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GEOMETRIC CIRCULAR GRAPHS

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ABSTRACT

*In this paper we study a class of graphs, which resemble a Circle in a plane in terms of diameter and radius. We introduce the term "**Geometric circular graphs**" for those graphs whose diameter is equal to twice the radius of the graph. Here we have studied some properties of geometric circular graphs. Also we have found some bounds in terms of the number of edges.*

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Key words and Phrases: *Diameter and Radius of a graph.*

1. Introduction

In this paper we introduce a class of graphs in which we bring the notion of a circle, along with radius and diameter, from Euclidian geometry. In literature, the concept of radius (and diameter) has been introduced on the basis of distance in graphs. It is well known the diameter of a circle is twice its radius in Euclidian geometry. But in case of graphs the diameter is bounded by radius at the lower end and twice radius of the graph on the upper end. Hence diameter assumes any value between these two extremes. The class of graphs for which radius is equal to

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diameter has been studied by [1], [3], under the title "self-centered graphs" or "equi-eccentric graphs". The class of graphs for which diameter is twice the radius of a graph, exactly resembles circle in a plane. Hence we term these graphs as **"Geometrical Circular Graphs"**, in short GCG.

Definition : A connected graph is said to be a geometric circular graph, GCG, if its diameter is twice radius, that is diam(G) = 2 rad(G).

For all definitions and terminology the reader may refer [4]. In particular, in this paper the diameter of a connected graph G is denoted by diam(G)=d(G) and the radius of the connected graph G is denoted by rad(G)=r(G).

2. Existing Results and Definitions

In this section we list some existing results, without proof, and definitions, which are helpful in proving the results of this paper.

Proposition1.1[5]: Suppose that all diametral paths of a graph G avoid the center, then $r(G) + 2 \leq d(G) \leq 2r(G) - 1$ and all pairs of values in this given range are attainable.

Proposition1.2[6]: If d(G) ≥ 3, then $d(\overline{G}) \leq 3$.

Proposition 1.3[4]: Every non-trivial self-complementary graph has diameter 2 or 3.

Proposition1.4[4]: Any graph G can be embedded in a supergraph H such that <C(H)> is isomorphic to G.

Proposition1.5 [8]: Let G and H be simple connected graphs. Then

$r(G \times H) = r(G) + r(H)$

$d(G \times H) = d(G) + d(H)$

Proposition 1.6[9]: For a graph G if $d(G)= 2r(G)$, then $P(G) \subseteq EC(G)$.

3. Results

In this section we prove some results on geometric circular graphs.

Lemma 2.1: A connected graph is geometric circular graph, GCG, if at least one diametral path contains a central vertex.

Proof: Let G be a connected graph. By Proposition 1.1[5], if all diametral paths avoid the center, then $d(G) \leq 2r(G) - 1$, holds. If at least one diametral path contains a central vertex, then $d(G) \geq 2r(G)$. Since G is connected $d(G) \leq 2r(G)$. Hence equality holds, making the graph G to be a GCG.

Proposition 2.2: If G is a GCG, then any diametral path cannot contain more than one central vertex.

Proof: For a GCG, G, let x-y be a diametral path. Suppose the x-y diametral path contains two central vertices, viz. u and v. Then,

$$d(x, y) \leq d(x, u) + d(u, y) + d(u, v)$$

$$\Rightarrow d(x, y) = d(G) \leq 2r(G) + d(u, v) \Rightarrow 2r(G) \leq 2r(G) + d(u, v) \Rightarrow d(u, v) = 0 \Rightarrow u = v.$$

Hence there lies exactly one central vertex on any diametral path.

Note: Clearly, if G is a GCG, then there does not exist a diametral path containing an edge of $<C(G)>$.

Proposition 2.3: In a GCG, every central vertex has at least two eccentric vertices.

Proof: Let G be a GCG. Let x be a central vertex of G. By Proposition 2.2 above, x lies on a diametral path, say, u-v path, i.e. $d(u,v)=d(G)=2\,r(G)$. If x is a midpoint of u-v path, then u,v will be the eccentric vertices of x.

If x is not a midpoint of u-v path, then, for the midpoint of u-v path, say, $x', e(x') \geq \frac{1}{2} d(G) = r(G)$

If $e(x') > \frac{1}{2} d(G)$, then $x' \notin C(G)$, where C(G) denotes the center of the graph G.

If $e(x') = \frac{1}{2} d(G)$, then $x' (= x) \in C(G)$. Hence, x′ will have u and v as its eccentric vertices.

Note: From the above result it is clear that every central vertex is a midpoint of some diametral path in a GCG. But the converse need not always be true.

Theorem 2.4: A connected graph G is GCG if and only if $P(G) = EC(G)$, where P(G) denotes the periphery of G and EC(G) denotes the set of all eccentric vertices of central vertices of G.

Proof: Let G be GCG. By Proposition1.6 [9], $P(G) \subseteq EC(G)$.

Conversely, if $x \in EC(G) \Rightarrow$ there exists a vertex $u \in C(G)$ such that $d(x,u) = r(G)$ and x is an eccentric vertex of u i.e. $d(x,u) = 1/2\ d(G)$. By the above proposition there exists at least one more eccentric vertex for u, say, y. Hence, $d(x,y) = d(x,u) + d(u,y) = 2\ r(G) = d(G)$. Hence, $x,y \in P(G)$.

So $EC(G) \subseteq P(G)$, making the two sets equal in case of geometric circular graphs.

Proposition 2.5: If a graph G is GCG, then its complement $\overline{G}$ is not.

Proof: Let G be a GCG. Hence, G is connected and has finite radius and diameter. If $d(G) \geq 3$, then by Proporosition1.2[6], $d(\overline{G}) \leq 3$ If possible assume that $\overline{G}$ is GCG, then $d(\overline{G}) = 2$ and $r(\overline{G}) = 1$. Hence, in $\overline{G}$ there exists a vertex, say, u, such that $deg_{\overline{G}}\ u = p - 1 \Rightarrow deg_G\ u = 0$, a contradiction to the connectedness of the graph G.

If d(G)<3 then d(G)≤2 and hence, $d(G)=2$, $r(G) = 1$. Similar argument as above leads to the contradiction to the connectedness of $\overline{G}$, if it were GCG. Hence, if G is GCG, then is not.

Note: If $\overline{G}$ is GCG, then G is not a GCG.

Theorem 2.6: There does not exist a self-complementary GCG.

Proof: By above proposition the proof follows.

Proposition 2.7: Any connected graph G can be embedded in some GCG.

Proof: Let G be a connected graph. We know that it it is easy to embed G as an induced subgraph in any supergraph H such that <C(H)> is isomorphic to G by [4]. Indeed, it is the sequential join

$K_1 + K_1 + \ldots.. + K_1(r - times) + G + K_1 + \ldots\ldots. + K_1\ (r - times)$. These graphs have $rad(H) = r$ and $diam(H) = 2\ r$, and hence these are geometric circular graphs.

Proposition 2.8: The Cartesian product of two graphs is GCG if, and only if both graphs are geometric circular.

Proof: Let G_1 and G_2 be any two geometric circular graphs. Let $G = G_1 \times G_2$ be the Cartesian product of G_1 and G_2. By Proposition 1.5[8], $r(G_1 \times G_2) = r(G_1) + r(G_1)$ and $d(G_1 \times G_2) = d(G_1) + d(G_2)$ Since both G_1 and G_2 are geometric circular, $d(G_1) = 2r(G_1)$ and $d(G_2) = 2\, r(G_2)$. Hence, $d(G_1 \times G_2) = d(G_1) + d(G_2) = 2r(G_1) + 2r(G_2) = 2([r(G_1) + r(G_2)] = 2r(G_1 \times G_2)$ making the Cartesian product to be a geometric circular graph.

For the converse, let the Cartesian product of two graphs be geometric circular, i.e. for any two graphs G_1 and G_2, let $G = G_1 \times G_2$, be GCG. Hence, $d(G_1 \times G_2) = 2r(G_1 \times G_2) \Rightarrow d(G_1) + d(G_2) = 2r(G_1) + 2r(G_2)$

$$\Rightarrow \quad r(G_1) = \frac{d(G_1) + d(G_2) - 2r(G_2)}{2}$$

If $d(G_2) = 2\, r(G_2)$ that is if G_2 is GCG, then G_1 is also GCG.

If $d(G_2) < 2r\,(G_2)$, then $d(G_1) < 2r(G_1)$. Hence $d(G_1 \times G_2) < 2r(G_1 \times G_2)$, a contradiction to the assumption that $G = G_1 \times G_2$ is a geometric circular graph.

So both G_1 and G_2 have to geometric circular if their Cartesian product is geometric circular.

Proposition 2.9: For a GCG, G, of order p, radius r, diameter d, size q, and the following holds true: $p - 1 \leq q \leq \frac{(p-d)^2 + 3p - d - 4}{2}$, and both the bounds are attainable.

Proof: Given G is GCG. Hence d is finite, so we need at least p-1 edges to cover all vertices. Hence, the lower bound follows and a tree with one central vertex attains it.

For the upper bound, we have $d(G) = 2r(G)$. Hence there exist at least one diametral path of length d. Let $v_1, v_2, \ldots, v_d, v_{d+1}$, be one of the diametral paths. Let the vertex set V(G) be partitioned into subsets $V_1, V_2, \ldots, V_d, V_{d+1}$ such that each $v_i \in V_i$ and all those vertices $x \in V_i$ satisfying the following two conditions: $d(x, v_1) \leq d$ and $d(x, v_{d+1}) \leq d$ otherwise the length of a path containing $x, v_1, v_2, \ldots, v_d, v_{d+1}$ exceeds the diameter. This partition is possible because each V_i contains at least one vertex viz., v_i itself. We can add maximum possible edges to each V_i among v_i's and among vertices of V_i and V_{i+1}, without disturbing d(G). The resulting graph is $K_{n_1}+K_{n_2}+\dots\dots+K_{n_{d+1}}$, where $\sum_{i=1}^{d+1} n_i = p$ This graph is maximal with respect to the diameter in the sense that by adding an additional edge to the graph the diameter is decreased by one. Hence, $q=\sum_{i=1}^{d+1}\binom{n_i}{2}+\sum_{i=1}^{d} n_i n_{i+1}$ So q is maximum when both summands are maximum.

For $\sum_{i=1}^{d+1}\binom{n_i}{2}\le\binom{\sum_{i=1}^{d+1} n_i}{2}=\binom{p-d}{2}$ as at least d-vertices are required to maintain the diameter.

But sum of the products $\sum_{i=1}^{d} n_i n_{i+1}$ is maximum whenever $n_i = n_{i+1}$.

As p-d vertices have already gone in first term there remain only d-vertices in second term. So, G would be isomorphic to $K_{p-d}+K_1+K_1+\dots..+K_1(d-times)$.

The size of the graph is maximum if

$G\cong K_1+K_1+\dots..+K_1(r\text{-}times)+K_{p\text{-}d}+K_1+K_1+\dots..+K_1(r\text{-}times)$. So,

$$q\le\binom{p-d}{2}+2(p-d)+2(r-1)=\frac{(p-d)(p-d-1)+4(p-d)+4(r-1)}{2}$$

$$q\le\frac{(p-d)(p-d-1+4)+(2d-4)}{2}=\frac{(p-d)^2+3p-d-4}{2}.$$

Hence, $G\cong K_1+K_1+\dots..+K_1(r\text{-}times)+K_{p\text{-}d}+K_1+K_1+\dots..+K_1(r\text{-}times)$ is the realizing graph for the upper bound of q.

In Theorem 1.6[9], it is proved that if a graph G has $d(G)=2\,r(G)$, i.e. if the graph G is GCG, then $P(G)\subseteq EC(G)$.In view of this result one may consider the size of the sets P(G) and EC(G). The following proposition gives a graph with given size of P(G) and EC(G).

Proposition 2.10: For any positive integers m and n, $2\le m\le n$, there exists a geometric circular graph G such that $|P(G)| = m$, $|EC(G)| = n$.

Proof: Consider a path $P_{2r}: v_1, v_2, \ldots, v_{r+1}, v_{r+2}, \ldots, v_{2r+1}$. And consider another path of length r whose end vertex is concatenated with v_{r+1}, and it is labeled as $v_{r+1}, u_1, u_2, \ldots, u_r$. Form a graph G by joining m-2 end vertices with v_2 and n-m-1 end vertices with u_{r-1}. Labeling these extra end vertices in forming G as $w_1, w_2, \ldots, w_{m-2}$, and $w'_1, w'_2, \ldots, w'_{n-m-1}$. Clearly, d(G) = 2 r(G). Hence G is a GCG.

Also it is not difficult to observe that $P(G) = \{v_1, w_1, w_2, \ldots, w_{m-2}, v_{2r+1}\}$and $EC(G) = \{v_r, w'_1, w'_2, \ldots, w'_{n-m-1}\} U \ P(G)$.
Therefore $|P(G)| = m$, $|EC(G)| = m + n - m - 1 + 1 = n$. Hence the result.

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