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On Estimates for Growth Rates of Unstable Azimuthal Disturbances in the Stability Problem of Swirling Flows with Radius-Dependent Density

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Abstract

Estimates for the growth rate of unstable two-dimensional disturbances to swirling flows with variable density are obtained and as a consequence it is proved that the growth rate tends to zero as the azimuthal wave number tends to infinity for two classes of basic flows.

Keywords: Hydrodynamic stability, Swirling flows, Inviscid flows, Variable density, Growth Rate.

1. Introduction

The stability of swirling flows has been studied extensively and for the vast literature on this problem one may be referred to the books Chandrasekhar [2], Chossat & Iooss [3] and Drazin & Reid [8]. For analytical studies on this problem one considers a basic flow with azimuthal and axial velocity components and general three-dimensional disturbances (see for example Howard and Gupta[12]). However the stability of basic flows with only an azimuthal velocity component to infinitesimal azimuthal disturbances has also been studied in many works (see for example

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[8]). In cylindrical polar coordinates (r, θ, z) the basic flow velocity is given by $(0, V(r), 0)$ and the basic flow density is $\rho_0(r)$ while the basic flow pressure $P_0(r)$ is calculated from the Euler equations. The angular velocity $\Omega(r) = \frac{V(r)}{r}$. The flow domain is the annular region between two infinite concentric cylinders with radii R_1 and R_2 where $0 < R_1 < R_2 < \infty$. If the disturbed flow is given by $(u, V + v, 0)$ and the disturbances are azimuthal disturbances, that is, disturbances are of the form $u = \hat{u}(r)e^{im(\theta - ct)}$ where m is an integer and it is called the azimuthal wave number and $c = c_r + ic_i$ is called the (complex) phase velocity. The boundary conditions satisfied by the disturbances is $\hat{u} = 0$ at $r = R_1, R_2$. Fung and Kurzweg[10] have found the linear stability equation for this problem and studied the instability of some specific basic flows. Then Fung[9] obtained some general analytical results for this problem. In particular Fung[9] has obtained a semicircular and semielliptical instability regions for basic flows satisfying the condition $ab(D\rho_0) \geq 0$ where, $a = min\Omega(r), b = max\Omega(r)$, and the differential operator D is defined by $D = \frac{d}{dr}$. Moreover, Fung[9] has defined the Richardson number J by $J = \frac{\Omega^2(D\rho_0)}{\rho_0 r(D\Omega)^2}$ and has found that a necessary condition for instability is that the minimum of the Richardson number is less than one quarter. Recently this problem has been studied in Dattu and Subbiah[5] where an improved instability region given by a generalized semiellipse theorem has been found for arbitrary angular velocity profiles. This improved instability region has also been used to find an estimate for the growth rate of unstable disturbances. Moreover, Dattu and Subbiah[5] have found an estimate for the growth rate of an arbitrary unstable disturbances given by

$$m^2 c_i^2 \leq \left(\frac{1}{4} - J_m\right)(D\Omega)^2{}_{max} R_2^2,$$

where, J_m is the minimum Richardson number and the subscript *max* stands for maximum over $[R_1, R_2]$.

The two-dimensional instability of the Rankine vortex with variable density has been studied recently in Dixit and Ramagovindarajan[7]. Since $D\Omega = 0$ in this case they have studied the instability with respect to the Atwood number rather than the

Richardson number. Also the instability of inviscid incompressible swirling flows with variable density with respect to two-dimensional disturbances has been studied asymptotically and numerically in Di Pierro and Abid[6]. For slowly varying velocity profiles Di Pierro and Abid [6] has studied the growth rate of an unstable disturbance in the limit $|m| \gg 1$ and found that their asymptotic result agrees with their numerical result. It may be noted here that the instability of swirling flows of homogeneous fluid has been studied in Le Dizes[13] with respect to azimuthal disturbances where a nonlinear critical layer analysis has been developed.

In the present paper we consider the stability problem of swirling flows with variable density of inviscid incompressible fluids confined between two infinite concentric cylinders at $r = R_1$and $r = R_2$ where $0 < R_1 < R_2 < \infty$ with respect to azimuthal disturbances. From Dattu and Subbiah[5]'s estimate for growth rate stated earlier it is found that the imaginary part of the complex phase speed c_i tends to zero as the wave number $m \to \infty$ for flows satisfying the condition of boundedness of $|D\Omega|$. In the present paper we have found a different estimate for the growth rate and from this estimate we prove that the growth rate $mc_i \to 0$ as the azimuthal wave number $m \to \infty$ for two classes of basic flows. The first class of flows consists of basic flow variables satisfying the condition of boundedness of $\lambda^2\ |D(\rho_0 Z)|^2$ where $Z = rD\Omega + 2\Omega$ is the vorticity of the basic flow, $\lambda^2 = \left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-\Omega_s)} - 1\right)^2 + 1$ and Ω_s is the value of Ω at the point where $D(\rho_0 Z)$ becomes zero. An example of a basic flow satisfying this condition is also given here. For homogeneous flows this result has been proved in Subbiah[16] and it may be remarked here that, in this case, a sharper result, namely, the existence of a critical wave number m_c such that $mc_i = 0$ whenever $m \geq m_c$ has been proved in Dattu and Subbiah[4] for a class of basic flows satisfying the condition of boundedness of the quantity $\frac{(DZ)r}{(\Omega-\Omega_s)}$. The second class of basic flows consists of flows with weak stratification, that is, flows satisfying the condition$\left|\Omega^2(D\rho_0)\right| \ll 1$. It may be remarked that the asymptotic results of Di Pierro and Abid[6] for the growth rate does not give this result of the growth rate tending to zero as the azimuthal wave number tending to infinity. But it should be noted that the basic flows considered in Di Pierro and Abid[6] are different from those considered in our analysis. In Di Pierro and Abid[6] the basic angular velocity is slowly varying while such a condition is not imposed in our analysis.

2. Eigenvalue Problem

The linear stability problem of inviscid incompressible but density stratified swirling flows between two infinite concentric co-axial cylinders at $r = R_1, R_2$ where $0 < R_1 < R_2 < \infty$ with respect to azimuthal disturbances of the form (function of r)$e^{im(\theta - ct)}$ is given by an eigen value problem consisting of the second order ordinary differential equation of Fung and Kurzweg[10]

$$\rho_0\left(D_*D - \frac{m^2}{r^2}\right)\phi + (D\rho_0)(D\phi) + \left\{\frac{\Omega^2(D\rho_0)}{r(\Omega-c)^2} - \frac{D(\rho_0 Z)}{r(\Omega-c)}\right\}\phi = 0, \qquad (2.1)$$

Where the operator D_* is defined to be $D_* = D + \frac{1}{r}$, and boundary conditions

$$\phi = 0 \text{ at } r = R_1, R_2. \qquad (2.2)$$

Here the function $\phi(r)$ is defined by $\phi(r) = \frac{iru(r)}{m}$ where $u(r)e^{im(\theta-ct)}$ is the axial disturbance velocity. In this equation the azimuthal wave number `m' appears as m^2 and so we can take $m > 0$ without loss of generality.

3. Estimate for Growth Rate of Unstable Disturbances

In this section we obtain an estimate for the growth rate of an arbitrary unstable disturbance and we deduce from this estimate that the growth rate of an unstable mode tends to zero as the azimuthal wave number tends to infinity for two classes of basic flows.

Theorem.3.1. If $r = r_s$ with $R_1 < r_s < R_2$ is the point at which $D(\rho_0 Z)$ vanishes and $\lambda^2|D(\rho_0 Z)|^2$ is bounded in $[R_1, R_2]$ then a necessary condition for the existence of non-trivial solution (ϕ, c, m^2) with $c_i > 0$ and $c_r = \Omega_s = \Omega(r_s)$ is that

$$m^2 c_i \le \frac{R_2}{\rho_{0min}} \max_{[R_1,R_2]} \{\lambda |D(\rho_0 Z)|\}$$

$$\text{where } \lambda = \left[\left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-\Omega_s)} - 1\right)^2 + 1\right]^{\frac{1}{2}}$$

Proof: The above equation (2.1) can be rewritten as

$$D_*(\rho_0 D\phi) - \frac{\rho_0 m^2 \phi}{r^2} + \left\{\frac{\Omega^2(D\rho_0)}{r(\Omega-\mathrm{c})^2} - \frac{D(\rho_0 Z)}{r(\Omega-c)}\right\}\phi = 0. \qquad (3.1)$$

Multiplying the above equation by $r\phi^*$(where * stands for complex conjugate) and integrating over (R_1, R_2) we have

$$\int_{R_1}^{R_2} D_*(\rho_0 D\phi) r\phi^*\, dr - \int_{R_1}^{R_2} \frac{\rho_0 m^2}{r} |\phi|^2\, dr + \int_{R_1}^{R_2} \left\{\frac{\Omega^2(D\rho_0)}{(\Omega-\mathrm{c})^2} - \frac{D(\rho_0 Z)}{(\Omega-c)}\right\} |\phi|^2 dr = 0 .$$

Using integration by parts formula and the boundary conditions (2.2) we have

$$\int_{R_1}^{R_2} \rho_0 \left(|D\phi|^2 + \frac{m^2}{r^2}|\phi|^2\right) r dr$$

$$+ \int_{R_1}^{R_2} \left\{\frac{D(\rho_0 Z)}{(\Omega-c)} - \frac{\Omega^2(D\rho_0)}{(\Omega-\mathrm{c})^2}\right\} |\phi|^2 dr = 0. \qquad (3.2)$$

Real part of the above equation gives

$$\int_{R_1}^{R_2} \rho_0 \left(|D\phi|^2 + \frac{m^2}{r^2}|\phi|^2\right) r dr + \int_{R_1}^{R_2} \left\{\frac{D(\rho_0 Z)(\Omega-c_r)}{|\Omega-c|^2} - \frac{\Omega^2(D\rho_0)\left((\Omega-c_r)^2 - c_i^2\right)}{|\Omega-c|^4}\right\} |\phi|^2 dr = 0. \qquad (3.3)$$

For $c_i > 0$ the imaginary part of the equation (3.2) gives

$$\int_{R_1}^{R_2} \left(\frac{D(\rho_0 Z)}{|\Omega-c|^2} - \frac{2\Omega^2(D\rho_0)(\Omega-c_r)}{|\Omega-c|^4}\right) |\phi|^2 dr = 0. \qquad (3.4)$$

Multiplying equation (3.1) by $\frac{r^3 D_*(\rho_0 D\phi^*)}{\rho_0}$ and integrating over (R_1, R_2) we have

$$\int_{R_1}^{R_2} \frac{r^3 |D_*(\rho_0 D\phi^*)|^2}{\rho_0}\, \mathrm{dr} - m^2 \int_{R_1}^{R_2} r\phi D_*(\rho_0 D\phi^*) dr +$$

$$\int_{R_1}^{R_2} \left\{\frac{\Omega^2\, (D\rho_0)}{r(\Omega-c)^2} - \frac{D(\rho_0 Z)}{r(\Omega-c)}\right\} \frac{r^3 \phi D_*(\rho_0 D\phi^*)}{\rho_0} dr = 0. \qquad (3.5)$$

Taking complex conjugate of both sides of equation (3.1) gives

$$D_*(\rho_0 D\phi^*) = \frac{\rho_0 m^2 \phi^*}{r^2} + \left\{\frac{D(\rho_0 Z)}{r(\Omega - c^*)} - \frac{\Omega^2 (D\rho_0)}{\mathrm{r}(\Omega - \mathrm{c}^*)^2}\right\}\phi^*. \tag{3.6}$$

Substituting (3.6) in (3.5), using integration by parts formula and the boundary conditions (2.2), we get the following integral relation:

$$\int_{R_1}^{R_2} \frac{r^3 |D_*(\rho_0 D\phi^*)|^2}{\rho_0}\,\mathrm{dr} + m^2 \int_{R_1}^{R_2} \rho_0 |D\phi|^2 r dr + m^2 \int_{R_1}^{R_2} \left\{\frac{\Omega^2\,(D\rho_0)}{r(\Omega - c)^2} - \frac{D(\rho_0 Z)}{r(\Omega - c)}\right\} |\phi|^2 r dr - \int_{R_1}^{R_2} \left|\frac{\Omega^2\,(D\rho_0)}{r(\Omega - c)^2} - \frac{D(\rho_0 Z)}{r(\Omega - c)}\right|^2 \frac{|\phi|^2 r^3}{\rho_0} dr = 0. \tag{3.7}$$

Real part of the above equation (3.7) gives

$$\int_{R_1}^{R_2} \frac{r^3 |D_*(\rho_0 D\phi^*)|^2}{\rho_0}\,\mathrm{dr} + m^2 \int_{R_1}^{R_2} \rho_0 |D\phi|^2 r dr +$$

$$m^2 \int_{R_1}^{R_2} \left\{\frac{\Omega^2\,(D\rho_0)\left((\Omega - c_r)^2 - c_i^2\right)}{r|\Omega - c|^4} - \frac{D(\rho_0 Z)(\Omega - c_r)}{r|\Omega - c|^2}\right\} |\phi|^2 r dr - \int_{R_1}^{R_2} \left|\frac{\Omega^2\,(D\rho_0)}{r(\Omega - c)^2} - \frac{D(\rho_0 Z)}{r(\Omega - c)}\right|^2 \frac{|\phi|^2 r^3}{\rho_0} dr = 0, \tag{3.8}$$

while for $c_i > 0$,the imaginary part of equation (3.7) gives

$$\int_{R_1}^{R_2} \left(\frac{2\Omega^2 (D\rho_0)(\Omega - c_r)}{|\Omega - c|^4} - \frac{D(\rho_0 Z)}{|\Omega - c|^2}\right) m^2 |\phi|^2 dr = 0. \tag{3.9}$$

Multiplying equation (3.3) by m^2 and adding the resultant equation to equation (3.8), we have the relation

$$\int_{R_1}^{R_2} \frac{r^3 |D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2 \int_{R_1}^{R_2} \rho_0 |D\phi|^2 r dr + m^4 \int_{R_1}^{R_2} \frac{\rho_0 |\phi|^2}{r} dr$$

$$- \int_{R_1}^{R_2} \left|\frac{\Omega^2 (D\rho_0)}{r(\Omega - c)^2} - \frac{D(\rho_0 Z)}{r(\Omega - c)}\right|^2 \frac{|\phi|^2 r^3}{\rho_0} dr = 0. \tag{3.10}$$

This is rewritten as

$$\int_{R_1}^{R_2} \frac{r^3 |D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2 \int_{R_1}^{R_2} \rho_0 |D\phi|^2 r dr + m^4 \int_{R_1}^{R_2} \frac{\rho_0 |\phi|^2}{r} dr$$

$$- \int_{R_1}^{R_2} \frac{\left(D(\rho_0 Z)\right)^2}{|\Omega - c|^2} \left|\frac{\Omega^2 (D\rho_0)}{D(\rho_0 Z)(\Omega - c)} - 1\right|^2 \frac{|\phi|^2 r}{\rho_0} dr = 0.$$

Since $|\Omega - c|^2 \geq c_i^2$, the above equation gives,

$$\int_{R_1}^{R_2} \frac{r^3 |D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2 \int_{R_1}^{R_2} \rho_0 |D\phi|^2 r dr + m^4 \int_{R_1}^{R_2} \frac{\rho_0 |\phi|^2}{r} dr$$

$$- \int_{R_1}^{R_2} \frac{\left(D(\rho_0 Z)\right)^2}{c_i^2} \left|\frac{\Omega^2 (D\rho_0)}{D(\rho_0 Z)(\Omega - c)} - 1\right|^2 \frac{|\phi|^2 r}{\rho_0} dr \leq 0;$$

i.e, $\int_{R_1}^{R_2} \frac{r^3|D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2\int_{R_1}^{R_2} \rho_0|D\phi|^2 r dr + m^4\int_{R_1}^{R_2} \frac{\rho_0|\phi|^2}{r} dr$

$$-\int_{R_1}^{R_2} \frac{(D(\rho_0 Z))^2}{c_i^2|\Omega-c|^2}\left\{\frac{\left(\Omega^2(D\rho_0)\right)^2}{\left(D(\rho_0 Z)\right)^2} + (\Omega-c_r)^2 + c_i^2 - \frac{2\Omega^2(D\rho_0)(\Omega-c_r)}{D(\rho_0 Z)}\right\}\frac{|\phi|^2 r}{\rho_0} dr \leq 0.$$

This can be rewritten as

$$\int_{R_1}^{R_2} \frac{r^3|D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2\int_{R_1}^{R_2} \rho_0|D\phi|^2 r dr + m^4\int_{R_1}^{R_2} \frac{\rho_0|\phi|^2}{r} dr$$

$$-\int_{R_1}^{R_2} \frac{(D(\rho_0 Z))^2}{c_i^2|\Omega-c|^2}\left\{(\Omega-c_r)^2\left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-c_r)} - 1\right)^2 + c_i^2\right\}\frac{|\phi|^2 r}{\rho_0} dr \leq 0.$$

Using the facts that $(\Omega-c_r)^2 \leq |\Omega-c|^2$ and $c_i^2 \leq |\Omega-c|^2$ in the above inequality we have,

$$\int_{R_1}^{R_2} \frac{r^3|D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2\int_{R_1}^{R_2} \rho_0|D\phi|^2 r dr + m^4\int_{R_1}^{R_2} \frac{\rho_0|\phi|^2}{r} dr$$

$$-\int_{R_1}^{R_2} \frac{(D(\rho_0 Z))^2}{c_i^2}\left\{\left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-c_r)} - 1\right)^2 + 1\right\}\frac{|\phi|^2 r}{\rho_0} dr \leq 0.$$

But, since $c_r = \Omega_s$ and $\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-\Omega_s)}$ is bounded in $[R_1, R_2]$ by hypothesis of the theorem, we have from the above inequality

$$\int_{R_1}^{R_2}\left(\frac{r^2|D_*(\rho_0 D\phi)|^2}{\rho_0} + 2m^2\rho_0(|D\phi|^2)\right) r dr + \int_{R_1}^{R_2} \frac{\left(\frac{m^4}{r^2}\rho_0^2 c_i^2 - \lambda^2(D(\rho_0 Z))^2\right)}{c_i^2}\frac{|\phi|^2}{\rho_0} r\, dr \leq 0, \qquad (3.11)$$

where
$$\lambda^2 = \left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-\Omega_s)} - 1\right)^2 + 1. \qquad (3.12)$$

From inequality (3.11) it follows that

$$\frac{m^4\rho_0^2 c_i^2}{r^2} \leq \lambda^2\left(D(\rho_0 Z)\right)^2,$$

atleast once in the flow domain and as a consequence we have

$$\frac{m^4\rho_{0min}^2 c_i^2}{R_2^2} \leq \left(\max_{[R_1,R_2]} \lambda|D(\rho_0 Z)|\right)^2$$

i.e, $$m^2 c_i \le \frac{R_2}{\rho_{0min}} \max_{[R_1,R_2]}\{\lambda |D(\rho_0 Z)|\} \tag{3.13}$$

So the theorem is proved.

As a corollary to the above theorem we have the following result.

Theorem.3.2. For basic flows satisfying the condition of boundedness of the quantity $\lambda^2 |D(\rho_0 Z)|^2$, we have

$$\lim_{m\to\infty} mc_i = 0. \tag{3.14}$$

Now we shall present an example of basic flow that satisfies the condition of Theorem 3.1.

Example. The angular velocity profile considered here is a particular case of the Shukhman[15] profile; namely, the one with

$$\Omega(r) = \tanh(\frac{\log r}{d}) \tag{3.15}$$

where $0 < d < 1$ is a constant and the density profile considered here is given by

$$\rho_0(r) = e^{-\tanh^3(\frac{\log(r-r_s+1)}{d})} \tag{3.16}$$

where r_s is the point where $D(\rho_0 Z)$ is zero.

We shall check that $\lambda^2\left(D(\rho_0 Z)\right)^2$ is bounded for this flow.

It is trivial to see that $D\Omega = \frac{1}{rd}\operatorname{sech}^2\left(\frac{\log r}{d}\right)$,

and that $Z = rD\Omega + 2\Omega = \frac{\operatorname{sech}^2\left(\frac{\log r}{d}\right)}{d} + 2\tanh\left(\frac{\log r}{d}\right)$.

It is seen that $D(\rho_0 Z) = 0$ at $r = r_s$ where $d - \tanh\left(\frac{\log r}{d}\right) = 0$, i.e,$\exists\, x^*$ such that $\tanh(x^*) = d$ where $x^* = \frac{\log r_s}{d}$, that is $r_s = e^{x^* d}$.

To check that $\lambda^2\left(D(\rho_0 Z)\right)^2$ is bounded, it is seen that

$$\lambda^2\left(D(\rho_0 Z)\right)^2 = \{\left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-\Omega_s)} - 1\right)^2 + 1\}\left(D(\rho_0 Z)\right)^2$$

$$= \{2 + \frac{\Omega^4 (D\rho_0)^2}{\left(D(\rho_0 Z)\right)^2(\Omega-\Omega_s)^2} - \frac{2\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-\Omega_s)}\}\left(D(\rho_0 Z)\right)^2$$

$$i.e,\ \lambda^2\left(D(\rho_0 Z)\right)^2 = \frac{2\left(D(\rho_0 Z)\right)^4(\Omega-\Omega_s)^2 + \Omega^4(D\rho_0)^2 - 2\Omega^2(D\rho_0)D(\rho_0 Z)(\Omega-\Omega_s)}{(\Omega-\Omega_s)^2} \tag{3.17}$$

The numerator is bounded in $[R_1, R_2]$ by continuity. Unboundedness will come only if the denominator is zero. It is seen that the denominator is zero of order 2 at $r = r_s$ and nowhere else. But it is found that the numerator is also zero at $r = r_s$ of order two or more. Therefore $\lambda^2\left(D(\rho_0 Z)\right)^2$ is bounded for this example of basic flow and so this example of basic flow satisfies the conditions of Theorem 3.1.

In the above theorems the density stratification is arbitrary but the angular velocity $\Omega(r)$ and the density $\rho_0(r)$ together satisfies the conditions of Theorem 3.1. However, if we consider basic flows with weak density stratification then we can prove the result of theorem 3.2. for arbitrary angular velocity profiles.

Theorem.3.3. If (ϕ, c, m^2) is a non-trivial solution of equation (2.1) and (2.2) with $c_i > 0$ and $|\Omega^2(D\rho_0)| \ll 1$ then $mc_i \to 0$ as $m \to \infty$.

Proof: Now consider,

$$\left|\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-c)} - 1\right|^2 = \left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-c)} - 1\right)\left(\frac{\Omega^2(D\rho_0)}{D(\rho_0 Z)(\Omega-c^*)} - 1\right)$$

$$= \frac{\left(\Omega^2(D\rho_0)\right)^2}{\left(D(\rho_0 Z)\right)^2|\Omega-c|^2} + 1 - \frac{2\Omega^2(D\rho_0)(\Omega-c_r)}{D(\rho_0 Z)|\Omega-c|^2}. \quad (3.18)$$

Under the weak stratification condition the first term on the right hand side is neglected and using the resultant relation in equation (3.8) we have the following integral relation:

$$\int_{R_1}^{R_2} \frac{r^3|D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2\int_{R_1}^{R_2} \rho_0|D\phi|^2 r dr + m^4\int_{R_1}^{R_2} \frac{\rho_0|\phi|^2}{r} dr$$

$$+\int_{R_1}^{R_2} \frac{2\Omega^2 D(\rho_0 Z)(\Omega-c_r)(D\rho_0)}{\rho_0|\Omega-c|^4}|\phi|^2 r dr - \int_{R_1}^{R_2} \frac{\left(D(\rho_0 Z)\right)^2}{\rho_0|\Omega-c|^2}|\phi|^2 r\, dr = 0. \quad (3.19)$$

Since $|\Omega - c|^2 \geq c_i^2$ the above equation becomes,

$$\int_{R_1}^{R_2} \frac{r^3|D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2\int_{R_1}^{R_2} \rho_0|D\phi|^2 r dr + m^4\int_{R_1}^{R_2} \frac{\rho_0|\phi|^2}{r} dr +$$

$$\int_{R_1}^{R_2} \frac{2\Omega^2 D(\rho_0 Z)(\Omega-c_r)c_i(D\rho_0)}{\rho_0|\Omega-c|^4 c_i}|\phi|^2 r dr - \int_{R_1}^{R_2} \frac{\left(D(\rho_0 Z)\right)^2}{\rho_0 c_i^2}|\phi|^2 r\, dr \leq 0. \quad (3.20)$$

Furthermore, since $(\Omega - c_r)^2 + c_i^2 \geq 2(\Omega - c_r)c_i$ and $D(\rho_0 Z) \geq -|D(\rho_0 Z)|$ we derive from equation (3.20), that,

$$\int_{R_1}^{R_2} \frac{r^3|D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2\int_{R_1}^{R_2} \rho_0|D\phi|^2 r dr + m^4\int_{R_1}^{R_2} \frac{\rho_0|\phi|^2}{r}dr$$

$$-\int_{R_1}^{R_2} \frac{\Omega^2|D(\rho_0 Z)|(D\rho_0)}{\rho_0|\Omega-c|^2 c_i}|\phi|^2 r dr - \int_{R_1}^{R_2} \frac{\left(D(\rho_0 Z)\right)^2}{\rho_0 c_i^2}|\phi|^2 r\,dr \le 0. \quad (3.21)$$

Using the fact that $|\Omega - c|^2 \ge c_i^2$ the above inequality can be rewritten as

$$\int_{R_1}^{R_2} \frac{r^3|D_*(\rho_0 D\phi)|^2}{\rho_0}\,\mathrm{dr} + 2m^2\int_{R_1}^{R_2} \rho_0|D\phi|^2 r dr +$$

$$\int_{R_1}^{R_2} \left\{\frac{m^2\rho_0^2}{r^2} - \frac{\left[\Omega^2|D(\rho_0 Z)|(D\rho_0)\right]}{c_i^3} - \frac{\left(D(\rho_0 Z)\right)^2 c_i}{c_i^3}\right\}\frac{|\phi|^2}{\rho_0} r dr \le 0. \quad (3.22)$$

The first two terms are non negative and so the integrand of the third term should be negative and since $c_i \le \frac{(b-a)}{2}$ by the semi circle theorem of Fung[9] we have the following inequality:

$$\frac{m^4\rho_{0min}^2}{R_2^2} - \frac{\left[\Omega^2|\mathrm{D}(\rho_0\mathrm{Z})|(\mathrm{D}\rho_0)\right]_{\max}}{c_i^3} - \frac{\left[\left(\mathrm{D}(\rho_0\mathrm{Z})\right)^2\right]_{\max}\frac{(\mathrm{b}-\mathrm{a})}{2}}{c_i^3} \le 0;$$

i.e, $$\frac{\rho_{0min}^2\, m^3 c_i^3}{R_2^2} \le \frac{\left[\Omega^2|\mathrm{D}(\rho_0\mathrm{Z})|(\mathrm{D}\rho_0)\right]_{\max}}{m} + \frac{\left[\left(\mathrm{D}(\rho_0\mathrm{Z})\right)^2\right]_{\max}\frac{(\mathrm{b}-\mathrm{a})}{2}}{m};$$

i.e, $$m^3 c_i^3 \le \left[\frac{\left[\Omega^2|\mathrm{D}(\rho_0\mathrm{Z})|(\mathrm{D}\rho_0)\right]_{\max}}{m} + \frac{\left[\left(\mathrm{D}(\rho_0\mathrm{Z})\right)^2\right]_{\max}\frac{(\mathrm{b}-\mathrm{a})}{2}}{m}\right]\frac{R_2^2}{\rho_{0min}^2}; \quad (3.23)$$

This implies that $mc_i \to 0$ as $m \to \infty$,and the theorem is proved.

Remark.3.4. In the context of the stability problem of density stratified inviscid shear flows it was first conjectured by Howard[11] that the growth rate of an unstable disturbance should tend to zero as the wave number tends to infinity. This Howard's conjecture has been proved for a class of basic flows called Garcia type flows in Banerjee et al[1] and for shear flows with weak density stratification in Shandil and Jagjit Singh[14]. So we may regard our theorems 3.2 and 3.3 as statement and proof of Howard's conjecture for two classes of swirling flows with variable density.

Concluding Remarks

In this paper we have considered the linear stability of inviscid incompressible but density stratified swirling flows to two-dimensional disturbances. We have found estimates for growth rate of unstable disturbances and as a consequence we have proved that the growth rate tends to zero as the azimuthal wave number tends infinity for two classes of basic flows. From the asymptotic result of Di Pierro and Abid[6] for the growth rate one cannot conclude that the growth rate tends to zero as the azimuthal wave number tends to infinity, but it should be noted here that the basic flow considered in their analysis should satisfy the condition of slow variation of the angular velocity and this is not supposed in our analysis.

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