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Solutions of Graph Equations Involving Line, Middle and Mycielski Graphs

H. P. Patil* and R. P. Raj†

Abstract

Let G be a graph with the vertex set $V = \{v_i: 1 \leq i \leq n\}$. The Mycielski graph of G denoted by $\mu(G)$ is the graph obtained from G by adding $n+1$ new vertices $V' = \{v_i' : 1 \leq i \leq n\}$ and u, then for $1 \leq i \leq n$, joining v_i' to the neighbours of v_i and to u. In this paper, we determine all pairs of graphs: (G, H) for which one of the graphs: $L(G)$, $\overline{L(G)}$, $M(G)$ and $\overline{M(G)}$ is isomorphic to the Mycielski graph$\mu(H)$.

Keywords: Line graph, middle graph, Mycielski graph, graph valued function.

1. Introduction

All graphs considered here are finite, undirected, without loops and without multiple edges. We follow the terminology of Harary[3]. For any graph G, let $\bar{G}$ and $L(G)$ denote the complement and the line graph of G, respectively. The end-edge graph G^+ is the graph obtained from G by adjoining an end-edge $u_i u_i'$ at each vertex u_i of G. Hamada *et al*. showed that the middle graph $M(G)$ is isomorphic to the line graph $L(G^+)$.[2] Let$V(G) = \{v_i : 1 \leq i \leq n\}$. The Mycielski graph $\mu(G)$ introduced in [1] is the graph obtained

* Pondicherry University; hpppondy@gmail.com
† Pondicherry University; pandiyarajmaths@gmail.com
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from G by inserting $(n+1)$ new vertices $\{v_i' : 1 \le i \le n\}$ and u to G by joining each v_i' to the neighbours of v_i and to u.

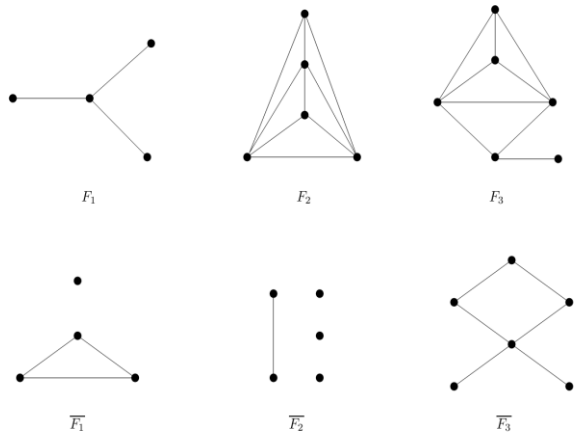

Fig 1

In this paper, we shall obtain all pairs of graphs(G, H), which satisfy the following four graph equations:

1. $L(G) = \mu(H)$
2. $M(G) = \mu(H)$
3. $\overline{L(G)} = \mu(H)$ and
4. $\overline{M(G)} = \mu(H)$.

Here, the equality sign, = means the isomorphism between the corresponding graphs. A pair of graphs (G, H), which satisfies a graph–equation is called its solution. Throughout our discussion, a pair (G, H) is always considered as a solution of a graph–equation mentioned above.

In order to determine the solutions of the above equations, we need the result of Beineke (Theorem 8.4; p. 74).[3] A graph G is a line graph if and only if G has none of nine specified graphs: G_i for $1 \le i \le 9$, as an induced subgraph. Among these, we depict here three forbidden subgraphs, call F_i $1 \le i \le 3$ and their complements $\overline{F_i}$ (Figure 1).

Further, it is noticed that for any two graphs G and H, if $\mu(H)$ has $\overline{F_i}$, $1 \leq i \leq 3$ as an induced subgraph, then $\overline{L(G)}$ (or $\overline{M(G)} = \mu(H)$) has no solution.

2. The solution of $L(G) = \mu(H)$

To solve this equation, we first observe that if $\mu(H)$ contains a subgraph isomorphic to F_1, then $\mu(H)$ cannot be the line graph $L(G)$ of G. Hence, the structure of H is as follows: $|H| \leq 2$, since otherwise for $|H| \geq 3$, $\mu(H)$ contains a subgraph isomorphic to $K_{1,3}$. Consequently, $\mu(H)$ contains a subgraph isomorphic to F_1. Next, we discuss two cases depending on whether or not H is connected.

Case 1

Suppose H is connected. Since, $|H| \leq 2$, it follows that H is either K_1or K_2. If $H = K_1$, then $L(G) = K_1 \cup K_2$. Hence, G must be $(K_2 \cup P_3)$. If $H = K_2$, then $L(G) = C_5$. Therefore, $G = C_5$.

Case 2

Suppose H is disconnected. Since, $|H| \leq 2$, it follows that H must be $\overline{K_2}$. Consequently, $L(G) = 2K_1 \cup P_3$, and so $G = 2K_2 \cup P_4$.

From the above discussion, we have the following result.

Theorem 2.1

For any two graphs G and$H$, the graph equation: $L(G) = \mu(H)$ holds if and only if (G, H) is one of the following pairs of graphs: $(K_2 \cup P_3,\ K_1)$; (C_5, K_2) and $(2K_2 \cup P_4,\ \overline{K_2})$.

3. The solutions of $M(G) = \mu(H)$

Theorem 2.1 provides three pairs of graphs: $(K_2 \cup P_3,\ K_1)$; (C_5, K_2) and $(2K_2 \cup P_4,\ \overline{K_2})$, which are the solutions of the equation: $L(G) = \mu(H)$. Among these pairs, only one pair of graphs $(2K_2 \cup P_4,\ \overline{K_2})$ is of the form: (G^+, H). Hence, the solution of the required equation: $L(G^+) = \mu(H)$ is $(G, H) = (2K_1 \cup K_2,\ \overline{K_2})$. Since, $L(G^+) = M(G)$, we have the following result.

Theorem 3.1

There is only one solution (G, H) of the graph equation, $M(G) = \mu(H)$, where $(G, H) = (2K_1 \cup K_2, \overline{K_2})$.

4. The solution of $\overline{L(G)} = \mu(H)$

Suppose $\mu(H)$ has one of the graphs: $\overline{F_1}, \overline{F_2}$ and $\overline{F_3}$ (Figure 1), as an induced subgraph. Then $\mu(H)$ cannot be the complement of the line graph $L(G)$ of G. Hence, the structure of H is such that H cannot have three or more components, since otherwise an induced subgraph $\overline{F_2}$ would appear in $\mu(H)$. This shows that the equation: $\overline{L(G)} = \mu(H)$ has no solution. Thus, H has at most two components. We discuss two cases depending on the connectivity of H:

Case 1

Suppose H is a component. Then, G is connected. Immediately, $\Delta(H) \leq 2$, since otherwise H has a vertex v of degree ≥ 3. Then any three edges of H incident with u form $K_{1,3}$ in H. But $\mu(K_{1,3})$ contains a forbidden subgraph isomorphic to $\overline{F_2}$, (see, Figure 2).

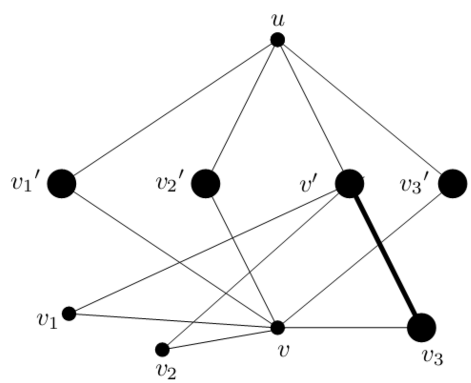

Fig 2

Since $\mu(K_{1,3})$ is a subgraph of $\mu(H)$, it follows that the equation: $\overline{L(G)} = \mu(H)$ has no solution. Thus, G is either a path or a cycle. However, we see that H cannot be a cycle. On the contrary, assume that $H = C_n$ for $n \geq 3$. There are two subcases depending on n:

Subcase 1.1. If $n = 3$, then H is a triangle K_3. It is easy to check that the forbidden subgraph $\overline{F_1}$ would appear in $\mu(K_3)$, and hence $\overline{L(G)} = \mu(H)$ has no solution.

Subcase 1.2 If $n \geq 4$, then H is a cycle C_n of length ≥ 4, which evidently contains a subgraph isomorphic to P_3. But $\mu(P_3)$ contains a forbidden subgraph isomorphic to $\overline{F_3}$, (see, Figure 3). Since $\mu(P_3)$ is a subgraph of $\mu(C_n)$, $\overline{L(G)} = \mu(H)$ has no solution. Consequently, H must be a path P_m for $m \geq 1$. Further, we see that $m \leq 2$; since otherwise Subcase (1.2) repeats. Thus, H is either K_1 or K_2.

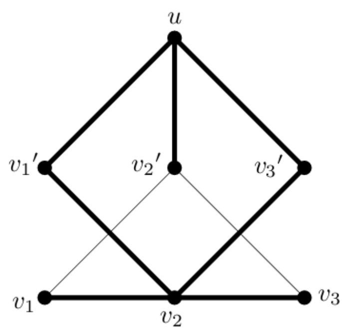

Fig 3

When $H = K_1$. Then $\mu(H) = K_1 \cup K_2$, and hence$L(G) = P_3$. This implies that $G = P_4$. In this case, (P_4, K_1) is the solution of the desired equation.

When$H = K_2$. Then $\mu(H) = C_5$, and hence $L(G) = C_5$. So, $G = C_5$. Consequently, (C_5, K_2) is the solution of the required equation.

Case 2

Suppose H has exactly two components with$|E(H)| \neq \emptyset$. Immediately, H contains a subgraph isomorphic to $(K_1 \cup K_2)$, and $\mu(H)$ contains a forbidden –subgraph isomorphic to $\overline{F_2}$. Hence there exists no solution to the required equation. Therefore, H must be $\overline{K_2}$. Consequently, $\mu(H) = \overline{K_2} \cup P_3$. Since $\mu(H) = \overline{L(G)}$, $L(G)$ is K_4 together with a vertex joined to two adjacent vertices. Therefore, $G = (K_{1,4} + e)$. Thus, $(K_{1,4} + e, \overline{K_2})$ is the solution of the required equation. Overall, the above discussion yields the following result.

Theorem 4.1

For any two graphs G and H, the graph equation: $\overline{L(G)} = \mu(H)$ holds if and only if (G, H) is one of the following pairs of graphs: (P_4, K_1); (C_5, K_2) and $(K_{1,4} + e,\ \overline{K_2})$.

5. The solution of $\overline{M(G)} = \mu(H)$

Now, we have determined the solutions (G, H) of the equation: $\overline{L(G)} = \mu(H)$ in theorem 4.1. Among these solutions, only one pair of graphs (P_4, K_1) is of the form: $(G^+,\ H)$. Therefore, the solution of the equation: $\overline{M(G)} = \mu(H)$ is $(G, H) = (K_2, K_1)$. Thus, we have the following result.

Theorem 5.1

There is only one solution $(\boldsymbol{G}, \boldsymbol{H})$ of the equation: $\overline{\boldsymbol{M}(\boldsymbol{G})} = \boldsymbol{\mu}(\boldsymbol{H})$. This is $(\boldsymbol{G}, \boldsymbol{H}) = (\boldsymbol{K_2},\ \boldsymbol{K_1})$.

6. Problem

For the application point of view, it is worth to solve the graph equation: $\mu(G) = \overline{\mu(H)}$. Then from the graph equation $\mu(G) = \overline{\mu(H)}$, we can get all the pairs: (G, H) for which one of the graphs: $L(G), \overline{L(G)}, M(G)$ and $\overline{M(G)}$ is isomorphic to the graph $\overline{\mu(H)}$.

References

[1] G. J. Chang, Huang and X. Zhu, "Circular Chromatic numbers of Mycielski's graphs," *Discrete Math.*, vol. 205, no. 1-3, 1999, pp. 23-37.

[2] T. Hamada and I. Yoshimura, "Traversability and connectivity of the middle graph of the graph," *Discrete Math.* vol.14 no.3, 1976, pp. 247-256.

[3] F. Harary, *Graph Theory*, MA: Addison –Wesley, 1969.