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Variable Viscosity Effects on Penetrative Convection in a Fluid Layer

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Abstract

The effect of variable viscosity on the onset of penetrative convection simulated via internal heating in a fluid layer. The upper surface of a fluid layer is assumed to be deformably free and dependence of viscosity is assumed to be exponential. The resulting eigen value problem is solved using a regular perturbation technique with wave number a as a perturbation parameter. The viscosity parameter, surface deformation and the presence of internal heat source play a decisive role on the stability characteristics of the system.

Keywords: Variable viscosity: Internal heat source

Nomenclature

a	horizontal wave number, $\sqrt{l^2+m^2}$
A	ratio of heat capacities
D	differential operator d/dz
d	thickness of the fluid layer
$\vec{g}$	acceleration due to gravity
k	permeability
l, m	wave number in x and y-directions respectively
M	Marangoni number $\sigma_T(T_0 - T_u)d/\mu\kappa$

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R	Rayleigh number in the fluid layer $\alpha g\left(T_0 - T_u\right)d^3/\nu\kappa$
Pr	Prandtl number for fluid layer, ν/κ
Ns	heat source strength $q\,d^2/2\kappa\left(T_0 - T_u\right)$
p	pressure
T	temperature
T_0	temperature at the interface
$\vec{V}$	velocity vector (u, v, w)
W	amplitude of perturbed vertical velocity

Greek symbols

∇^2	Laplacian operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$
κ	thermal diffusivity
Θ	amplitude of perturbed temperature
μ	fluid viscosity
ρ_0	fluid density
σ	temperature dependent surface tension
ν	kinematic viscosity μ/ρ_0

1. Introduction

The mechanism of internal heating in a flowing fluid is relevant to the thermal processing of liquid foods through ohmic heating, where the internal heat generation serves for the pasteurization/sterilization of the food [1]. Other important applications of flows with internal heat generation are relative to nuclear reactors, as well as to the geophysics of the earth's mantle. In both cases, the internal heating is due to the radioactive decay. For nuclear reactors, processes of natural convection with internal heating are extremely important in the analysis of severe accident conditions. As pointed out by [2], flows with volumetric heating are relevant for the physics of the atmosphere, in connection with the absorption of solar radiation. Due to the wide range of industrial and geophysical applications, extensive literature has been recently produced on this subject; see e.g. (Carr 2004, Carr and

Putter 2003, Hill 2004, Straughan 2008, Straughan and Walker 1996, Tse and Chasnov 1998, and Zhang and Schubert 2002).

In this paper, the stationary Benard-Marangoni instability in a variable viscosity fluid layer with internal heat generation will be studied using linear stability analysis. The upper surface of a fluidlayer is assumed to be deformably free and boundaries are considered to be insulated to temperature perturbations. A regular perturbation technique with wave number a as a perturbation parameter is used to solve the eigen value problem in a closed form. The influences of temperature-dependent viscosity and internal heating on the stability limit will be analyzed by developing explicit solution.

2. Mathematical Formulation

We consider penetrative convection via internal heating in a system consisting of an infinite horizontal fluid layer of thickness d and the z-axis pointing vertically upwards opposing the direction of gravity. The temperatures of the lower and upper boundaries are taken to be uniform and equal to T_l and T_u respectively, with $T_l > T_u$. The upper free surface of fluid layer is free of deformities with its position being $z = d + \Omega(x, y, t)$.

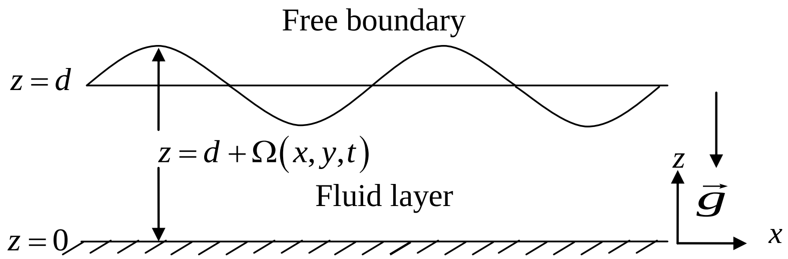

Fig. 1 Physical configuration

The governing equations for the fluid layer are:

$$\nabla\cdot\vec{V}=0 \tag{1}$$

$$\rho_0\left(\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\nabla\right)\vec{V}\right)=-\nabla p+\rho_0\vec{g}\left[1-\alpha\left(T-T_0\right)\right]+2\nabla.\left[\mu\left(\nabla.\vec{V}+\nabla.\vec{V}^T\right)\right] \tag{2}$$

$$\frac{\partial T}{\partial t}+\left(\vec{V}\cdot\nabla\right)T=k\nabla^2T+q\,. \tag{3}$$

where

$$\mu=\mu_0\,\exp\left[-A\left(T-T_0\right)\right] \tag{4}$$

and μ_0 is the dynamic viscosity corresponding to a temperature equal to the mean of temperature at the boundaries.

In the above equations, $\vec{V}=(u,v,w)$ is the velocity vector, p is the pressure, T is the temperature, q is the heat source in the fluid layer, κ is the thermal diffusivity, α is the thermal expansion coefficient and ρ_0 is the reference fluid density.

The basic state is quiescent and is of the form

$$\left(u,v,w,p,T\right)=\left[0,0,W_0,p_b\left(z\right),T_b\left(z\right)\right] \tag{5}$$

The basic steady state is assumed to be quiescent and temperature distributions are found to be

$$T_b\left(z\right)=T_0-\left[\left(\frac{\left(T_0-T_u\right)}{d}-\frac{q\,d}{2\kappa}\right)z+\frac{q}{2\kappa}z^2\right] \tag{6}$$

Where T_0 is the interface temperature. In order to investigate the stability of the basic solution, infinitesimal disturbances are introduced in the form

$$\vec{V}=\vec{V}',\quad T=T_b+T',\quad p=p_b+p',\quad \rho=\rho_b+\rho',\quad \mu=\mu_b+\mu' \tag{7}$$

where the primed quantities are the perturbations and assumed to be small. Eq.(7)is substituted in Eqs. (1)-(3) and linearized in the usual manner. The pressure term is eliminated from Eq. (2) by taking curl twice on these two equations and only the vertical component is retained. The variables are then nondimensionalized using $d, d^2/\kappa, \kappa/d$ and $T_0 - T_u$ as the units of length, time, velocity, and temperature in the fluid layer and the non-dimensional disturbance equations are now given by

$$\frac{1}{pr}\frac{\partial}{\partial t}\nabla^2 w = \tilde{f}\,\nabla^4 w + 2\frac{\partial \tilde{f}}{\partial z}\nabla^2\frac{\partial w}{\partial z} + \frac{\partial^2 \tilde{f}}{\partial z^2}\left(\nabla^2 w - 2\nabla_h^2 w\right) + R\nabla_h^2 T \tag{8}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T = w\left[1 - Ns(1-2z)\right] \tag{9}$$

where $R = \alpha g\left(T_0 - T_u\right)d^3/\nu\kappa$ is the Rayleigh number, $Ns = q\,d^2/2\kappa\left(T_0 - T_u\right)$ is the dimensionless heat source strength and $\nabla^2 = \nabla_h^2 + \partial^2/\partial z^2$ is the Laplacian operator with $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The function $\tilde{f}$ representing the temperature dependence of viscosity, is defined as

$$\tilde{f} = \exp\left[B\left(z - \frac{1}{2}\right)\right], \quad B = \left(\frac{\nu_{\max}}{\nu_{\min}}\right). \tag{10}$$

The appropriate boundary conditions are

$$\frac{\partial \Omega}{\partial t} = w, \frac{\partial T}{\partial z} + Bi\left[T - \left(Ns + 1\right)\Omega\right] = 0 \qquad at \quad z = 1 \tag{11}$$

$$\tilde{f}\left(\frac{\partial^2}{\partial z^2} - \nabla_h^2\right)w = M\,\nabla_h^2\left[T - \left(1 + Ns\right)\Omega\right] \qquad at \quad z = 1 \tag{12}$$

$$\tilde{f}\,Cr\left[\frac{1}{\Pr}\frac{\partial}{\partial t} + \left(\frac{\partial^2}{\partial z^2} + 3\nabla_h^2\right)\right]\frac{\partial w}{\partial z} + \left(B_0 - \nabla_h^2\right)\nabla_h^2\Omega = 0 \qquad at \quad z = 1 \tag{13}$$

Since the principle of exchange instabilities holds for holds good even for the present configuration as well. Hence, the time derivatives will be dropped conveniently from Eqs. (8)and(9).Then performing a normal mode expansion of the dependent variables as

$$\left(w,T\right)=\left[W\left(z\right),\Theta\left(z\right)\right]\exp\left[i\left(lx+my\right)\right] \tag{14}$$

and substituting them in Eqs. (8) and (9) (with $\partial/\partial t=0$), we obtain the following ordinary differential equations

$$\tilde{f}\left(D^2-a^2\right)^2 W+2D\tilde{f}\left(D^2-a^2\right)DW+D^2\tilde{f}\left(D^2+a^2\right)=Ra^2\Theta \tag{15}$$

$$\left(D^2-a^2\right)\ \Theta=-W\left[1-Ns\left(1-2z\right)\right] \tag{16}$$

The linearized boundary conditions are:

$$W=D\Theta+Bi\left[\Theta-\left(1+Ns\right)Z\right]=0 \qquad at \quad z=1 \tag{17}$$

$$\tilde{f}\left(D^2+a^2\right)W+Ma^2\left[\Theta-\left(1+Ns\right)Z\right]=0 \qquad \text{at } z=1 \tag{18}$$

$$\tilde{f}\,Cr\left(D^2-3a^2\right)DW=\left(B_0+a^2\right)a^2 Z \qquad \text{at } z=1 \tag{19}$$

$$W=0, \qquad DW=0 \text{ and } D\Theta=0 \qquad \text{at } z=0. \tag{20}$$

3. Method of Solution

Since the critical wave number is exceedingly small for the assumed temperature boundary conditions (Nield and Bejan 2006) the eigen value problem is solved using a regular perturbation technique with wave number a as a perturbation parameter. Accordingly, the dependent variables are expanded in powers of a^2 in the form

$$\left(W,\Theta\right)=\sum_{i=0}^{N}\left(a^2\right)^i\left(W_i,\Theta_i\right) \tag{21}$$

Substitution of Eq. $\left(21\right)$ into Eqs. $\left(15\right)-\left(16\right)$ and the boundary conditions $\left(17\right)-\left(20\right)$

$$\tilde{f}\,D^4W_0+2D\tilde{f}\,D^3W_0+D^2\tilde{f}\,D^2W_0=0 \tag{22}$$

$$D^2\theta_0=-f(z)W_0 \tag{23}$$

where

$$f(z)=\left[1-Ns\left(1-2z\right)\right] \tag{24}$$

The boundary conditions (17)-(20) become

$$W_0=DW_0=D\Theta_0 \;=0 \quad at \;\; z=0 \tag{25}$$

$$\tilde{f}(1)D^2W_0=D\Theta_0 \;=0 \quad at \;\; z=1 \tag{26}$$

Then solutions to above equations are

$$W_0=0 \quad \text{and} \quad \Theta_0=1 \tag{27}$$

First- order equations $(15)-(16)$ become

$$D^4W_1+2B\;D^3W_1+B^2\;D^2W_1=R\,Exp[-B\left(z-1/2\right)] \tag{28}$$

$$D^2\Theta_1=1-f(z)W_1. \tag{29}$$

The boundary conditions $(17)-(20)$ become

$$W_1=DW_1 \;=0 \quad at \;\; z=0 \tag{30}$$

$$\tilde{f}(1)D^2W_0+Ma^2\left[\Theta-\left(1+Ns\right)Z\right]=0 \quad at \;\; z=1 \tag{31}$$

$$\tilde{f}(1)D^3W_1-\frac{B_0}{Cr}Z_0=0 \;\; at \;\; z=1 \tag{32}$$

The general solution of (28) is

$$W_1=R\left[C_1+C_2z+C_3e^{-Bz}+C_4\,ze^{-Bz}+\frac{z^2}{2B^2}Exp[-B\left(z-1/2\right)]\right] \tag{33}$$

Where C_1,C_2,C_3 and C_4 are constants and they have to determined using the appropriate boundary conditions.

$$C_1=-\frac{e^{-B/2}}{2B^2}\left(\frac{e^{-B}+B-7}{1-2e^{-B}+e^{-2B}-B^2e^{-B}}\right),$$

$$C_2=\frac{e^{-B/2}}{-2B^2}\left(\frac{2-2e^{B}+B+Be^{B}}{e^{2B}-B^2e^{B}-1}\right)$$

$$C_3 = \frac{e^{B/2}}{2B^2}\left(\frac{e^B - B - 1}{1 + e^{2B} - 2e^B - B^2 e^B}\right),$$
$$C_4 = \frac{e^{B/2}}{2B^2}\left(\frac{2 - 2e^B + 2B + B^2}{2e^B - e^{2B} + B^2 e^B - 1}\right).$$

The differential Equation (29) involving $D^2\Theta_1$ provide the solvability requirement which is given by

$$\int_0^1 f(z) W_1 \, dz = 1 \tag{34}$$

The expressions for W_1 is back substituted into Eq. (34) and integrated to yield an relation for the critical Rayleigh number and Marangoni number is obtained

$$\frac{R}{\Delta_1 + \Delta_2} + \frac{M}{\Delta_3 + \Delta_4} = 1 \tag{35}$$

where

$$\Delta_1 = -\frac{e^{-B/2}}{2B^2}\left(\frac{e^{-B} - B - 1}{1 - 2e^{-B} + e^{-2B} - B^2 e^{-B}}\right) + \frac{e^B}{2B^2}\left(\frac{2 - 2e^{-B} + B + Be^B}{e^{-2B} + 2e^{-B} - B^2 e^{-B} - 1}\right),$$

$$\Delta_2 = \frac{e^{-B} - B - 1}{(\mathrm{B_0}\text{-Cr M}(1+Ns)) + (2\,\mathrm{B_0} - 3\,\mathrm{CrM}(1+Ns) + 6\mathrm{B_0}}$$

$$\Delta_3 = \frac{1}{2B^3}\left[(2B - 2)e^{B/2} + 2e^{-B/2} + Ns\left((2 - 2B)e^{-B/2} - 2e^{B/2}\right)\right].$$

$$\Delta_4 = \frac{2 - e^{B/2}B + B^2}{e^{2B} + B^2 e^{-B} - 1} + \frac{e^{-B} - B - 1}{2\,\mathrm{B_0} - 3\,\mathrm{CrM}(1+Ns) + 6\mathrm{B_0}}$$

In the limit absence of internal heating (i.e., $Ns \to 0$) and constant viscosity (i.e., $B \to 0$),we recover the know result.

$$\frac{R}{320} + \frac{M}{48} = 1 \tag{36}$$

4. Results and Discussion

The effect of internal heat generation on the criterion for the onset of Benard-Marangoni instability in a variable viscosity with upper surface of a fluid layer is deformably free is investigated theoretically. The resulting eigenvalue problem is solved using a regular perturbation technique with wave number a as a perturbation parameter.

In considering pure Benard convection(i.e $M = 0$), the critical Rayleigh number R_c is a function of the viscosity parameter B and the internal heat source strength Ns. Figure 2 shows the variation of critical Rayleigh number R_c with the internal heat source strength Ns for different values of B, it is observed that in the absence of variable viscosity ($B = 0$) the critical Rayleigh number R_c increases initially, with Ns reaches maximum and then decreases with further increase in the value of Ns. As a result of Fig. 2 some unusual behaviours are observed namely, (i) increasing variable viscosity parameter shows some destabilizing effect and (ii) increasing internal heat source strength causes stabilizing effect initially. Figure 3 depicts the perturbed vertical velocity eigen functions W for different values of internal heat source strength Ns for $M = 0$ and $B = 1$. It is noted that the convection occurs maximum at the upper part of fluid layer as increasing in heat source strength.

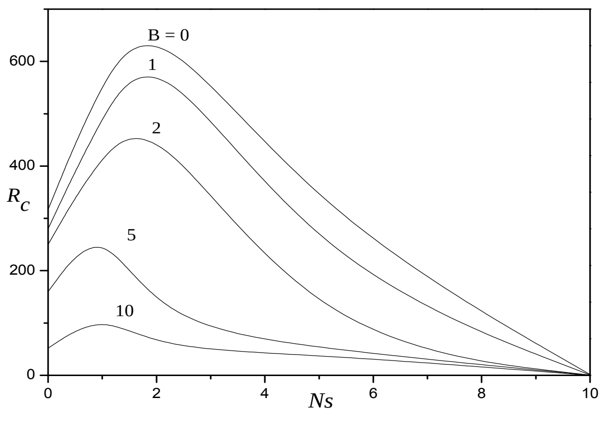

Fig.2 Critical Rayleigh number versus Ns for different values of B for

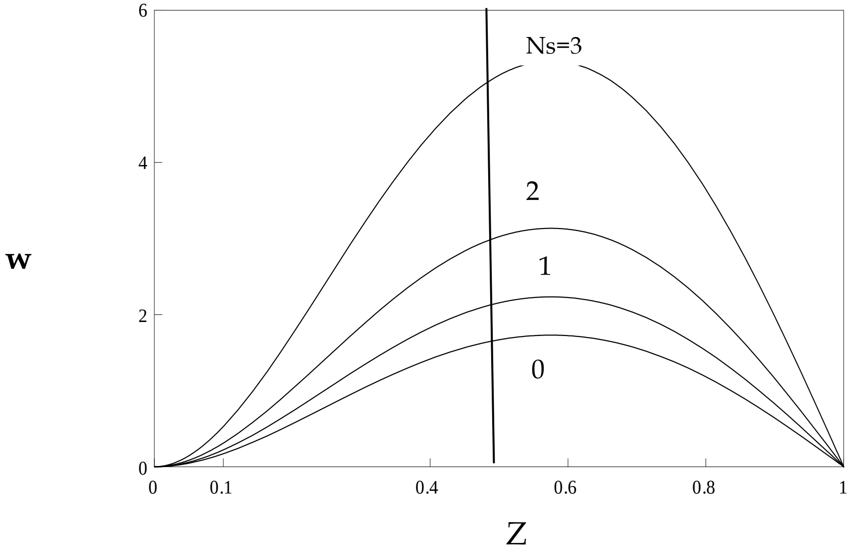

Fig.3Perturbed velocity eigen functions W for different values of Ns with $B=1$

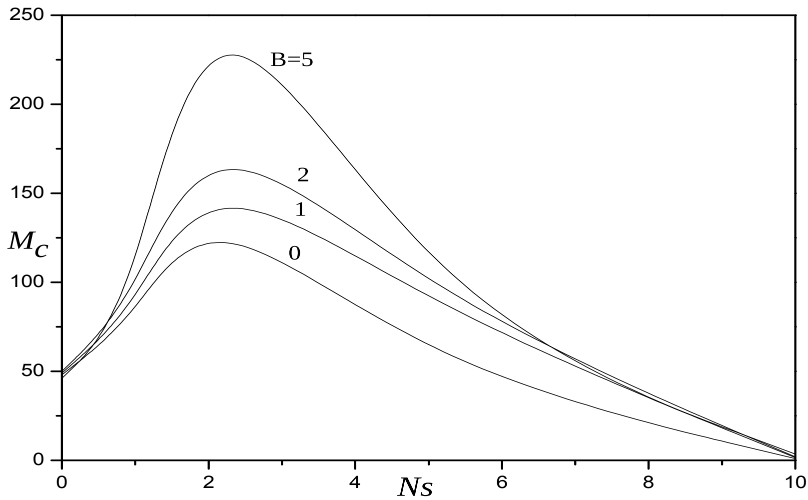

Fig.4 Critical Marangoni number versus Ns for different values of B for

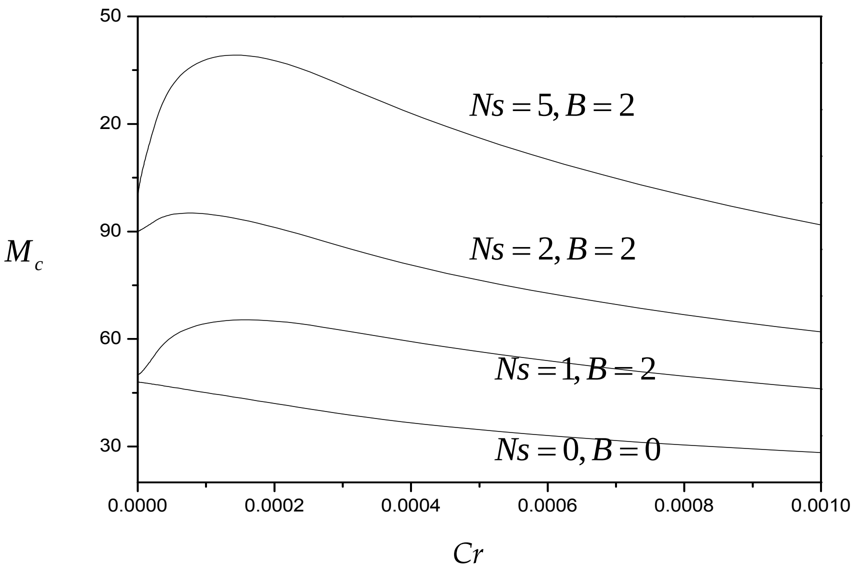

Fig.5 Critical Marangoni number versus Cr for different values of Ns.

In the absence of thermal buoyancy (i.e $R = 0$) we merely consider the Marangoni convective instability at the upper free surface. Figure 4 shows the variation of critical Marangoni number M_c with the internal heat source strength Ns for different values of B, it is observed that in the absence of variable viscosity ($B = 0$) the critical Marangoni number M_c increases initially, with Ns reaches maximum and then decreases with further increase in the value of Ns. As a result of Fig.4 some unusual behaviours are observed namely, (i) increasing variable viscosity parameter shows some stabilizing effect and (ii) increasing internal heat source strength causes stabilizing effect initially.

In Fig.5 the critical Marangoni number M_c is plotted against the Crispation number Cr for different values of Ns. it is observed that in the absence of variable viscosity and Ns the critical Marangoni number M_c decreases as Cr increases while in the presence of variable viscosity critical Marangoni number M_c increases initially, with Cr and Ns reaches maximum and then decreases with further increase in the value of Ns reaches maximum and then decreases with further increase in the value of Cr. It is observed that the critical Marangoni number M_c decreases with an increase of the Crispation number for $Cr < 10^{-4}$ and thus making system more unstable. The reason being that an increase in Cr is to increase the deflection of the upper free surface, which in turn, promotes instability much faster.

The effect of the Bond number B_0 on the critical Marangoni number M_c different values of Ns with $B = 2$ is shown in Fig.6. It is observed that increase in the value of B_0 makes the system more stable. The reason for this may be attributed to the fact that an increase in the gravity effect, which keep the free surface flat against the effect of surface tension, which forms a meniscus on the free surface, and hence an increase in B_0 makes the system more stable.

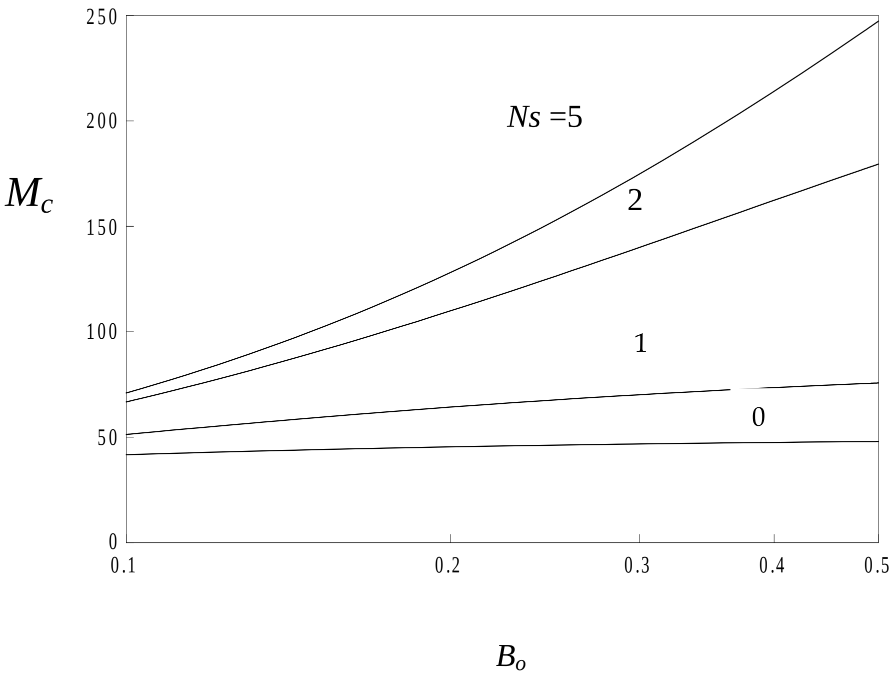

Fig.6Critical Marangoni number versus B_0 for different values of Ns for $B = 2$.

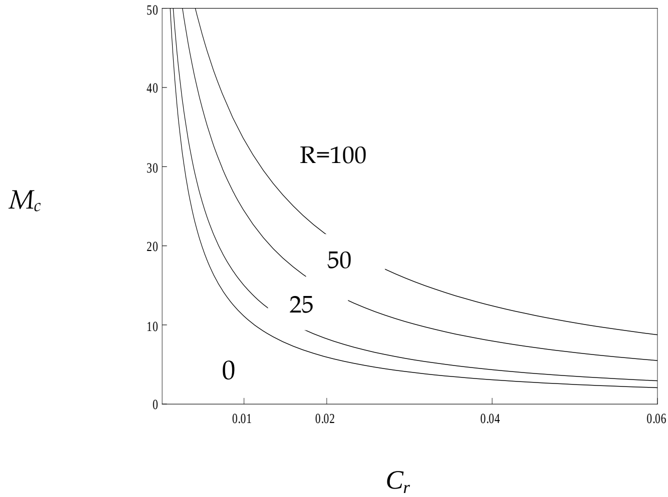

Fig.7Critical Marangoni number versus Cr for different values of R_c for $B = 2 = Ns.$

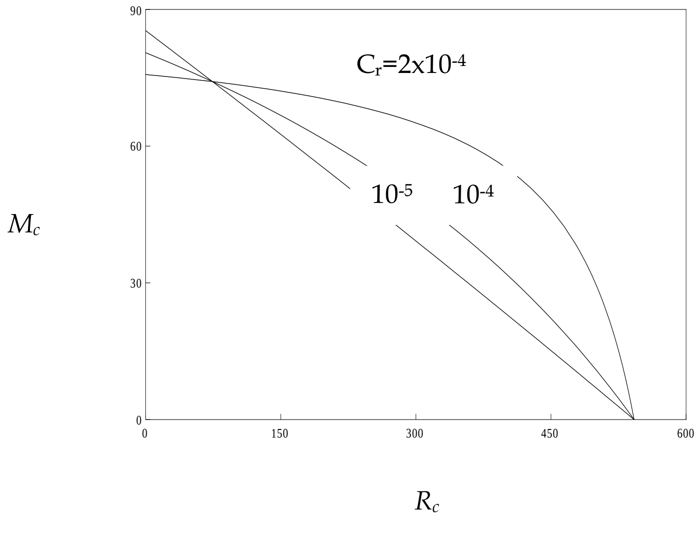

Fig.8 Critical Marangoni number versus Critical Rayleigh number for different values of Cr for $B = 2 = Ns.$

The plot of the critical Marangoni number M_c verses different values of Cr with $B = 2$ and $Ns = 2$ is shown in Fig.7. It is observed that the Cricipation number Cr decreases as R^c increase; the instability of the system is dominated by surface R_c tension.

A plot of M_c as a function of R^c is shown in Fig.8 for a several values of Cr for $B = 2$ and $Ns = 2$. We notice from figure that when $M_c = 0$, the curve trend toward $R^c = 538$. This shows that the thermal buoyancy dominates the system over the effect of surface tension. On other hand when $Cr < 10^{-4}$, which increases the inverse effect of surface tension, the system will be easily destabilised because of thermal buoyancy become surface tension. It is evident from figure that the effect of thermal buoyancy increases so that the system is under the domination of the thermal mode.

5 Conclusions

The onset of penetrative convection via internal heating in fluid layer with variable viscosity has been studied theoretically and following result are obtained.

a. Critical Rayleigh number R_c increases initially, with simultaneous effects of Ns and B and reaches maximum and then decreases with further increase in the value of Ns and B. It is noted that the appearance of newly formed sub layer, which first occurs at the maximum critical Rayleigh number R_c with associated viscosity parameter, continues to manifest itself after then, becoming dominant at the critical state. As B is further increased, the viscously suppressive effects of main fluid layer above shorten the depth of sub layer and R_c then decreases with B.

b. Increasing viscosity parameter on Marangoni convection shows the stabilizing effect initially.

c. Increasing in the value of Bond number and decreasing the Cricipation number makes the system more stable.

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