



MODIFIED SECOND ORDER SLOPE ROTATABLE DESIGNS USING SYMMETRICAL UNEQUAL BLOCK ARRANGEMENTS WITH TWO UNEQUAL BLOCK SIZES

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ABSTRACT

In this paper, a new method of construction of modified second order slope rotatable designs using symmetrical unequal block arrangements with two unequal sizes are constructed.

1. Introduction

Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces. Das and Narasimham (1962) constructed rotatable designs through balanced incomplete block designs (BIBD). Das et al. (1999) constructed response surface designs, symmetrical and asymmetrical, rotatable and modified.

The study of rotatable designs mainly emphasized on the estimation of absolute response. Estimation of differences in response at two different points in the factor space will often be of great importance. If differences at two points close together,

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estimation of local slope (rate of change) of the response is of interest. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in an animal etc., Hader and Park (1978) introduced slope rotatable central composite designs. Park (1987) introduced a class of multifactor designs for estimating the slope of response surfaces. Victorbabu and Narasimham (1991) studied general second order slope rotatable designs (SOSRD) and constructed SOSRD using BIBD. Jang and Park (1993) studied a measure and graphical method for evaluating slope rotatability in response surface designs. Victorbabu (2002a) suggested a note on the construction of four and six level SOSRD. Victorbabu (2002b) constructed SOSRD using symmetrical unequal block arrangements with two unequal sizes. Victorbabu (2003) studied SOSRD using incomplete block designs. Victorbabu (2005) studied modified SOSRD using pairwise balanced designs.

2. Conditions for Modified SOSRD

A second order response surface design $D = ((x_{iu}))$ for fitting,

$$Y_u = b_0 + \sum_{i=1}^v b_i x_{iu} + \sum_{i=1}^v b_{ii} x_{iu}^2 + \sum \sum b_{ij} x_{iu} x_{ju} + e_u \quad (2.1)$$

where x_{iu} denotes the level of the i^{th} factor ($i = 1, 2, \dots, v$) in the u^{th} run ($u = 1, 2, \dots, N$) of the experiment, e_u 's are uncorrelated random errors with mean zero and variance σ^2 .

SOSRD: A second order response surface design D is said to be a SOSRD, if the variance of the estimate of first order partial derivative $(\partial \hat{Y}_u / \partial x_i)$ with respect to

each of independent variables (x_i) is only a function of the distance $(d^2 = \sum_i x_{iu}^2)$ of the point $(x_{1u}, x_{2u}, \dots, x_{vu})$ from the origin (centre) of the design. Such a spherical variance function for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions (cf. Hader and Park (1978), Victorbabu and Narasimham (1991)).

$$\sum_{u=1}^N \prod_{i \in I} x_{iu}^{\alpha_i} = 0 \text{ if any } \alpha_i \text{ is odd, for } \sum \alpha_i \leq 4 \quad (2.2)$$

$$(i) \quad \sum_{u=1}^N x_{iu}^2 = \text{constant} = N\lambda_2$$

$$(ii) \quad \sum_{i=1}^N x_{iv}^4 = \text{constant} = cN\lambda_4, \text{ for all } i \quad (2.3)$$

$$\sum_{i=1}^N x_{iv}^2 x_{iw}^2 = \text{constant} = N\lambda_4, \text{ for } i \neq j \quad (2.4)$$

$$(c+v-1)\lambda_4 > v\lambda_2^2 \quad (2.5)$$

$$\lambda_4 [v(5-c) - (c-3)^2] + \lambda_2^2 [v(c-5) + 4] = 0 \quad (2.6)$$

where c, λ_2 and λ_4 are constants and the summation is over the design points.

The variances and co-variances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{\lambda_4 (c+v-1) \sigma^2}{N[\lambda_4 (c+v-1) - v\lambda_2^2]}$$

$$V(\hat{b}_1) = \frac{\sigma^2}{N\lambda_2}$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_4}$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\lambda_4} \left[\frac{\lambda_4 (c+v-2) - (v-1)\lambda_2^2}{\lambda_4 (c+v-1) - v\lambda_2^2} \right]$$

$$\text{Cov}(\hat{b}_0, \hat{b}_{ii}) = \frac{-\lambda_2 \sigma^2}{N[\lambda_4 (c+v-1) - v\lambda_2^2]}$$

$$\text{Cov}(\hat{b}_{ii}, \hat{b}_{jj}) = \frac{(\lambda_2^2 - \lambda_4) \sigma^2}{(c-1)N\lambda_4 [\lambda_4 (c+v-1) - v\lambda_2^2]}$$

and other co-variances are zero.

$$V\left(\frac{\partial \hat{y}}{\partial x_i}\right) = V(\hat{b}_1) + 4x_i^2 V(\hat{b}_{ii}) + \sum_{i \neq j} x_i^2 x_j^2 V(\hat{b}_{ij}) = \frac{1}{N} \left[\frac{\lambda_4 + \lambda_2 d^2}{\lambda_2 \lambda_4} \right] \sigma^2$$

The usual method of construction of SOSRD is to take combinations with unknown constants, associate a 2^v factorial combinations or a suitable fraction of it with factors each at +1 and -1 levels to make the level codes equidistant. All such combinations form a design. Generation of design points this way ensures satisfaction of all the conditions even though the design points contain unknown levels.

Alternatively by putting some restrictions indicating some relation among $\sum x_{iw}^2$, $\sum x_{iw}^4$ and $\sum x_{iw}^2 x_{iw}^2$ some equations involving the unknowns are obtained and their solution gives the unknown levels. In SOSRD the restriction used is $V(b_{ij}) = 4V(b_{ij})$ viz. equation (2.6). Other restrictions are also possible though, it seems, not yet exploited. We shall investigate the restriction $(\sum x_{iw}^2)^2 = N \sum x_{iw}^2 x_{iw}^2$ i.e., $(N\lambda_2)^2 = N(N\lambda_4)$ to get modified SOSRD. By applying the new modification of Das et al. (1999) restriction in equation (2.6), we get $c=5$. Further,

$$V(\hat{b}_0) = \frac{(v+4)\sigma^2}{4N}$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\sqrt{\lambda_4}}$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4}$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{4N\lambda_4}$$

$$\text{Cov}(\hat{b}_0, \hat{b}_{ii}) = \frac{-\sigma^2}{4N\sqrt{\lambda_4}} \text{ and other co-variances are zero.}$$

$$V\left(\frac{\partial \hat{Y}}{\partial x_i}\right) = \left[\frac{\sqrt{\lambda_4} + d^2}{N\lambda_4}\right] \sigma^2$$

3. Construction Of Modified SOSRD Using Symmetrical Unequal Block Arrangements With Two Unequal Block Sizes

The method of construction of modified SOSRD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes is given in theorem (3.1). Here we use the notations of Raghavarao (1971), pp. 291-292 and 298-300, Victorbabu and Narasimham (1991), pp. 2467-2472.

SUBA with Two Unequal Block Sizes (c.f. Raghavarao, 1962)

The arrangement of v -treatments in b blocks where b_1 blocks of size k_1 and b_2 blocks of size k_2 is said to be a symmetrical unequal block arrangement with two unequal block sizes, if

- (i) every treatment occurs $\frac{b_i k_i}{v_{nu}}$ blocks of size k_i ($i = 1, 2$), and
- (ii) every pair of first associate treatments occurs together in u blocks of size k_1 and in $(\lambda - u)$ blocks of size k_2 while every pair of second associate treatments occurs together in λ blocks of size.

From (i) each treatment occurs in $\left(\frac{b_1 k_1}{v_{nu}}\right) + \left(\frac{b_2 k_2}{v_{nu}}\right) = r$ blocks in all. ($v, b, r, k_1, k_2, b_1, b_2, \lambda$) are known as the parameters of the SUBA with two unequal block sizes.

Let $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$, $k = \sup(k_1, k_2)$ and $b_1 + b_2 = b$ be a SUBA with two unequal block sizes. $2^{(k)}$ denotes a resolution V fractional factorial of 2^k in $+1$ levels, such that no interaction with less than five factors is confounded.

$[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]$ denote the design points generated from the transpose of incidence matrix of SUBA with two unequal block sizes, $[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{(k)}$ are the $b2^{(k)}$ design points generated from SUBA with two unequal block sizes by multiplication (c.f. Raghavarao (1971), pp. 298-300), $(\alpha, 0, 0, \dots, 0)2^1$ denote the design points generated from $(\alpha, 0, 0, \dots, 0)$ point set, and U denotes combination of the design points generated from different sets of points. Repeat this set of additional design points say n_0 times when $r < 5\lambda$. Let $(\alpha, \alpha,$

$\dots, a)2^{(v)}$ denote the design points generated from (a, a, \dots, a) point set. Repeat this set of additional design points say n_0 times when $r > 5\lambda$. Let n_0 be the number of central points in modified SOSRD.

Theorem (3.1): Let $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ be a SUBA with two unequal block sizes,

Case (i): If $r < 5\lambda$, the design points,

$[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{(k)} \cup n_0 (a, 0, \dots, 0)2^1 \cup (n_0)$ give a v -dimensional

Modified SOSRD in $N = \frac{(r2^{(k)} + 2n_0a^2)^2}{\lambda 2^{(k)}}$ design points, if, (3.1)

$$a^4 = \frac{(5\lambda - r)2^{(k)-1}}{n_0}, \quad (3.2)$$

$$n_0 = \frac{(r2^{(k)} + 2n_0a^2)^2}{\lambda 2^{(k)}} - [b2^{(k)} + 2n_0v] \quad (3.3)$$

and n_0 turns out to be an integer.

Case (ii): If $r = 5\lambda$, then the design points,

$[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{(k)} \cup n_0$ give a three level v -dimensional modified

SOSRD in $N = \frac{(r2^{(k)})^2}{\lambda 2^{(k)}}$ design points if, (3.4)

$$n_0 = \frac{(r2^{(k)})^2}{\lambda 2^{(k)}} - b2^{(k)} \text{ and } n_0 \text{ turns out to be an integer.} \quad (3.5)$$

Case (iii): If $r > 5\lambda$, then the design points,

$[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]2^{(k)} \cup n_0 (a, a, a, \dots, a)2^{(v)} \cup n_0$ give a v -dimensional

modified SOSRD in $N = \frac{(r2^{(k)} + n_0 2^{(v)} a^2)^2}{\lambda 2^{(k)} + n_0 2^{(v)} a^4}$ design points if, (3.6)

$$a^4 = \frac{(r - 5\lambda)2^{(k)-(v)-2}}{n_0} \quad (3.7)$$

$$n_0 = \frac{(r2^{l(k)} + n_0 2^{l(v)} \alpha^2)^2}{\lambda 2^{l(k)} + n_0 2^{l(v)} \alpha^4} - (b2^{l(k)} + n_0 2^{l(v)}) \quad (3.8)$$

and n_0 turns out to be an integer.

Proof: Case (i): Let $r < 5\lambda$, from conditions of modified SOSRD, we have

$$\sum x_{iw}^2 = r2^{l(k)} + 2n_0 \alpha^2 = N\lambda_2 \quad (3.9)$$

$$\sum x_{iw}^4 = r2^{l(k)} + 2n_0 \alpha^4 = 5N\lambda_4 \quad (3.10)$$

$$\sum x_{iw}^2 x_{iw}^2 = \lambda 2^{l(k)} = N\lambda_4 \quad (3.11)$$

The modified condition $(\sum x_{iw}^2)^2 = N \sum x_{iw}^2 x_{iw}^2$ leads to N given in equation (3.1). (Alternatively N may be obtained directly as $N = b2^{l(k)} + 2vn_0 + n_0$, where n_0 is given in equation (3.3)). Equations (3.10) and (3.11) leads to α^4 given in equation (3.2).

Case (ii): Let $r = 5\lambda$, from conditions of modified SOSRD, we have,

$$\sum x_{iw}^2 = r2^{l(k)} = N\lambda_2 \quad (3.12)$$

$$\sum x_{iw}^4 = r2^{l(k)} = 5N\lambda_4 \quad (3.13)$$

$$\sum x_{iw}^2 x_{iw}^2 = \lambda 2^{l(k)} = N\lambda_4 \quad (3.14)$$

The modified condition $(\sum x_{iw}^2)^2 = N \sum x_{iw}^2 x_{iw}^2$, leads to N given in equation (3.4). (Alternatively N may be obtained directly as $N = b2^{l(k)} + n_0$, where n_0 is given in equation (3.5)). Equations (3.13) and (3.14), we have $r2^{l(k)} = 5\lambda 2^{l(k)}$ implies $r = 5\lambda$.

Case (iii): Let $r > 5\lambda$, from conditions of modified SOSRD, we have,

$$\sum x_{iw}^2 = r2^{l(k)} + n_0 2^{l(v)} \alpha^2 = N\lambda_2 \quad (3.15)$$

$$\sum x_{iw}^4 = r2^{l(k)} + n_0 2^{l(v)} \alpha^4 = 5N\lambda_4 \quad (3.16)$$

$$\sum x_{iw}^2 x_{iw}^2 = \lambda 2^{l(k)} + n_0 2^{l(v)} \alpha^4 = N\lambda_4 \quad (3.17)$$

The modified condition $(\sum x_{ij}^2)^2 = N \sum x_{ij}^2 x_{jv}^2$, leads to N given in equation (3.6). (Alternatively N may be obtained directly as $N = b2^{(k)} + na2^{(v)} + n_0$, where n_0 is given in equation (3.8)). Equations (3.16) and (3.17), leads to a^4 given in equation (3.7).

Corollary: If $k_1 = k_2 = k$, then theorem (3.1) reduces to the method of construction of modified SOSRD using BIBD.

Example (i): Here we construct a modified SOSRD for $v = 6$ factors with the help of a SUBA with two unequal block sizes ($v=6, b=7, r=3, k_1=2, k_2=3, b_1=3, b_2=4, \lambda=1$). The design points,

$[1 - (6, 7, 3, 2, 3, 3, 4, 1)]2^3 \cup n_0 (a, 0, \dots, 0)2^1 \cup (n_0)$. Let us fix $n_0=2$ Here $r < 5\lambda$. Equations (3.2) $\Rightarrow a^2=2.00$, (3.1) $\Rightarrow N=128$ and (3.3) $\Rightarrow n_0=48$. Thus with $c=5, a^2=2.00, n_0=48$, we get a 6-factor modified SOSRD in $N=128$ design points.

Example (ii): Consider the SUBA with two unequal block sizes, $[1 - (v=9, b=18, r=5, k_1=2, k_2=3, b_1=9, b_2=9, \lambda=1)]2^3 (n_0)$. Here $r=5\lambda$. Equations (3.5) $\Rightarrow n_0=56$, (3.4) $\Rightarrow N=200$. Thus with $c=5, n_0=56$, we get a 9-factor three level modified SOSRD in $N=200$ design points.

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