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A Study on Wiener Polynomial for Steiner n - distance of some graphs

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Abstract

In this paper, the Wiener polynomial $\boldsymbol{W_n(G;x)}$ for some graphs such as complete, bipartite and star graphs are studied. The Wiener polynomial for Steiner n-distance of Corona and Complement graphs are derived. An attempt is made to obtain the Wiener polynomial for Steiner n-distance of Prism graphs.

Keywords: Prism graph, corona graph, complement graph, Steiner distance, wiener polynomial

1. Introduction

A graph G(V, E) consists of a finite nonempty set V=V(G) of p vertices (or points) together with a set E=E(G) of q unordered pairs of vertices of V which are known as edges (or lines). In this paper, we use non – trivial, finite, undirected connected graph without loops and multiple edges.

The distance $d_G(u,v)$ between the vertices u and v is the length of the shortest path in G connecting u and v. The eccentricity e(u)= max{d(u,v): v∈V(G)}. The radius r(G) and the diameter d(G) of the

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graph G are defined by r(G)=min{e(u):u∈V(G)} and d(G)= max{e(u):u∈V(G)}, respectively.

For general notation and terminology, we follow Harary.**[6]**

The Steiner distance for a non empty set S ⊆ V(G), denoted by $d_G(S)$, is the size of the smallest connected subgraph H(S) containing S.**[3]** If the minimal subgraph H(S) is a tree of G, it is called the Steiner tree of S. Thus, Steiner distance of the set S of n distinct vertices is the minimum number of edges in a connected sub graph that contains S. If |S| = 2 then the Steiner distance is the distance between the two vertices.

If $2 \leq n \leq p$ and |S|=n, then Steiner distance of S is called Steiner n-distance of S and is denoted by $d_G(S)$. The Steiner n-diameter of a graph G denoted by $diam_n(G)$ or $\delta_n(G)$ is defined as the maximum Steiner distance of S of n vertices of V(G).The total Steiner n-distance $d_n(G)$ is defined by $d_n(G) = \Sigma\{ d_G(S) \mid S \subseteq V(G), |S| = n \}$.

The average Steiner n-distance $\mu_n(G)$ of a connected graph G is the average distance over all subsets S of n vertices in G, $\mu_n(G)= \frac{d_n(G)}{\binom{p}{n}}$, $\binom{p}{n}$ is the number of subsets having n elements.

The problem of finding $\mu_n(G)$ is NP complete if $2 < n < p$.**[8]** The sharp bounds for $\mu_n(G)$ are already obtained.**[4]**

2. Preliminaries

The concept of the Wiener polynomial $W(G;x)$ of a graph G was put forward by Hosoya. **[7]**

Definition 2.1

The *Wiener polynomial* of a graph G is defined as $W(G;x)=\sum_{k=0}^{\delta(G)} C(G,k)x^k$ where $C(G;k)$ is the number of pairs of vertices in G that are distance k apart and $\delta(G)$ is the diameter of the graph G.**[9]**

Gutman **[5]** established some basic properties of $W(G;x)$. Saeed **[9]** obtained the Wiener polynomial for several classes of graphs and studied some properties of the sequence $\{C(G;k)\}$ which generates the polynomial $W(G;x)$.

Ali and Said defined the Wiener polynomial of Steiner n-distance of connected graph G and derived the same for some special graphs.**[1]**

Definition 2.2

The Wiener polynomial of Steiner n-distance of a connected graph G is defined as $W_n(G;x)=\sum_{k=n-1}^{\delta n(G)} C_n(G,k)\, x^k$ where $2 \leq n \leq p$, $C_n(G,k)$ is the number of subsets S of n distinct vertices with Steiner distance k in the graph G, and $\delta_n(G)$ is the Steiner n-diameter of G.**[1]** For n=2, $W_2(G;x) = W(G;x) - p$.

Results 2.3

For a complete graph of order p, $W_n(K_p;x) = \binom{p}{n} x^{n-1}$.**[1]**

Results 2.4

If $K_{r,s}$ is the complete bipartite graph of order $r+s$, then $W_n(K_{r,s};x) = \left[\binom{r}{n} + \binom{s}{n}\right]x^n + \left[\sum_{i=1}^{n-1}\binom{r}{i}\binom{s}{n-i}\right]x^{n-1} = \left[\binom{r}{n} + \binom{s}{n}\right]x^n + \left[\binom{r+s}{n} - \binom{r}{n} - sn\right]x^{n-1}$; if a and b are positive integers and a < b, then $ab = 0$.**[1]**

Results 2.5

For S_p, a star graph of order p, $W_n(S_p;x) = \binom{p-1}{n}x^n + \binom{p-1}{n-1}x^{n-1}$.**[1]**

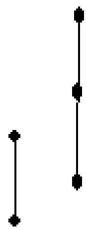
3. Wiener polynomial of Steiner n-distance of corona graph

Definition 3.1

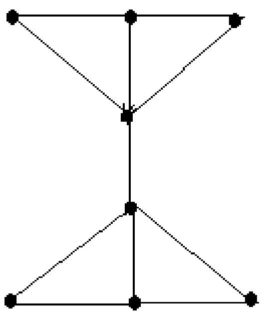
Let G_1 and G_2 be two simple, connected graphs. The corona $G_1 0 G_2$ of two graphs G_1 and G_2 was defined by Frucht and Harary as the graph obtained by taking one copy of G_1(which has p_1 points) and p_1 copies of G_2, and then joining the i^{th} copy of G_2 with G_1. **[6]** $G_1 0$-G_2 has $p_1(1+p_2)$ vertices and $q_1+p_1q_2 + p_1p_2$ edges.

Example 3.2 Let $G_1 = K_2$ and G_2=$K_{1,2}$

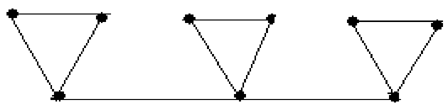
$\boldsymbol{G_1}$ $\boldsymbol{G_2}$



$\boldsymbol{G_{1}\circ G_{2}}$



$\boldsymbol{G_{2}\circ G_{1}}$



v(G_1)=2; v(G_2)=3; e(G_1)=1; e(G_2)=2; v($G_{1}\circ G_2$)=2+3×2=8; v($G_{2}\circ$ G_1)=3+3×2=9.

e($G_{1}\circ G_2$)=1+2×3+2×3=13; e($G_{2}\circ G_1$)=2+3×1+3×2=11.

Theorem 3.3

For $3 \leq n \leq p_1 + p_1p_2$, $W_n(G_{1}\circ G_2;x) = W_n(G_1;x)+ p_1.W_n(G_2;x) + \sum_{r=1}^{n-1}\prod_{i=1}^{p_1} W_{r_i}(G_2;x).W_{n-\sum r_i}(G_1;x)$

Proof

Let S be a subset of V($G_{1}\circ G_2$) containing n vertices such that $d_{G_1 \circ G_2}$(S) = k. We consider the following cases:

(i) S⊆ V(G_1). The number of such n- subsets S is C_n $(G_1;k)$ and this produces the polynomial W_n $(G_1;x)$.

(ii) S$\subseteq$ V(G_2). The number of such n- subsets S is C_n $(G_2; k)$ and this produces the polynomial W_n $(G_2; x)$.

In $G_{1\circ}G_2$, we have $p_1$copies of v(G_2) and this produces the polynomial p_1 W_n $(G_2; x)$.

(iii) S$\subseteq$ {V(G_1)$\cup p_1.V(G_2)$},where $p_1.V(G_2)$ stands for $p_1$copies of V(G_2) so that |S|=|S_0|+$\sum_{i=1}^{p_i}|S_i|$=n ,where S_0 contains vertices of $G_1$and S_i contains vertices of G_2 for i=1,2,….p_1(p_1 copies).

If k is the Steiner distance of S, then k = $k_0 + \sum_{i=1}^{p_i} k_i$ where k_i are Steiner distance of each copy of $S_i, i = 1,2, \ldots . p_1$. It is clear that $1 \leq d_{G_1}(S_0) \leq k_0$ and $1 \leq d_{G_2}(S_i) \leq k_i$ for i=1,2,….p_1. The number of such $S_0$is $C_{n-(\sum_1^{p_1} r_i)}(G_1; k_0)$ and the number of such S_i is $C_{r_i}(G_2; k_i)$ for i=1,2, ….p_1, r_i=1,2, ….n-1.

The coefficient of x^k is $\sum_{i=r}^{k-1} C_{n-(\sum_1^{p_1} r_i)}(G_1; k_0).\prod_{i=1}^{p_1} C_{r_i}(G_2; k_i)$. Summing over k, n-1 ≤ k≤ $\delta_n(G_{1\circ}G_2)$ and then over r, 1≤ r ≤ n-1, we get $\sum_{r=1}^{n-1} \prod_{i=1}^{p_1} W_{r_i}(G_2; x) \,.\, W_{n-\sum r_i}(G_1; x)$

Hence adding the polynomials obtained in the above three cases, we get the Wiener polynomial of corona graph $G_{1\circ}G_2$.Similarly the Wiener polynomial of the corona graph $G_{2\circ}G_1$ is $W_n(G_{2\circ}G_1\ ; x) = W_n(G_2; x) + p_2. W_n(G_1; x) + \sum_{r=1}^{n-1} \prod_{i=1}^{p_2} W_{r_i}(G_1; x) \,.\, W_{n-\sum r_i}(G_2; x)$.

4. Wiener polynomial of Steiner n-distance of the complement graph

Definition 4.1

The *complement* graph $\bar{G}$ of a graph G is defined to be the graph which has V(G) as its vertex set and two points are adjacent in $\bar{G}$ if and only if they are not adjacent in G.**[6]**

Definition 4.2

The graph G is said to be a *self-complementary graph* if G is isomorphic to $\bar{G}$. **[6]**

Definition 4.3

Two graphs G_1= (V_1,E_1) and G_2= (V_2,E_2) are said to be *isomorphic* if there exists a bijection f: $V_1 \rightarrow V_2$ such that u, v are adjacent in G_1 if and only if f(u), f(v) are adjacent in G_2.This relation is denoted by $G_1 \cong G_2$. The map f is called an *isomorphism* of G_1 to G_2.**[6]**

Remark 4.4

The procedure for obtaining Wiener polynomial of Steiner distance of the complement graph $\bar{G}$ is same as that of G, if $\bar{G}$ is a connected graph.

Remark 4.5

If the complement graph $\bar{G}$ is disconnected, Wiener polynomial of Steiner distance of the complement graph $\bar{G}$ does not exist. Hence the relation between the Wiener polynomial of G and $\bar{G}$ cannot be found.

Example 4.6

If the graph G is given by fig (1), its complement graph $\bar{G}$ is given by fig (2)

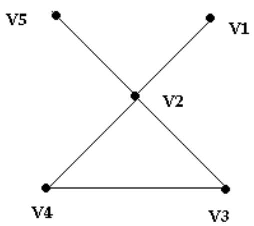


Fig.1

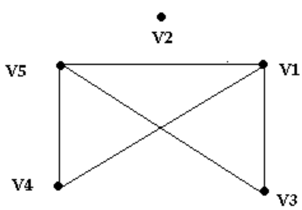


Fig.2

$W_n(G;x) = 5x + 5x^2$ and $W_n(\bar{G};x)$ does not exist.

Remark 4.7

If G is a complete graph K_n, Wiener polynomial of Steiner distance of the complement graph $\bar{G}$ does not exist as the complement of a complete graph is totally disconnected for which Wiener polynomial of Steiner distance cannot be defined.

Remark 4.8

In a connected graph G, if a vertex v is adjacent to every other vertex of G, then the Wiener polynomial of Steiner distance of the complement graph $\bar{G}$ does not exist.

5. Wiener polynomial of Steiner n-distance of Prism graph

Definition 5.1

A *prism graph* Y_N is a graph corresponding to the skeleton of an n-prism. It is also called a circular ladder graph and denoted by CL_N. Prism graphs are both planar and polyhedral. An N-prism graph has 2N nodes and 3N edges.**[10]**

The prism graph Y_3 is the line graph of the complete bipartite graph $K_{2,3}$.

Definition 5.2

If G is a graph, $\bar{G}$ is its complement and π is a bijection $\pi: V(G) \to V(\bar{G})$, the *complementary prism* $G\bar{G}$ is the graph obtained by taking disjointed copies of G and $\bar{G}$ and adding the edge $\{v, \pi(v)\}$ for each $v \in V(G)$. The complementary prism of a graph G is obtained from a copy of G and its complement $\bar{G}$ by adding a perfect matching between the corresponding vertices of G and $\bar{G}$.**[10]**

Theorem 5.3

The Wiener polynomial of Steiner n-distance of Prism graph Y_3 is $W_n(Y_3\,; x) = \sum_{k=1}^{\delta_n} C_n(Y_3; x)x^k$ for $2 \le n \le N$.

Proof

Let us find the Wiener polynomial of Steiner n-distance of Prism graph Y_3. The vertices of inner graph are u_1, u_2, u_3 and the vertices of the outer graph are v_1, v_2, v_3. Let us consider subsets $S \subseteq V(Y_3)$ with n vertices, n = 2,3,…2N. For $2 \le n \le N$, the following three cases may be considered.

(i) vertices of the inner graph,

(ii) vertices of the outer graph and

(iii) vertices of both inner and outer graphs.

Hence the Wiener polynomial of Steiner n-distance may be computed separately for the three cases and the sum will be the required polynomial.

$$W_2(Y_3\,; x) = 3x + 3x + \{3x + 6x^2\} = 9x + 6x^2;$$

$$W_3(Y_3\,;x) = x^2 + x^2 + \{12x^2 + 6x^3\} = 14x^2 + 6x^3;$$

$$W_4(Y_3\,;x) = 15x^3;$$

$$W_5(Y_3\,;x) = 6x^4\,;$$

$$W_6(Y_3\,;x) = x^5.$$

In general $W_n(Y_3\,;x) = \sum_{k=1}^{\delta_n} C_n(Y_3;x)x^k$ for $2 \leq n \leq N$.

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