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Further Results on Sum Labelling of Split Graphs

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Abstract

A *sum labelling* is a mapping λ from the vertices of G into the positive integers such that, for any two vertices $u, v \in$ V(G) with labels λ(u) and λ(v), respectively, *uv* is an edge iff λ(u) + λ(v) is the label of another vertex in V(G). Any graph supporting such a labelling is called a *sum graph*. It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as *isolates* and the labelling scheme that requires the fewest isolates is termed *optimal*. The number of isolates required for a graph to support a sum labelling is known as the *sum number* of the graph. In this paper, we obtain *optimal sum labelling scheme* for path union of split graph of star, $K_{1,m} \odot Spl(P_n)$ and $K_{1,m} \odot Spl(K_{1,n})$.

Keywords: Sum labelling, sum graph, sum number, split graph, path union.

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1. Introduction

All the graphs considered here are simple, finite and undirected. For all terminologies and notations we follow Harary [1] and graph labelling as in [2]. Sum labelling of graphs was introduced by Harary [3] in 1990. Following definitions are useful for the present study.

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Definition 1.1

Sum Labelling is a mapping λ from the vertices of G into the positive integers such that, for any two vertices u, $v \epsilon$ V(G) with labels λ(u) and λ(v), respectively, uv is an edge iff λ(u)+λ(v) is the label of another vertex in V(G). Any graph supporting such a labelling is called a *sum graph*.

Definition 1.2

It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as *isolates* and the labelling scheme that requires the fewest isolates is termed *optimal*.

Definition 1.3

The number of isolates required for a graph G to support a sum labelling is known as the *sum number* of the graph. It is denoted as $\sigma(G)$.

Definition 1.4

(Shiama [4]) For a graph G the split graph is obtained by adding to each vertex v, a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G. The resultant graph is called the *split graph* denoted by Spl(G).

Definition 1.5

(Shee and Ho [5]) Let $G_1, G_2, \ldots\ldots, G_n$, $n \geq 2$ be n copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_{i+1} for i =1, 2,..., n-1 is called *path union* of G.

2. Sum Labelling for Split Graphs

In [6], Gerard *et al.* has proved that split graph of path, star are sum graph with sum number 1 and bi–star is sum graph with sum number 2.

Theorem 2.1

Path union of split graph of star $K_{1,n}$ is a sum graph with sum number 1.

Proof

Consider a star $K_{1,n}$ with (n+1) vertices. Let G be the split graph of star, Spl($K_{1,n}$). Let G* be the path union of m copies of G. Let v_1, v_{11}, v_{12}, ……, v_{1n}, v_2, v_{21}, v_{22}, ……,v_{2n}, ………, v_m, v_{m1}, v_{m2}, ……, v_{mn} be the vertices of m copies of the star $K_{1,n}$. Let u_1, u_{11}, u_{12}, ……u_{1n}, u_2, u_{21}, u_{22}, ……,u_{2n}, ………, u_m, u_{m1}, u_{m2}, ……, u_{mn} be the vertices corresponding to v_1, v_{11}, v_{12}, ……, v_{1n}, v_2, v_{21}, v_{22}, ……,v_{2n}, ………, v_m, v_{m1}, v_{m2}, ……, v_{mn} of m copies of the star $K_{1,n}$ which are added, to obtain the split graph of m copies of star. G* has $3nm$ vertices and $3nm + (m - 1)$ edges. Let x be the isolated vertex.

Define f: V (G*) → N

$$f(v_1) = 1 \qquad f(v_2) = 2$$
$$f(v_i) = f\big(v_{(i-1)}\big) + f\big(v_{(i-2)}\big)\ for\ 3 \le i \le m$$
$$f(v_{11}) = f(v_m) + f\big(v_{(m-1)}\big)$$
$$for\ 1 \le i \le m$$
$$\begin{cases} f\big(v_{ij}\big) = f\big(v_{i(j-1)}\big) + f(v_i) \quad for\ \ 2 \le j \le n \\ f(u_i) = f(v_{in}) + f(v_i) \\ f(u_{i1}) = f(u_i) + f(v_{i1}) \\ f\big(u_{ij}\big) = f\big(u_{i(j-1)}\big) + f(v_i) \quad for\ \ 2 \le j \le n \\ f\big(v_{(i+1)1}\big) = f(u_{in}) + f(v_i) \quad if\ \ i \ne m \end{cases}$$
$$f(x) = f(u_{mn}) + f(v_m)$$

Thus, Path union of Split graph of star $K_{1,n}$ is a sum graph with sum number 1.

Theorem 2.2

$K_{1,m} \odot Spl(P_n)$ is a sum graph with sum number 1.

Proof

Let c, v_1, v_2,……, v_m be the vertices of $K_{1,m}$ where c is the centre of the star. Let v_{11}, v_{12}, ……, v_{1n}, u_{11}, u_{12}, ……u_{1n}, v_{21}, v_{22}, ……,v_{2n}, u_{21}, u_{22}, ……,u_{2n},……, v_{m1}, v_{m2}, ……, v_{mn}, u_{m1}, u_{m2}, ……, u_{mn} be the vertices of the m copies of the split graph of path P_n. The vertices v_{11}, v_{21},……, v_{m1} are attached to the vertices v_1, v_2,……, v_m respectively. Let G = $K_{1,m} \odot Spl(P_n)$. Therefore the vertex set of G, V(G) = {c, v_{11}, v_{12}, ……, v_{1n}, u_{11}, u_{12}, ……u_{1n}, v_{21}, v_{22}, ……,v_{2n}, u_{21}, u_{22}, ……,u_{2n},……, v_{m1}, v_{m2}, ……, v_{mn}, u_{m1}, u_{m2}, ……, u_{mn}}. G has $2nm + 1$ vertices and $3n(n-1) + m$ edges. Let x be the isolated vertex.

Define f: V (G) → N

$$f(c) = 1 \qquad f(v_{11}) = 2$$

$$for\ 1 \le i \le m$$

$$\begin{cases} f(v_{i2}) = f(v_{i1}) + 1 \\ f(v_{ij}) = f(v_{i(j-1)}) + f(v_{i(j-2)}) \ for\ 3 \le j \le n \\ f(u_{i1}) = f(v_{i(n-1)}) + f(v_{in}) \\ f(u_{i2}) = f(u_{i1}) + 1 \\ f(u_{ij}) = f(u_{i(j-1)}) + f(v_{i(j-2)}) \ for\ 3 \le j \le n \\ f(v_{(i+1)1}) = f(u_{in}) + f(v_{i(n-1)}) \quad if\ i \ne m \end{cases}$$

$$f(x) = f(u_{mn}) + f(v_{m(n-1)})$$

Hence, $K_{1,m} \odot Spl(P_n)$ is a sum graph with sum number 1.

Theorem 2.3

$K_{1,m} \odot Spl(K_{1,n})$ is a sum graph with sum number 1.

Proof

Let c, $v_1, v_2, \ldots, v_m$ be the vertices of $K_{1,m}$ where c is the centre of the star. Let $c_1, v_{11}, v_{12}, \ldots, v_{1n}, u_1, u_{11}, u_{12}, \ldots u_{1n}, c_2, v_{21}, v_{22}, \ldots, v_{2n}, u_2, u_{21}, u_{22}, \ldots, u_{2n}, \ldots, c_m, v_{m1}, v_{m2}, \ldots, v_{mn}, u_m, u_{m1}, u_{m2}, \ldots, u_{mn}$ be the vertices of the m copies of the split graph of star $K_{1,n}$. The vertices $c_1, c_2, \ldots, c_m$ are attached to the vertices $v_1, v_2, \ldots, v_m$ of $K_{1,n}$ respectively. Let G = $K_{1,m} \odot Spl(K_{1,n})$. Therefore the vertex set of G, V(G) = {c, $c_1, v_{11}, v_{12}, \ldots, v_{1n}, u_1, u_{11}, u_{12}, \ldots u_{1n}, c_2, v_{21}, v_{22}, \ldots, v_{2n}, u_2, u_{21}, u_{22}, \ldots, u_{2n}, \ldots, c_m, v_{m1}, v_{m2}, \ldots, v_{mn}, u_m, u_{m1}, u_{m2}, \ldots, u_{mn}$ }. G has $2nm + 1$ vertices and $3n(n-1) + m$ edges. Let x be the isolated vertex.

Define f: V (G) → N

$$f(c) = 1 \qquad f(c_1) = 2$$

$$f(c_i) = f(c_{(i-1)}) + 1 \quad for\ 2 \le i \le m$$

$$f(v_{11}) = f(c_m) + 1$$

$$for\ 1 \le i \le m$$

$$\begin{cases} f(v_{ij}) = f(v_{i(j-1)}) + f(v_i) \ for\ 2 \le j \le n \\ f(u_i) = f(v_{in}) + f(c_i) \\ f(u_{i1}) = f(u_i) + f(v_{i1}) \\ f(u_{ij}) = f(u_{i(j-1)}) + f(c_i) \quad for\ 2 \le j \le n \\ f(v_{(i+1)1}) = f(u_{in}) + f(c_i) \quad if\ i \ne m \end{cases}$$

$$f(x) = f(u_{mn}) + f(c_m)$$

Thus, $K_{1,m} \odot Spl(K_{1,n})$ is a sum graph with sum number 1.

Illustration: Sum labelling for path union of split graph of star $K_{1,n}$ is given in figure 2.1

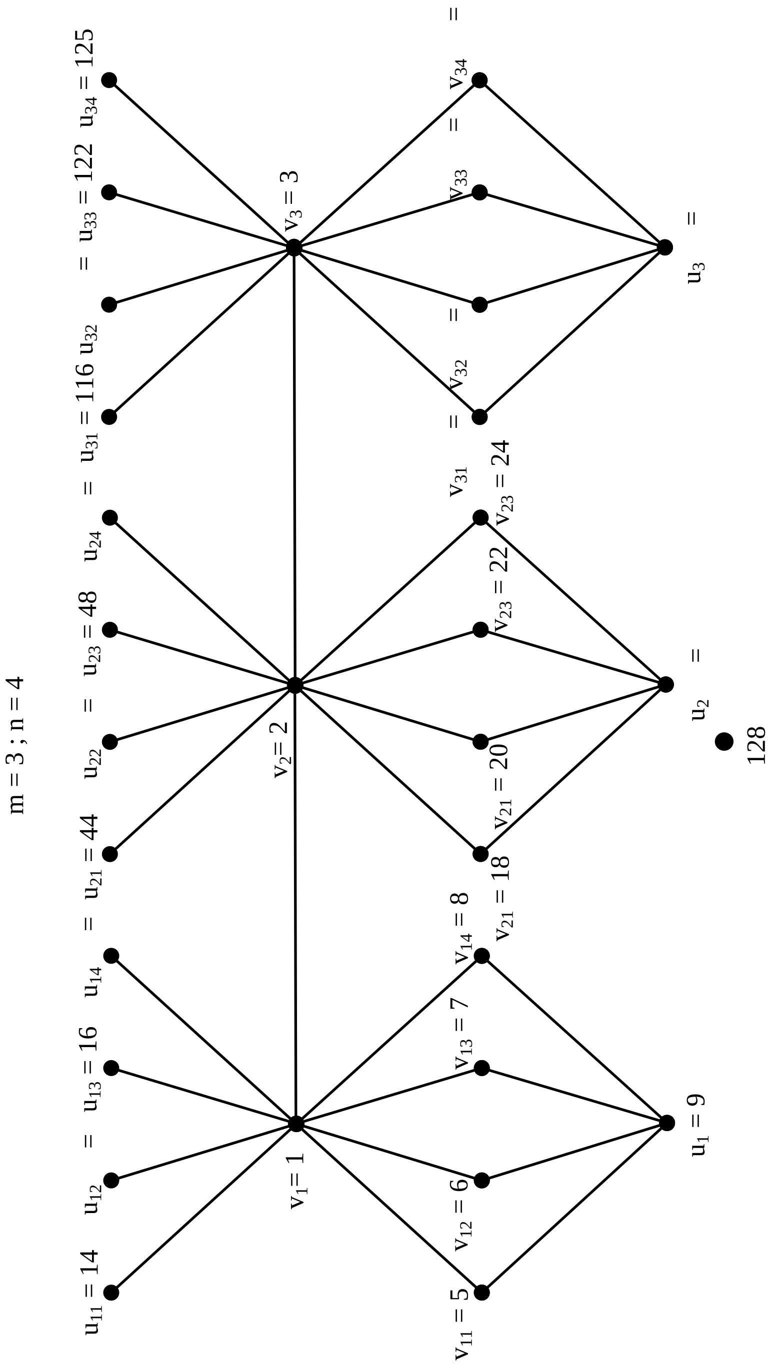

Fig. 2.1

$\boldsymbol{K_{1,m} \odot Spl(P_n)}$ $K_{1,m} \odot Spl(P_n)$ is obtained by attaching a copy of $Spl(P_n)$ to each pendent vertex of $K_{1,m}$.

Illustration *Sum labelling for* $K_{1,m} \odot Spl(P_n)$ *is given in figure 2.2*

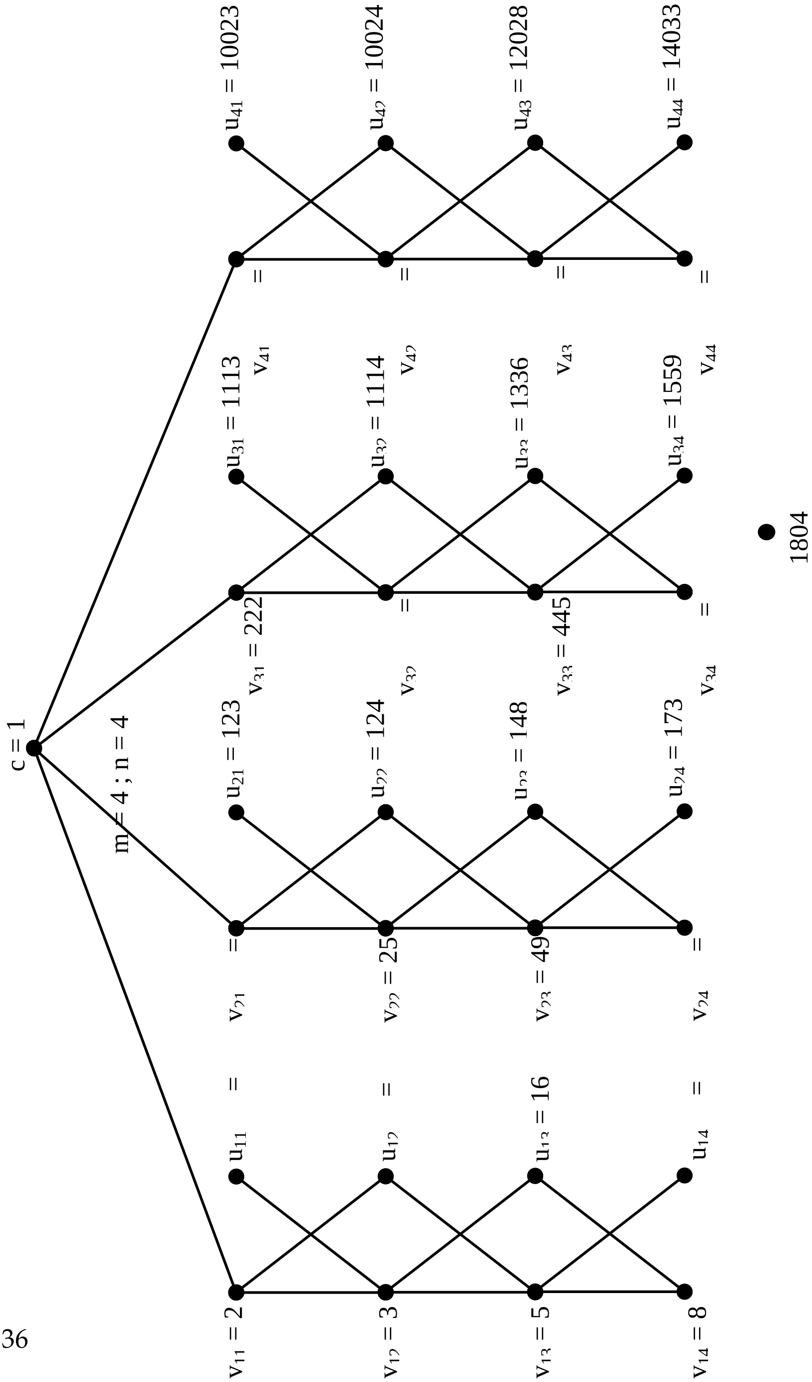

Fig 2.2

$\boldsymbol{K_{1,m} \odot Spl(K_{1,n})}$ $K_{1,m} \odot Spl(K_{1,n})$ is obtained by attaching a copy of $Spl(K_{1,n})$ to each pendent vertex of $K_{1,m}$.

Illustration *Sum labelling for* $K_{1,m} \odot Spl(K_{1,n})$ *is given in figure 2.3*

m = 3; n = 2

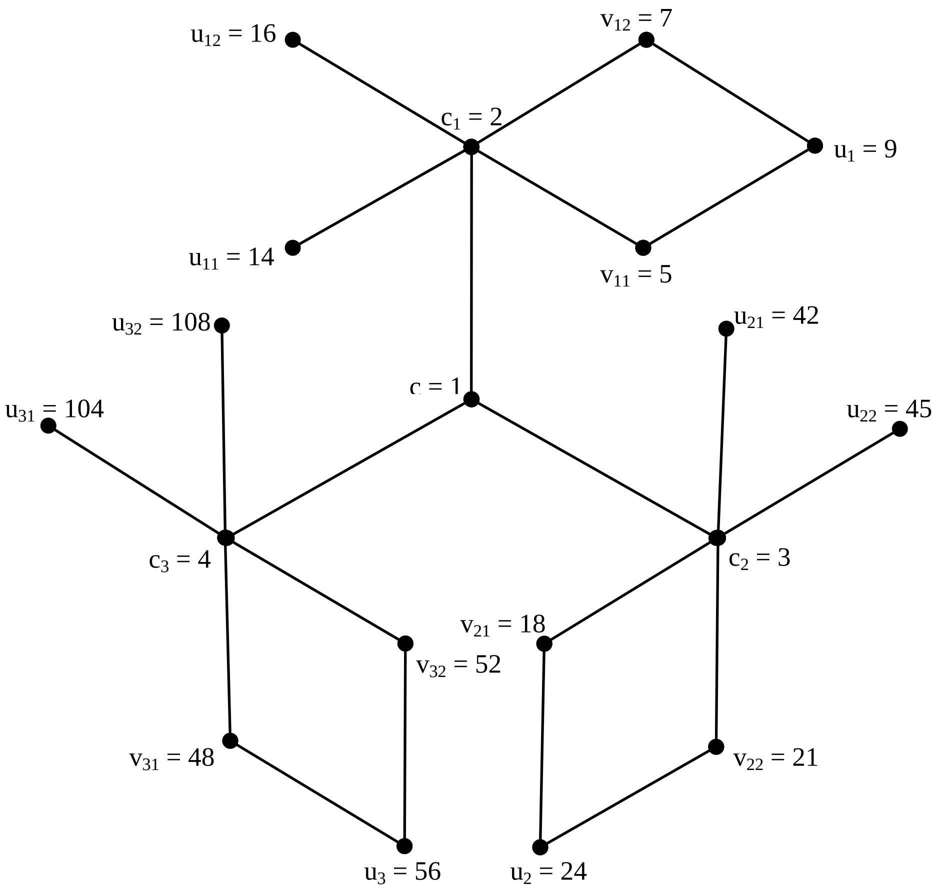

Fig 2.3

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