweyh [9]. In this section we shall extend the concept of neighborhoods to meromorphic univalent functions.

For $\delta \geq 0$, $-1 \leq B < A \leq 1$ and $f(z)=\frac{1}{z}+\sum_{k=1}^{\infty} a_k z^k \in M$. We define neighborhood of $f(z)$ by

$$N_\delta(f) = N_\delta(f,\alpha,\lambda,A,B)$$

$$= \left\{ g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \in M : \sum_{k=1}^{\infty} \frac{(k(1-|B|\alpha))}{\alpha(A-B)} B_k(\lambda)\,|b_k - a_k| \leq \delta \right\}$$

Theorem 4. *Let $\delta > 0$ and $\alpha > 0$. If $f(z)=\frac{1}{z}+\sum_{k=1}^{\infty} a_k z^k \in M$ satisfies*

$$\frac{f(z)+\varepsilon z^{-1}}{1+\varepsilon} \in M^*(\alpha,\lambda,A,B) \tag{3.1}$$

for any complex number ε such that $|\varepsilon| < \delta$, then $N_\delta(f) \subset M^*(\alpha,\lambda,A,B)$.

Proof. It is easily seen from (1.2) that $g(z) \in M^*(\alpha,\lambda,A,B)$ if and only if for any complex number σ with $|\sigma| = 1$

$$\left|\frac{z^2(D^\lambda g(z))' + 1}{Bz^2(D^\lambda g(z))' + A}\right| \neq \sigma\alpha \quad (z \in U)$$

which is equivalent to

$$\frac{g(z) * h(z)}{z^{-1}} \neq 0 \quad (z \in E) \tag{3.2}$$

where

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} c_k z^k = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(k-1)(1-B\sigma\alpha)}{\alpha\sigma(B-A)} B(\lambda) z^{k-1} \tag{3.3}$$

From (3.3) we have

$$|c_k| = \left|\frac{(k-1)(1-B\sigma\alpha)}{\sigma\alpha(B-A)}B_k(\lambda)\right| \leq \frac{(k-1)(1+|B|\alpha)}{\sigma\alpha(A-B)}B_k(\lambda) \; .$$

If $f(\bar{z})$ of the form (1.1) and belonging to the class M satisfies the condition (3.1) and (3.2) yields

$$\left|\frac{f(z) * h(z)}{z^{-1}}\right| \geq \delta \quad (z \in E)$$

Now we let $p(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \in N_\delta(f)$, then so that

$$\left|\frac{(p(z) - f(z)) * h(z)}{z^{-1}}\right| = \left|\sum_{k=1}^{\infty}(b_k - a_k)c_k z^k\right|$$

$$\leq |z| \sum_{k=1}^{\infty} \frac{k(1-|B|\alpha)}{\alpha(A-B)} B_k(\lambda) \; |(b_k - a_k)| < \delta$$

Thus for any complex number σ such that $|\sigma| = 1$, we have

$$\frac{p(z) * h(z)}{z^{-1}} \neq 0 \; (z \in E)$$

which implies that $p(z) \in M^* (\alpha, \lambda, A, B)$. The proof is complete.

In the following theorem we obtain partial sums for the general class of meromorphic univalent functions with positive coefficients.

Theorem 5. *Let $f \in M$ be given by (1.1) and define the partial sums $s_1(z)$ and $s_n(z)$ by*

$$s_1(z) = \frac{1}{z} \quad \text{and} \quad s_n(z) = \frac{1}{z} + \sum_{k=1}^{n-1} a_k z^k \; (n \in N \setminus \{1\}). \tag{3.4}$$

Suppose also that

$$\sum_{k=1}^{\infty} d_k | a_k | \leq 1, \tag{3.5}$$

where

$$d_k := \frac{k(1-|B|\alpha)}{(B-A)} B_k(\lambda)$$

Furthermore,

$$\operatorname{Re}\left\{\frac{f(z)}{s_n(z)}\right\} > 1 - \frac{1}{d_n} \quad (z \in U, n \in N) \tag{3.6}$$

and

$$\operatorname{Re}\left\{\frac{s_n(z)}{f(z)}\right\} > \frac{d_n}{1+d_n} \quad (z \in U, n \in N) \tag{3.7}$$

Each of the bounds in (3.6) and (3.7) is the best possible for $n \in N$.

Proof. It is easily seen that $\frac{1}{z} \in M(\alpha, \lambda, A, B)$. Thus from Theorem 4 of the hypothesis (3.5) we have

$$N_1\left(\frac{1}{z}\right) \subset M(\alpha; \lambda, A, B) \tag{3.8}$$

which shows that $f \in M(\alpha, \lambda, A, B)$ as asserted by Theorem 5.

Next for the coefficient d_k given by (3.5), it is not difficult to verify that

$$d_{k+1} > d_k > 1 \quad (k = 1, 2, 3, \ldots) \tag{3.9}$$

Therefore, we have

$$\sum_{k=1}^{n-1} | a_k | + d_n \sum_{k=n}^{\infty} | a_k | \leq \sum_{k=1}^{\infty} d_k | a_k | \leq 1 \tag{3.10}$$

by using the hypothesis (3.5) again by setting,

$$g_1(z) = d_n \left\{ \frac{f(z)}{s_n(z)} - \left(1 - \frac{1}{d_n}\right) \right\}$$

$$= 1 + \frac{d_n \sum_{k=n}^{\infty} a_k z^k}{1 + \sum_{k=1}^{n-1} a_k z^k} \tag{3.11}$$

and applying (3.10), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{d_n \sum_{k=n}^{\infty} |a_k|}{2 - 2\sum_{k=1}^{n-1} |a_k| - d_n \sum_{k=1}^{\infty} |a_k|} \leq 1, \quad z \in U, \tag{3.12}$$

which readily yields the assertion (3.6) of Theorem 5.

If we take

$$f(z) = \frac{1}{z} - \frac{z^n}{d_n} \tag{3.13}$$

then

$$\frac{f(z)}{s_n(z)} = \frac{\frac{1}{z} - \frac{z^n}{d_n}}{\frac{1}{z} + \sum_{k=1}^{n-1} a_k z^{k-1}} = 1 - \frac{z^n}{d_n} \to 1 - \frac{1}{d_n} \text{ as } z \to 1,$$

which shows that the bound in (3.7) is the best possible for each $n \in N$. Similarly, we put

$$g_2(z) = (1 + d_n) \left\{ \frac{s_n(z)}{f(z)} - \frac{d_n}{1 + d_n} \right\}$$

$$= 1 - \frac{(1 + d_n) \sum_{k=n}^{\infty} a_k z^k}{1 + \sum_{k=1}^{\infty} a_k z^k} \tag{3.14}$$

and making use of (3.10), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_n)\sum_{k=n}^{\infty}|a_k|}{2-2\sum_{k=1}^{n-1}|a_k| + (1-d_n)\sum_{k=n}^{\infty}|a_k|} \leq 1. \tag{3.15}$$

which leads us immediately to the assertion (3.7) of Theorem 5.

The bound in (3.7) is sharp for each $n \in N$, with the extremal function $f(z)$ given by (3.13). The proof of the Theorem is thus completed.

References

1. M.K. Aouf, On a certain class of meromorphic univalent functions with positive coefficients, *Rent. Math.*, (7) 11 (1991), 209-219.

2. S.K. Bajpai, A note on a class of meromorphic univalent functions, *Rev. Roumaine Math. Pure & Appl.*, 22 (1977), 295 - 297.

3. J. Clunie, On meromorphic Schlicht function, *J. London Math. Soc.*, 34 (1959), 215 - 216.

4. A.W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, 8 (1957), 598 - 601.

5. J.L. Liu and H.M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal & Appl.*, 259 (2001), 566 - 581.

6. J.L. Liu, Properties of some families of meromorphic p-valent functions, *Math Japonica.*, 52 No. 3 (2000), 425 - 434.

7. Ch. Pommerenke, On meromorphic starlike functions, *Pacific J. Math.*, 13 (1963), 212 - 235.

8. O. Sangkwon and N.E. Cho, Hadamard product of certain meromorphic univalent functions, *Tamkang J. Math.*, Vol. 28, No. 4 Winter 1997.

9. S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, 81 (1981), 521 - 527.