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Split and Non-Split Dominator Chromatic Numbers and Related Parameters

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Abstract

A proper graph coloring is defined as coloring the nodes of a graph with the minimum number of colors without any two adjacent nodes having the same color. Dominator coloring of G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. In this paper, new parameters, namely strong split and non-split dominator chromatic numbers and block, cycle, path non-split dominator chromatic numbers are introduced. These parameters are obtained for different classes of graphs and also interesting results are established.

Keywords: domination, dominator, split domination, non-split domination, strong split domination, block non-split domination, path non-split domination, block non-split domination, cycle non-split domination

1. Preliminaries

In our study, we consider only simple and undirected graphs. In this section, we review the notions of domination, coloring and dominator coloring [1, 2].

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Definition 1.1 Let G = (V, E) be a graph. A subset D of V is called a *dominating set* of G if every vertex in V – D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

Definition 1.2 A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$, is the minimum number of colors required for a proper coloring of G. A *Color class* is the set of vertices, having the same color. The color class corresponding to the color i is denoted by C_i.

Definition 1.3 A *dominator coloring* of a graph G is a proper coloring in which every vertex of G dominates every vertex of at least one color class. The convention is that if {v} is a color class, then v dominates the color class {v}. The dominator chromatic number $\chi_d(G)$ is the minimum number of colors required for a dominator coloring of G.

2. Strong Split Dominator chromatic number

In this section, we extend the notion of strong split domination [3], to strong split dominator chromatic number $\chi_{ssd}(G)$ and obtain $\chi_{ssd}(G)$ for the classes of graphs: trees, cycles, paths, wheels, complete graphs and bipartite graphs.

Definition 2.1 Consider a graph G and its dominator coloring with $\chi_d(G)$ colors. The *strong split dominator chromatic number* of G is minimum number of color classes whose removal from G leaves the remaining graph totally disconnected and is denoted by $\chi_{ssd}(G)$.

Example 2.2 Let G be the graph with V = {v_1, v_2, v_3, v_4, v_5, v_6} in Figure 1.

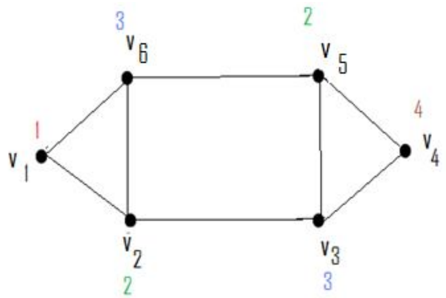

Figure 1

The color classes of G are: C_1 = {v_1}, C_2 = {v_2, v_5}, C_3 = {v_3, v_6} and C_4 = {v_4}.

Here, V–{C_2, C_3} is totally disconnected. Hence $\chi_{ssd}(G) = 2$.

Theorem 2.3 For cycle graphs C_n of order n ≥ 3,

$$\chi_{ssd}(C_n) = \begin{cases} 1 & \text{when} \quad n = 4, 6, 8, 10 \\ 2 & \text{when} \quad n = 3, 5, 7 \\ 2 & \text{when} \quad n = 3k \text{ or } 3k+2, k \geq 3 \\ 3 & \text{when} \quad n = 3k+1, k \geq 4 \end{cases}$$

Proof: Let C_n be the cycle graph of order n ≥ 3 and let v_1, v_2, …, v_n be the label of its vertices. When n = 3, 5 or 7, the vertices of C_n are colored by the coloring sequence (1, 2, 3), (1, 2, 1, 3, 2), (1, 2, 1, 3, 1, 4, 5) to obtain a dominator coloring. Hence $\chi_{ssd}(C_n) = 2$, when n = 3, 5, or 7.

Case (i) When n = 4, 6, 8 or 10.

Color the odd subscripted vertices v_1, v_3, v_5, … by color 1 and the even subscripted vertices v_2, v_4, v_6, … by colors 2, 3, 4, … respectively, resulting in a dominator coloring. Removal of color class 1 results in a totally disconnected graph. Hence $\chi_{ssd}(C_n) = 1$, when n = 4, 6, 8 or 10.

Case (ii) When $n = 3k$, where $k \geq 3$

Color the vertices $v_1, v_4, v_7, \ldots$ by color 1, $v_2, v_5, v_8, \ldots$ by color 2 and $v_3, v_6, v_9, \ldots$ by colors 3, 4, 5, …respectively. Removal of color classes 1 and 2 results in totally disconnected graph. Hence $\chi_{ssd}(C_n) = 2$ in this case.

Case (iii) When $n = 3k + 1$, $k \geq 4$

Color the vertices $v_1, v_4, v_7, \ldots, v_{n-3}$ by color 1, $v_2, v_5, v_8, \ldots, v_{n-2}$ by color 2, $v_3, v_6, v_9, \ldots, v_{n-1}$ by colors 3, 4, 5, … (k+2) respectively and v_n by color (k+3). Removal of color classes 1, 2 and (k+3) results in totally disconnected graph. Hence $\chi_{ssd}(C_n) = 3$ in this case.

Case (iv) When $n = 3k + 2$, $k \geq 3$

Color the vertices $v_1, v_4, v_7, \ldots, v_{n-1}$ by color 1, $v_2, v_5, v_8, \ldots, v_{n-3}$ by color 2, $v_3, v_6, v_9, \ldots, v_{n-2}$ by colors 3, 4, 5, … (k+2) respectively and v_n by color (k+3). Removal of color classes 1, 2 results in totally disconnected graph. Hence $\chi_{ssd}(C_n) = 2$ in this case.

From the above cases, we conclude that

$$\chi_{ssd}(C_n) = \begin{cases} 1 & \text{when} \quad n = 4, 6, 8, 10 \\ 2 & \text{when} \quad n = 3, 5, 7 \\ 2 & \text{when} \quad n = 3k \text{ or } 3k+2, k \geq 3 \\ 3 & \text{when} \quad n = 3k+1, k \geq 4 \end{cases}$$

Theorem 2.4 For path graphs P_n of order $n \geq 2$, $\chi_{ssd}(P_n) = 1$.

Proof: Let P_n be a path of order $n \geq 2$ with vertex labels $v_1, v_2, \ldots, v_n$. A dominator coloring of P_n is obtained by coloring the odd subscripted vertices $v_1, v_3, v_5, \ldots$ by color 1 and even subscripted vertices $v_2, v_4, v_6, \ldots$, respectively by colors 2, 3, 4, …. Removal of color class 1 results in a totally disconnected graph. Hence $\chi_{ssd}(P_n) = 1$.

Theorem 2.5 For wheel graphs W_n of order $n \geq 4$,

$$\chi_{ssd}(W_n) = \begin{cases} 3 & \text{when } n \text{ is even} \\ 2 & \text{when } n \text{ is odd} \end{cases}$$

Proof: Let W_n be a wheel graph of order $n \geq 4$. Let the vertices of the wheel graph W_n, $n \geq 4$ be labeled as follows. The vertex at the centre is labeled by v_1 and the vertices on the rim be labeled consecutively by $v_2, v_3, \ldots, v_n$.

A dominator coloring of W_n is by coloring v_1 by color 1 and coloring the vertices in the rim alternatively by colors 2 and 3 from vertex v_2.

When n is odd, the vertices v_{n-1} and v_n are colored by 2 and 3 and when n is even, the vertices v_{n-2}, v_{n-1} and v_n are colored by colors 2, 3 and 4. Now it is easy to see that the removal of color classes 1 and 2 in the case of n is odd and the color classes 1, 2 and 4 in the case of n is even results in a totally disconnected graph.

From the above two cases, we conclude that

$$\chi_{ssd}(W_n) = \begin{cases} 3 & \text{when } n \text{ is even} \\ 2 & \text{when } n \text{ is odd} \end{cases}$$

Theorem 2.6 For star graphs S_n of order $n \geq 3$, $\chi_{ssd}(S_n) = 1$.

Proof: Let S_n be a star graph of order $n \geq 3$. A dominator coloring of S_n is by coloring the centre vertex by color 1 and the remaining vertices by color 2. Removal of color class 1 results in a totally disconnected graph. Hence $\chi_{ssd}(S_n) = 1$.

Theorem 2.7 For complete graphs K_n of order $n \geq 2$, $\chi_{ssd}(K_n) = n-1$.

Proof: A dominator coloring of K_n is by coloring its vertices by colors 1, 2, 3, …, n respectively. Therefore $\chi_{ssd}(K_n) = n - 1$.

Theorem 2.8 For complete bipartite graphs $K_{m,n}$, $\chi_{ssd}(K_{m,n}) = 1$, $\forall$ m, $n \geq 2$.

Proof: For $K_{m,n}$, color the vertices in one of the partition by color 1 and the vertices in the other partition by color 2. This gives a dominator coloring of $K_{m,n}$. Removal of color class either 1 or 2 results in a totally disconnected graph. Hence $\chi_{ssd}(K_{m,n}) = 1$.

3. Strong non-split dominator chromatic number

In this section, the concept of strong non-split domination [4], is extended to strong non-split dominator chromatic number $\chi_{snd}(G)$ and find $\chi_{snd}(G)$ for various classes of graphs.

Definition 3.1 Let G be a graph. Consider its dominator coloring with $\chi_d(G)$ colors. The *strong non-split dominator chromatic number* of G is the minimum number of color classes whose removal from G leaves the remaining graph complete and is denoted by $\chi_{snd}(G)$.

Example 3.2 Let G be the graph with V = {v_1, v_2, v_3, v_4, v_5, v_6} in Fig. 2. The color classes of G are: V_1 = {v_1, v_4, v_6}, V_2 = {v_2}, V_3 = {v_3} and V_4 = {v_5}. Here, V−{V_1} is complete. Hence $\chi_{snd}(G)$ = 1.

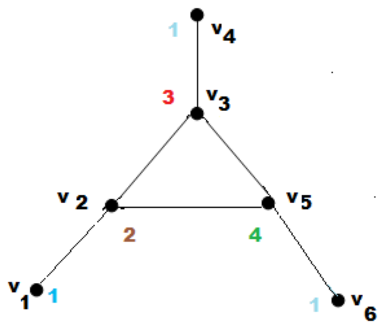

Figure 2

Theorem 3.3 For cycle graphs C_n,

$$\chi_{snd}(C_n) = \begin{cases} \lceil n/3 \rceil & \text{when } n = 5, 6 \text{ or } 3k+1, k \geq 2 \\ \lceil n/3 \rceil + 1 & \text{when } n = 3k-1 \text{ or } 3k, k \geq 3 \end{cases}$$

Proof: Consider the dominator coloring of C_n as in Theorem 2.3. The removal of the required number of color classes reduces the graph to either K_1 or K_2, proving the claim in the theorem.

Note 3.1 For the cycle graph C_3, $\chi_{snd}(C_3) = 0$.

Theorem 3.4 For path graphs P_n of order $n \geq 3$,

$$\chi_{snd}(P_n) = \begin{cases} \lceil n/3 \rceil & \text{when } n = 3, 4, 5, 7 \\ \lceil n/3 \rceil + 1 & \text{otherwise} \end{cases}$$

Proof: Consider the dominator coloring of P_n as in Theorem 2.4. The result can be seen easily.

Note 3.2 For the path of order 1 and 2, $\chi_{snd}(P_1) = \chi_{snd}(P_2) = 0$.

Theorem 3.5 For wheel graphs W_n of order $n \geq 5$, $\chi_{snd}(W_n) = 2$.

Proof: Consider the dominator coloring of W_n as in Theorem 2.5. The removal of color classes 2 and 3, for any n, results in a complete graph. Hence, $\chi_{snd}(W_n) = 2$.

Note 3.3 For the wheel graph W_4, $\chi_{snd}(W_4) = 0$.

Theorem 3.6 For the star graph S_n of order $n \geq 3$, $\chi_{snd}(S_n) = 1$.

Proof: Consider the dominator coloring of S_n as in Theorem 2.6. Removal of color class 2 results in a complete graph. Hence $\chi_{snd}(S_n) = 1$.

Note 3.4 For complete graphs K_n of order $n \geq 1$, $\chi_{snd}(K_n) = 0$.

4. Block dominator chromatic number

In this section, block non-split domination [5] is extended to block dominator chromatic number $\chi_{bd}(G)$ and $\chi_{bd}(G)$ is obtained for various classes of graphs.

Definition 4.1 Consider a graph G and its dominator coloring with $\chi_d(G)$ colors. The *block dominator chromatic number* of G is the minimum number of color classes to be removed so that the remaining graph is a union of blocks in G and is denoted by $\chi_{bd}(G)$. We note that for a graph G, which is a block, $\chi_{bd}(G) = 0$.

Example 4.2 For the graph in Fig. 2, V-{V_1} is a block. Hence $\chi_{bd}(G) = 1$.

Note 4.1 We observe the following facts:

- For the cycle C_n of order $n \geq 3$, $\chi_{bd}(C_n) = 0$.
- For the wheel graph W_n of order $n \geq 4$, $\chi_{bd}(W_n) = 0$.
- For the complete graph K_n of order $n \geq 2$, $\chi_{bd}(K_n) = 0$.
- For the bipartite graph $K_{m, n}$ of order m, $n \geq 2$, we have $\chi_{bd}(K_{m, n}) = 0$.

5. Cycle non-split dominator chromatic number

In this section, the notion of cycle non-split domination is extended to cycle non-split dominator chromatic number $\chi_{cnd}(G)$ and $\chi_{cnd}(G)$ is obtained for W_n.

Definition 5.1 Consider a graph G and its dominator coloring with $\chi_d(G)$ colors. The *cycle non-split dominator chromatic number* of G is the minimum number of color classes that should be removed so that the remaining graph is a cycle and is denoted by $\chi_{cnd}(G)$.

Example 5.2 Consider the graph in Fig. 2, and its corresponding color classes. We observe that V-{V_1} is a cycle. Hence $\chi_{cnd}(G) = 1$.

Theorem 5.3 For the wheel graph W_n of order $n \geq 4$, $\chi_{cnd}(W_n) = 1$.

Proof: Consider the dominator coloring of W_n as in Theorem 2.5. It can be seen that the removal of color class 1 for any n results in a cycle graph. Hence $\chi_{cnd}(C_n) =1$.

Theorem 5.4 For the wheel graph W_n of order $n \geq 4$, we have $\chi_{cnd}(W_n) + \Delta(W_n) = n$.

Observation 5.1 For the complete graph C_n, $\chi_{cnd}(K_n) = n-3$.

Observation 5.2 For the cycle graph C_n, $\chi_{cnd}(C_n) = 0$.

Observation 5.3 For the complete bipartite graph $K_{m, n}$, $\chi_{cnd}(K_{m, n}) = 0$, when $m = n = 2$.

6. Path non-split dominator chromatic number

In this section, we extend the notion of path non-split domination, to path non-split dominator chromatic number $\chi_{pnd}(G)$ and obtain $\chi_{pnd}(G)$ for various classes of graphs.

Definition 6.1 Consider a graph G and its dominator coloring with $\chi_d(G)$ colors. The *Path non-split dominator chromatic number* of G is the minimum number of color classes that should be removed so that the remaining graph is a path and is denoted by $\chi_{pnd}(G)$.

Example 6.2 Let G be the graph with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ in Figure 4.

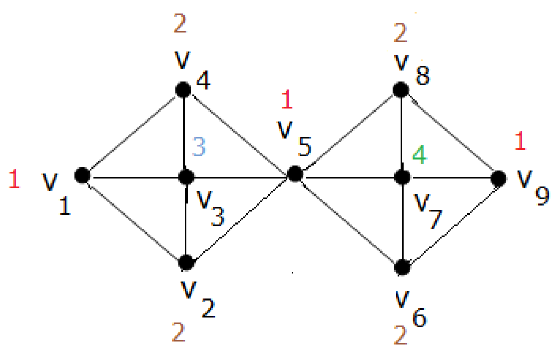

Figure 4

The color classes of G are, $V_1 = \{v_1, v_5, v_9\}$, $V_2 = \{v_2, v_4, v_6, v_8\}$, $V_3 = \{v_3\}$ & $V_4 = \{v_7\}$.

Here, $V-\{V_2\}$ is a path. Hence $\chi_{pnd}(G) = 1$.

Theorem 6.3 For the cycle graph C_n of order, $\chi_{pnd}(C_n) = 1$, when n = 3, 5, 6,7,…

Proof: Consider the dominator coloring of C_n as in Theorem 2.3. The removal of the last color class reduces the graph to a path of order n-1. Hence $\chi_{pnd}(C_n) = 1$.

Theorem 6.4 For the wheel graph W_n of order $n \geq 4$,

$$\chi_{pnd}(W_n) = \begin{cases} 2 & \text{when } n = 4, 6, 7, 8, 9, \ldots \\ 1 & \text{when } n = 5 \end{cases}$$

Proof: Consider the dominator coloring of W_n as in Theorem 2.5. It can be seen that the removal of color class 2 and 3 when n = 4, 6, 7, … results in a path. When n = 5, removal of either the color class 2 or 3 results in a path.

Therefore, we have

$$\chi_{pnd}(W_n) = \begin{cases} 2 & \text{when } n = 4, 6, 7, 8, 9, \ldots \\ 1 & \text{when } n = 5 \end{cases}$$

Theorem 6.5 For the complete graph K_n of order $n \geq 2$, $\chi_{pnd}(K_n) = n - 2$.

Proof: A dominator coloring of K_n , is by coloring its vertices by colors 1, 2, 3, …, n respectively. Removal of n-2 color classes results in a path of order 2. Therefore $\chi_{pnd}(K_n) = n-2$.

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