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A STUDY OF STREAMLINES IN SECOND GRADE FLUID FLOWS

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ABSTRACT

We obtain solutions for second grade fluid in (ϕ,ψ) net where $\phi(x, y)$= constant, an arbitrary family of curves and $\psi(x, y)$=constant, stream lines. Further exact solutions are determined when the stream line patterns are of the form

$$\frac{y-g(x)}{f(x)} = \text{constant} \quad \text{or} \quad \frac{x-k(y)}{m(y)} = \text{constant}.$$

Keywords: *Streamlines, Second grade fluid, Incompressible.*

1. Introduction

Non-Newtonian fluids have gained more and more important industrially, over the past decades. Polymer solutions and polymer melts are the most common examples of non-Newtonian fluids. The equations of motion of such fluids are highly non-linear and one order higher than the Navier-Stokes equations. Martin [1] has used a natural curvilinear co-ordinate system (ϕ,ψ) in the physical plane (x, y) where ψ = constant are the streamlines and ϕ=constant is an arbitrary family of curves

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to study plane viscous flows. C.S. Bagewadi and Siddabasappa [6] studied the plane rotating viscous MHD flows by using differential geometry techniques. O.P. Chandna and Labropulu [2] obtained the exact solutions for steady plane viscous flows by taking the arbitrary family of curves ϕ=constant to be x=constant. Rajagopal [3] found some interesting exact solutions of unsteady unidirectional second grade fluid flows. More recently, Labropulu [4] studied generalized Beltrami flows and other closed form solutions of an unsteady viscoelastic fluid.

In the present work we first use Martin's [1] method to decompose the basic equations of non-Newtonian fluids and next following the work of Labropulu and O.P. Chandna [5], A.M. Siddiqui, P.N. Kaloni and O.P. Chandna [7], C.S. Bagewadi and S. Bhagya [9] and Erikson J.L [11], we study whether the second grade fluid flow along a given family of curves $\frac{y-g(x)}{f(x)}$ = constant, where $f(x) \neq 0$ can exist?

The plan of this paper is as follows; in section 2, we employ Martin's [1] approach and recast the flow equations. This section also contains the recasting of the equations in a new form by employing some results from differential geometry. In section 3, we outline the method of determining whether a given family of curves can be the streamlines. In section 4, we deal with the particular examples.

2. Equation of Motions

The flow of a homogeneous incompressible second grade fluid flow, neglecting thermal effects and body forces, is governed by [10]

$$\text{div}\, V = 0 \quad (1)$$

$$\text{div}\, T = \rho V \quad (2)$$

where T is Cauchy stress which describes second grade fluids given by B.D. Coleman and W. Noll [8],

$$T = -p\,I + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \quad (3)$$

The following nomenclatures are used; V the velocity vector field, p the fluid pressure function, ρ the constant fluid density, μ the constant coefficient of viscosity and α_1 and α_2 are the normal stress moduli.

The Rivlin-Ericksen tensors A_1 and A_2 are defined as

$$A_1 = (\text{grad}\, V) + (\text{grad}\, V)^t$$

$$A_2 = A_1 + (\text{grad}\, V)^t A_1 + A_1(\text{grad}\, V) \quad (4)$$

Here (grad V)' denotes the transpose of grad V. If we substitute (3) in (2) and make use of (4) we get

$$-\text{grad}\, p + \mu\nabla^2 V + \alpha_1 [\nabla^2 V_t + \nabla^2 (\nabla \times V) \times V + \text{grad}\, (V.\nabla^2 V + \frac{1}{4} | A_1 |^2)$$

$$+ (\alpha_1 + \alpha_2)\ \text{div}\ A_1^2 = \rho V \tag{5}$$

where ∇^2 denotes the Laplacian, V_t denotes the partial derivative of V with respect to time t and

$$| A_1 |^2 = \text{tr} A_1 A_1^T.$$

The decomposition of equations (1) and (2) gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\mu}{\rho u_0}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\alpha_1}{\rho}\left\{\frac{\partial}{\partial x}\left(2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 u}{\partial x \partial y} + 4\left(\frac{\partial u}{\partial x}\right)^2\right.\right.$$

$$\left.\left. + 2\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\ \right) + \frac{\partial}{\partial y}\left[\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]\right\}$$

$$+ \frac{\alpha_2}{\rho}\left\{\frac{\partial}{\partial x}\left(4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right)\right\} \tag{7}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\mu}{\rho u_0}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 xz}{\partial y^2}\right) + \frac{\alpha_1}{\rho}\left\{\frac{\partial}{\partial x}\left[\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right.\right.$$

$$\left.\left. + 2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\ \right] + \frac{\partial}{\partial y}\left(2u\frac{\partial^2}{\partial x \partial y} + 2v\frac{\partial^2 v}{\partial y^2} + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right)\right\}$$

$$+ \frac{\alpha_2}{\rho}\left\{\frac{\partial}{\partial y}\left[4\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\ \right]\right\} \tag{8}$$

We introduce the Vorticity function ω and energy function h as

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \tag{9}$$

and $$h=p+\frac{1}{2}\rho\left(u^2+v^2\right)-\alpha_1\left(u\nabla^2 u+v\nabla^2 v\right)-\left(\frac{3\alpha_1+2\alpha_2}{4}\right)|A_1|^2 \tag{10}$$

where $$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

and $$|A_1|^2=4\left(\frac{\partial u}{\partial x}\right)^2+4\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2$$

We write the equations (7) to (10) as system of second order equations in non dimensional variables by using (9) and (10) in (7) and (8) and taking non dimensional quantities for velocity and pressure

$\frac{V}{u_0}=(u(x,y),\ v(x,y),\ 0),\quad p(x,y)=\frac{p}{\rho u_0^2}$ as follows:

$$\frac{\partial h}{\partial x}=v\omega-\frac{1}{R}\frac{\partial \omega}{\partial y}-\frac{We}{R}v\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right) \tag{11}$$

$$\frac{\partial h}{\partial y}=-u\omega+\frac{1}{R}\frac{\partial \omega}{\partial x}+\frac{We}{R}u\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right) \tag{12}$$

$$h=p+\frac{1}{2}\left(u^2+v^2\right)-\frac{We}{R}\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]$$

$$-\left(\frac{3+2\upsilon}{2}\right)\frac{We}{R}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right] \tag{13}$$

Here u, v are the velocity components and p the pressure function of x, y and $R=\frac{\rho u_0}{\mu}$ the Reynolds number, $We=\frac{\alpha_1 u_0}{\mu}$ is the Weissenberg number, $\upsilon=\frac{\alpha_2}{\alpha_1}$ is the ratio of normal stress moduli and u_0 the characterstic velocity. Thus (6), (9), (11), (12) and (13) are five equations in five unknown functions u, v, ω, h and p of x, y.

The continuity equation (6) implies the existence of a stream function $\psi(x, y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{14}$$

We introduce a curvilinear coordinate system in the physical plane in which the curves $\psi(x, y)$ = constant are the streamlines and the curves $\phi(x, y)$ = constant are arbitrary.

Let

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi) \tag{15}$$

define a curvilinear net in the (x, y) plane with the squared element of arc length along any curves given by

$$ds^2 = E(\phi,\psi)\,d\phi^2 + 2F(\phi,\psi)\,d\phi\,d\psi + G(\phi,\psi)\,d\psi^2 \tag{16}$$

where

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2, \quad F = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi}, \quad G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 \tag{17}$$

Equation (15) can be solved to obtain $\phi = \phi(x, y)$, $\psi = \psi(x, y)$ such that

$$\frac{\partial x}{\partial \phi} = J\frac{\partial \psi}{\partial y}, \qquad \frac{\partial x}{\partial \psi} = -J\frac{\partial \phi}{\partial y}$$

$$\frac{\partial y}{\partial \phi} = -J\frac{\partial \psi}{\partial x}, \qquad \frac{\partial y}{\partial \psi} = J\frac{\partial \phi}{\partial x} \tag{18}$$

provided $0 < |J| < \infty$, and J is Jacobian given by

$$J = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi} = \pm\sqrt{EG - F^2} = \pm W \tag{19}$$

Denoting by α the local angle of inclination of the tangent to the coordinate line ψ= constant, directed in the sense of increasing ϕ, we have from differential geometry the following [1]:

$$\frac{\partial x}{\partial \phi}=\sqrt{E}\cos\alpha, \qquad \frac{\partial y}{\partial \phi}=\sqrt{E}\sin\alpha \tag{20}$$

$$\frac{\partial x}{\partial \psi}=\frac{F}{\sqrt{E}}\cos\alpha-\frac{J}{\sqrt{E}}\sin\alpha$$

$$\frac{\partial y}{\partial \psi}=\frac{F}{\sqrt{E}}\sin\alpha+\frac{J}{\sqrt{E}}\cos\alpha \tag{21}$$

$$\frac{\partial \alpha}{\partial \phi}=\frac{J}{E}\Gamma^2_{11}, \qquad \frac{\partial \alpha}{\partial \psi}=\frac{J}{E}\Gamma^2_{12} \tag{22}$$

$$K=\frac{1}{W}\left[\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma^2_{11}\right)-\frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma^2_{12}\right)\right]=0 \tag{23}$$

where

$$\Gamma^2_{11}=\frac{1}{2W^2}\left[-F\frac{\partial E}{\partial \phi}+2E\frac{\partial F}{\partial \phi}-E\frac{\partial E}{\partial \psi}\right]$$

$$\Gamma^2_{12}=\frac{1}{2W^2}\left[E\frac{\partial G}{\partial \phi}-F\frac{\partial E}{\partial \psi}\right]$$

$$\Gamma^2_{22}=\frac{1}{2W^2}\left[E\frac{\partial G}{\partial \psi}-2F\frac{\partial F}{\partial \psi}+F\frac{\partial G}{\partial \phi}\right] \tag{24}$$

and K is the Gaussian curvature.

We transform equations (9) to (13) governing our flow into new forms in the new variables ϕ, ψ.

Equation of continuity and Vorticity

Martin [1] has obtained the necessary and sufficient condition for the flow of a fluid, along the co-ordinate lines ψ = constant of a curvilinear co-ordinate system (15) with ds^2 given by (16) to satisfy the principle of conservation of mass to be

$$Wq=\sqrt{E}, \qquad u+iv=\frac{\sqrt{E}}{W}e^{i\alpha} \tag{25}$$

where

$$\alpha=\int\frac{\partial\alpha}{\partial\phi}d\phi+\frac{\partial\alpha}{\partial\psi}d\psi$$

$$=\int\frac{J}{E}\left(\Gamma_{11}^{2}d\phi+\Gamma_{12}^{2}d\psi\right)$$

Then equation (12) becomes

$$\omega=\frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right)-\frac{\partial}{\partial\psi}\left(\frac{E}{W}\right)\right] \quad (26)$$

In this work, we consider that the fluid towards higher or lower parameter values ϕ accordingly as J is positive or negative and speed q of the fluid flow is given in (25).

Linear Momentum Equation

On employing (14) in the linear momentum equations (11), (12) and making use of (18), we have

$$RJ\frac{\partial h}{\partial\phi}=F\frac{\partial\omega}{\partial\phi}-E\frac{\partial\omega}{\partial\psi} \quad (27)$$

$$RJ\left(\frac{\partial h}{\partial\psi}+\omega\right)=G\frac{\partial\omega}{\partial\phi}-F\frac{\partial\omega}{\partial\psi}+WeJ\nabla^{2}\omega \quad (28)$$

Energy Equation

Employing (14) in the energy equation (13), transforming to (ϕ, ψ) net and using (18), we have

$$h=p+\frac{E}{2W^{2}}-\frac{We}{RW^{2}}\left(F\frac{\partial\omega}{\partial\phi}-E\frac{\partial\omega}{\partial\psi}\right)-\left(\frac{3+2\upsilon}{2}\right)\frac{We}{R}\left\{\omega^{2}+\frac{4}{W\sqrt{E}}\left[\frac{\partial}{\partial\psi}\left(\frac{\sqrt{E}}{W}\right)\Gamma_{11}^{2}\right.\right.$$

$$\left.\left.-\frac{\partial}{\partial\phi}\left(\frac{\sqrt{E}}{W}\right)\Gamma_{12}^{2}\right]\right\} \quad (29)$$

Summing up, we have

Theorem 1: If the streamlines $\psi(x, y)$=constant of a steady plane motion of an incompressible second grade fluid are taken as one set of co-ordinate lines in a curvilinear co-ordinate system ϕ, ψ in the physical plane, then the flow is governed by the system of equations (25) to (29).

These are five equations in six unknown functions E, F, G, ω, h and p of ϕ, ψ. Here

$$\nabla^2\omega=\frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{G\frac{\partial\omega}{\partial\phi}-F\frac{\partial\omega}{\partial\psi}}{W}\right)+\frac{\partial}{\partial\psi}\left(\frac{-F\frac{\partial\omega}{\partial\phi}+E\frac{\partial\omega}{\partial\psi}}{W}\right)\right] \tag{30}$$

Eliminating h from the linear momentum equations by using the integrability conditions $\frac{\partial^2 h}{\partial\phi\partial\psi}=\frac{\partial^2 h}{\partial\psi\partial\phi}$, we have

$$W\nabla^2\omega\mp R\frac{\partial\omega}{\partial\phi}\pm We\frac{\partial}{\partial\phi}\left(\nabla^2\omega\right)=0 \qquad \text{(Integrability)}$$

$$\omega=\frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right)-\frac{\partial}{\partial\psi}\left(\frac{E}{W}\right)\right] \qquad \text{(Vorticity)}$$

$$\frac{\partial}{\partial\psi}\left(\frac{W}{E}\Gamma_{11}^{2}\right)-\frac{\partial}{\partial\phi}\left(\frac{W}{E}\Gamma_{12}^{2}\right)=0 \qquad \text{(Gauss)} \tag{31}$$

of three equations for E, F, G and ω as function of ϕ, ψ. The continuity equation (25) determines the speed $q = q(\phi, \psi)$, the linear momentum equations determines

$$h=h(\phi,\psi)=\int\frac{\partial h}{\partial\phi}d\phi+\frac{\partial h}{\partial\psi}d\psi$$

and the energy equation provides the pressure function $p=p(\phi, \psi)$.

3. A Study of a Family of Curves in Van Mises Co-ordinate System

In this section we study the properties of streamlines in Von Mises co-ordinate system that is in (x, ψ) and (y, ψ) coordinate net.

3.1. For the (x, ψ) Co-ordinate Net

To analyze whether a given family of curves $\frac{y-g(x)}{f(x)}$ = constant can or cannot be the streamlines, we assume the affirmative so that there exists some function $M(\psi)$ such that

$$\frac{y-g(x)}{f(x)} = M(\psi), \qquad M^1(\psi)\neq 0 \tag{32}$$

where $M^1(\psi)$ is the derivative of unknown function $M(\psi)$. The system of equations (31) can be now determined for these flows by setting $\phi(x, y) = x =$ constant, so that the curvilinear co-ordinate net is the Von Mises net x, ψ. Employing equation (32) in (16) and simplifying the resulting equation, we obtain

$$ds^2 = \{1+[g(x)+f(x)\ M(\psi)]^2 d\phi^2 + 2[g(x) + f(x)M\ (\psi)]\ f(x)\ M\ (\psi) d\phi d\psi$$
$$+ f^2(x)\ M^2(\psi) d\psi^2$$

Equations (17) and (19) become

$$E = 1 + [g^1(x) + f^1(x)\ M(\psi)]^2$$
$$F = |g^1(x) + f^1(x)\ M(\psi)\ |\ f(x)\ M^1(\psi),$$
$$G = f^2(x)\ M^{12}(\psi) \quad and \quad J = W = f(x)\ M^1(\psi) \tag{33}$$

so that

$$E = 1 + \frac{F^2}{G}$$

when the fluid is assumed to be flowing in the direction of increasing x along the streamlines. Employing (33) in (31), Gauss equation is identically satisfied. If the curves (32) constitute a streamline pattern for a steady plane flow of an incompressible second grade fluid, then the flow must satisfy the Vorticity function

$$\omega=\frac{1}{f^2(x)M^1(\psi)}\Big\{ f(x)g^{11}(x)-2g^1(x)f^1(x)+\big[f(x)f^{11}(x)-2f'^2(x)\big]M(\psi)$$

$$+\big[1+g'^2(x)\big]\frac{M''(\psi)}{M'^2(\psi)}+2g'(x)f'(x)\frac{M(\psi)M''(\psi)}{M'^2(\psi)}+f'^2(x)\frac{M^2(\psi)M''(\psi)}{M'^2(\psi)}\Big\} \qquad (34)$$

and

$$f(x)M'(\psi)\frac{\partial^2\omega}{\partial x^2}-2\big[g'(x)+f'(x)M(\psi)\big]\frac{\partial^2\omega}{\partial x\partial\psi}+\frac{1}{f(x)M'(\psi)}\big[1+f'^2(x)$$

$$+2f'(x)g'(x)M(\psi)+f'^2(x)M^2(\psi)\big]\frac{\partial^2\omega}{\partial\psi^2}+\frac{1}{f(x)}\{2g'(x)f'(x)-f(x)g''(x)+$$

$$+\big[2f'^2(x)-f(x)f''(x)\big]M(\psi)-\big[1+g'^2(x)\big]\frac{M''(\psi)}{M'^2(\psi)}-2g'(x)f'(x)\frac{M(\psi)M''(\psi)}{M'^2(\psi)}$$

$$-f'^2(x)\frac{M^2(\psi)M''(\psi)}{M'^2(\psi)}\}\frac{\partial\omega}{\partial\psi}-R\frac{\partial\omega}{\partial x}+\frac{We}{f(x)M'(\psi)}\Big(f(x)M'(\psi)\frac{\partial^3\omega}{\partial x^3}-2[g'(x)\Big)$$

$$+f^l(x)M(\psi)\big]\frac{\partial^2\omega}{\partial x^2}\frac{\partial\omega}{\partial\psi}+\frac{1}{f(x)M'(\psi)}\big[1+g'^2(x)+2g'(x)f'(x)M(\psi)$$

$$+f'^2(x)M^2(\psi)\big]\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial\psi^2}+\frac{1}{g(x)}\{4g'(x)f'(x)-3f(x)g''(x)+\big[4f'^2(x)$$

$$-3f(x)f''(x)\big]M(\psi)-\big[1+g'^2(x)\big]\frac{M''(\psi)}{M'^2(\psi)}-2f'(x)g'(x)\frac{M(\psi)M''(\psi)}{M'^2(\psi)}$$

$$-f'^2(x)\frac{M^2(\psi)M''(\psi)}{M'^2(\psi)}\Big\}\frac{\partial^2\omega}{\partial x\partial\psi}+\frac{2}{f^2(x)M'(\psi)}\{f(x)g'(x)g''(x)-f'(x)\big[1+g'^2(x)\big]$$

$$+\big[f(x)f'(x)g''(x)+f(x)g'(x)f''(x)-2g'(x)f'^2(x)\big]M(\psi)+f'(x)[f(x)f''(x)$$

$$-f'^2(x)]M^2(\psi)\}\frac{\partial^2\omega}{\partial\psi^2}+\frac{1}{f^2(x)}\{f(x)[3f'(x)g''(x)+2g'(x)f''(x)]-4g'(x)f'^2(x)$$

$$-f^2(x)g'''(x)+\big[5f(x)f'(x)f''(x)-4f'^3(x)-f^2(x)f'''(x)\big]M(\psi)+2\,|\,f'(x)g'^2(x)$$

$$+f'(x)-f(x)g'(x)g''(x)]\frac{M''(\psi)}{M'^2(\psi)}+2[2g'(x)f'^2(x)-f(x)g'(x)f''(x)$$

$$-f(x)f'(x)g''(x)]\frac{M(\psi)M''(\psi)}{M'^2(\psi)}+2f'(x)\left[f'^2(x)-f(x)f''(x)\right]\frac{M^2(\psi)M''(\psi)}{M'^2(\psi)}\Bigg\}\frac{\partial\omega}{\partial\psi}\Bigg)=0 \quad (35)$$

and $M(\psi)$ is some function of such that $M^1(\psi)\neq 0$.

Conculsion: A given family of curves $\frac{y-g(x)}{f(x)}$ = constant is a permissible family of streamlines in second grade fluids if and only if the solution obtained for $M(\psi)$ is such that $M^1(\psi)\neq 0$.

3.2. For the (y, ψ) co-ordinate net

To analyze whether a given family of curves $\frac{x-k(y)}{m(y)}$ = constant can or cannot be streamlines, we assume the affirmative so that there exists some function $N(\psi)$ such that

$$\frac{x-k(y)}{m(y)}=N(\psi)\,, \qquad N^1(\psi)\neq 0. \qquad (36)$$

where $N^1(\psi)$ is the derivative of the unknown function $N(\psi)$ and we take the co-ordinate lines ϕ = constant to be y = constant. Employing equation (36) in (16) and simplifying the resulting equation, we obtain

$$ds^2=\left[1+(k'(y)+m'(y)N(\psi))^2\right]d\phi^2+2\left[k'(y)+m'(y)N(\psi)\right]m(y)N'(\psi)d\phi d\psi$$
$$+m^2(y)N'^2(\psi)d^2\psi \qquad (37)$$

Equations (17) and (19) become

$$E=\{1+(k'(y)+m'(y)N(\psi))^2\}, \qquad F=\left[k'(y)+m'(y)N(\psi)\right]m(y)N'(\psi)$$

$$G=m^2(y)N'^2(\psi) \qquad W=\sqrt{EG-F^2}=m(y)N'(\psi) \qquad (38)$$

when the fluid is assumed to be flowing in the direction of increasing y - along the streamlines. Employing (38) in (31), Gauss equation is identically satisfied.

If the curves (36) constitute a streamline pattern for a steady plane flow of an incompressible second grade fluid, then the flow must satisfy

$$\omega=\frac{1}{m^2(y)N'(\psi)}\Big[\ m(y)k''(y)-2k'(y)m'(y)+(m(y)m''(y)2\,2m'^2(y))N(\psi)$$

$$+\left[1+k'^2(y)\right]\frac{N''(\psi)}{N'^2(\psi)}+2k'(y)m'(y)\frac{N(\psi)N''(\psi)}{N'^2(\psi)}+m'^2(y)\frac{N^2(\psi)N''(\psi)}{N'^2(\psi)}\Big] \qquad (39)$$

and

$$m(y)N'(\psi)\frac{\partial^2\omega}{\partial y^2}-2[k'(y)+m'(y)N(\psi)]\frac{\partial^2\omega}{\partial y\partial\psi}+\frac{1}{m(y)N'(\psi)}\Big[1+k'^2(y)+2k'(y)m'(y)N(\psi)$$

$$+m'^2(y)N^2(\psi)\Big]\left\{\frac{\partial^2\omega}{\partial\psi^2}+\frac{1}{m(y)}2k'(y)m'(y)-m(y)k''(y)+\left[2m'^2(y)-m(y)\ m(y)m''(y)\right]N(\psi)\right.$$

$$\left.-\left[1+k'^2(y)\right]\frac{N''(\psi)}{N'^2(\psi)}-2k'(y)m'(y)\frac{N(\psi)N''(\psi)}{N'^2(\psi)}-m'^2(y)\frac{N^2(\psi)N''(\psi)}{N'^2(\psi)}\right\}\frac{\partial\omega}{\partial\psi}$$

$$-R\frac{\partial\omega}{\partial y}+\frac{We}{m(y)N'(\psi)}\left(m(y)N'(\psi)\frac{\partial^2\omega}{\partial y^2}-2[k'(y)+m'(y)N(\psi)]\frac{\partial^2\omega}{\partial y^2}\frac{\partial\omega}{\partial\psi}\right.$$

$$+\frac{1}{m(y)N'(\psi)}\left[1+k'^2(y)+2k'(y)m'(y)N(\psi)+m^2(y)N^2(\psi)\right]\frac{\partial\omega}{\partial y}\frac{\partial^2\psi}{\partial\psi^2}+\left\{\frac{1}{m(y)}4k'(y)m'(y)\right.$$

$$-3m(y)k''(y)+\left[4m'^2(y)-3m(y)m''(y)\right]N(\psi)-\left[1+k'^2(y)\right]\frac{N''(\psi)}{N'^2(\psi)}$$

$$\left.-2k'(y)m'(y)\frac{N(\psi)N''(\psi)}{N'^2(\psi)}-m'^2(y)\frac{N^2(\psi)N''(\psi)}{N'^2(\psi)}\right\}\frac{\partial^2\omega}{\partial y\partial\psi}+\frac{2}{m^2(y)N'(\psi)}\{m(y)k'(y)k''(y)$$

$$-m'(y)\left[1+k'^2(y)\right]+\left[m(y)m'(y)k''(y)+m(y)k'(y)m''(y)-2k'(y)m'^2(y)\right]N(\psi)$$

$$+m'(y)\left[m(y)m''(y)-m'^2(y)\right]N^2(\psi)\}\frac{\partial^2\omega}{\partial\psi^2}+\frac{1}{m^2(y)}\Big\{\ m(y)[3m'(y)k''(y)+2k'(y)m''(y)]$$

$$-4k'(y)m'^2(y)-m^2(y)k''(y)+\left[5m(y)m'(y)m''(y)-4m'^3(y)-m^2(y)m'''(y)\right]N(\psi)$$

$$+2\left[m'(y)k'^2(y)+m'(y)-m(y)k'(y)k''(y)\right]\frac{N''(\psi)}{N'^2(\psi)}+2\left[2k'(y)m'^2(y)-m(y)k'(y)m''(y)\right.$$

$$\left.-m(y)m'(y)k''(y)\right]\frac{N(\psi)N''(\psi)}{N'^2(\psi)}+2m'(y)\left[m'^2(y)-m(y)m''(y)\right]\frac{N^2(\psi)N''(\psi)}{N'^2(\psi)}\Bigg\}\frac{\partial\omega}{\partial\psi}\Bigg)=0 \quad (40)$$

and $N(\psi)$ is some function of ψ such that $N'(\psi) \neq 0$.

Conclusion: A given family of curves $\frac{x-k(y)}{m(y)}$ = constant is a permissible family of streamlines in second grade fluids if and only if the solution obtained for $N(\psi)$ is such that $N'(\psi) \neq 0$.

4. Applications:

Example 1. (Flow with $y - C_1x^2 - C_2x + 1$ = constant as streamlines). We assume

$$y = C_1x^2 + C_2x + M(\psi); \qquad M'(\psi) \neq 0$$

where C_1, C_2 are arbitrary constants and comparing this y with (32), we have

$$g(x) = C_1x^2 + C_2x, \quad f(x) = 1$$

The streamline pattern for this flow is shown in Figure 1.

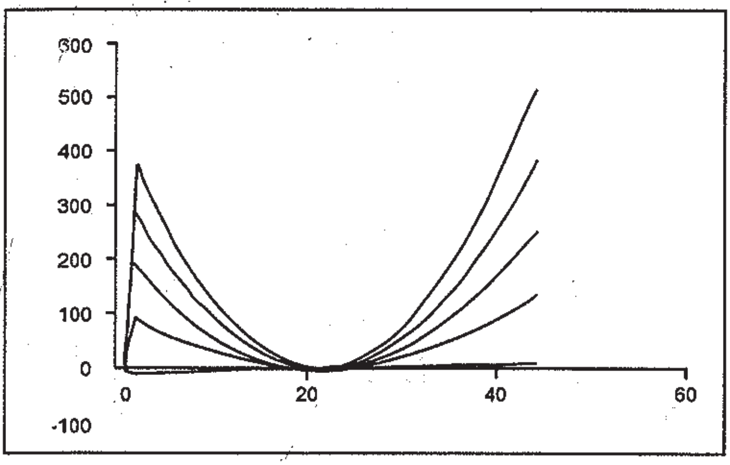

Figure 1. Streamline pattern for $y - C_1x^2 - C_2x + 1$ = constant.

Example 2. (Flow with $y - C_1e^x - C_2x$ = constant as streamlines). We assume

$$y = C_1e^x + C_2x + M(\psi); \qquad M'(\psi) \neq 0$$

where C_1, C_2 are arbitrary constants and therefore, we have

$$g(x) = C_1e^x + C_2x, \qquad f(x) = 1$$

The streamline pattern for this flow is shown in Figure 2.

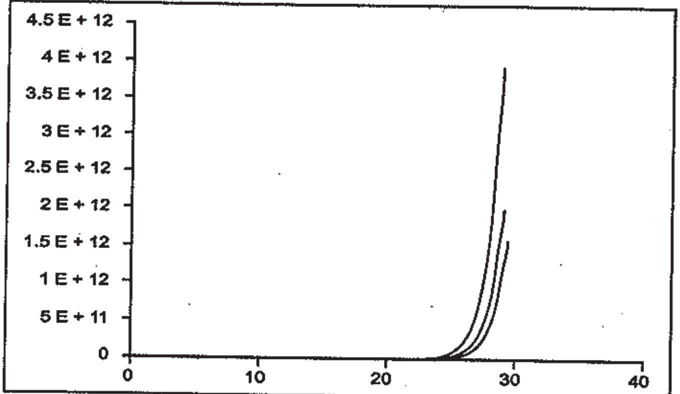

Figure 2. Streamline pattern for $y - C_1e^x - C_2x$ = constant.

Example 3. (Flow with $x(y - C_1x - C_2)$ = constant as streamlines). We assume

$$y = C_1x + C_2 + M(\psi)\,/\,x, \qquad M'(\psi) \neq 0$$

where C_1, C_2 are arbitrary constants and therefore, we have

$$g(x) = C_1x + C_2, \qquad f(x) = 1/x$$

The streamline pattern for this flow is shown in Figure 3. (C_1=0)

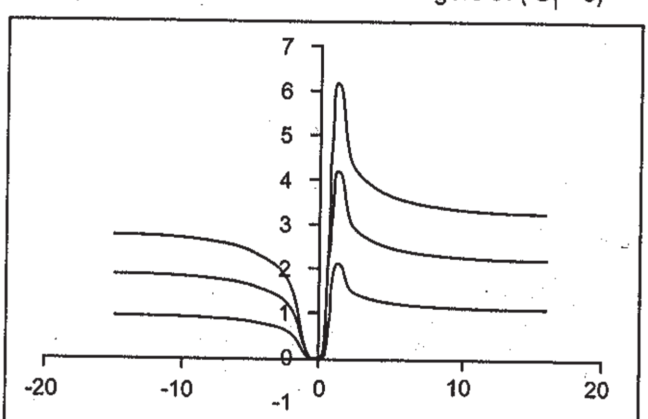

Figure 3. Streamline pattern for $x(y - C_1x - C_2)$ = constant.

Example 4. (Flow with $y - C_1x^3 - C_2x^2 + C_3$ = constant as streamlines). We assume

$$y = C_1x^3 + C_2x^2 + 1 + M(\psi); \qquad M'(\psi) \neq 0$$

where C_1, C_2 and C_3 are arbitrary constants and therefore, we have

$$g(x) = C_1x^3 + C_2x^2 + C_3, \qquad f(x) = 1.$$

The streamline pattern for this flow is shown in Figure 4.

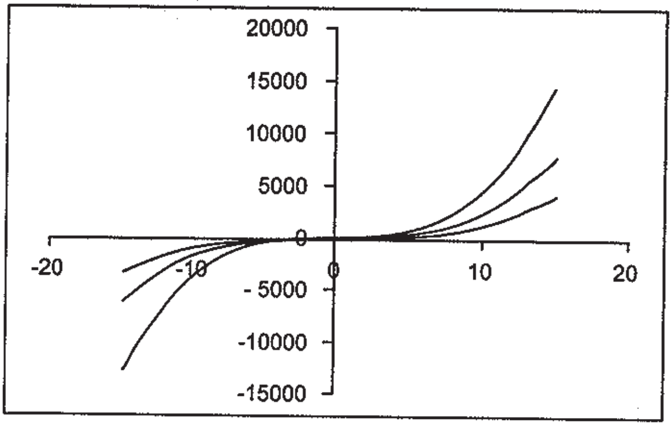

Figure 4. Streamline pattern for $y - C_1x^3 - C_2x^2 + C_3$ = constant.

Remark: Similarly we can discuss applications in (y, ψ) net.

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