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On SD-Harmonious Labeling

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Abstract

A graph G is said to be SD-harmonious labeling if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, q\}$ such that the induced function $f^* : E(G) \to \{0, 2, \cdots, 2q-2\}$ defined by $f^*(uv) = S + D\ (mod\ 2q)$ is bijective, where $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$, for every edge uv in $E(G)$. A graph which admits SD-harmonious labeling is called SD-harmonious graph. In this paper, we investigate SD-harmonious labeling of path related graphs, tree related graphs, star related graphs and disjoint union of graphs.

Keywords: Harmonious labeling, SD-harmonious labeling

Mathematics Subject Classification (2010): 05C78

1. Introduction

Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order $|V(G)| = p$ and size $|E(G)| = q$. All notations not defined in this paper can be found in [1]. An injective function $f : V(G) \to \{1, 2, \cdots, q\}$ is called a graceful labeling of G if all the edge labels of G given by $f(uv) = |f(u) - f(v)|$ for every $uv \in E$ are distinct. This concept was first introduced by Rosa in 1967 [2]. For all detailed survey of graph labeling we refer to Gallian [3].

G.C. Lau *et al.* introduced the concept of SD-prime labeling in [5, 6]. Given a bijection $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$, we associate 2 integers $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$ with every edge uv in E.

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Definition 1.1. [6] *A bijection* $f : V(G) \rightarrow \{1, 2, \cdots, |V(G)|\}$ *induces an edge labeling* $f' : E(G) \rightarrow \{0, 1\}$ *such that for any edge* uv *in G,* $f'(uv) = 1$ *if* $gcd(S, D) = 1$*, and* $f'(uv) = 0$ *otherwise. We say* f *is SD-prime labeling if* $f'(uv) = 1$ *for all* $uv \in E(G)$*. Moreover, G is SD-prime if it admits SD-prime labeling.*

In 1980, Graham and Sloane [4] introduced harmonious labeling in connection with error-correcting codes and channel assignment problems.

Definition 1.2. [4] *A graph G with q edges is said to be harmonious if there exists an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label* $f(x) + f(y)$ $(mod\ q)$*, the resulting edge labels are distinct.*

Motivated by the concept of SD-prime labeling and harmonious labeling, we introduce the new concept SD-harmonious labeling.

Definition 1.3. *A graph G is said to be SD-harmonious labeling if there exists an injection* $f : V(G) \rightarrow \{0, 1, 2, \cdots, q\}$ *such that the induced function* $f^* : E(G) \rightarrow \{0, 2, \cdots, 2q-2\}$ *defined by* $f^*(uv) = S + D\ (mod\ 2q)$ *is bijective, where* $S = f(u) + f(v)$ *and* $D = |f(u) - f(v)|$*, for every edge uv in* $E(G)$*. A graph which admits SD-harmonious labeling is called SD-harmonious graph.*

In this paper, we investigate SD-harmonious labeling for path related graphs, tree related graphs, star related graphs and disjoint union of graphs. We use the following definitions in the subsequent sections.

Definition 1.4. $P_m@P_n$ *is a graph obtained by identifying the pendant vertex of a copy of the path* P_n *at each vertex of the path* P_m*.*

Definition 1.5. *The corona* $G_1 \odot G_2$ *of two graphs* $G_1(p_1, q_1)$ *and* $G_2(p_2, q_2)$ *is defined as the graph obtained by taking one copy of* G_1 *and* p_1 *copies of* G_2 *and then joining the* i^{th} *vertex of* G_1 *to all the vertices in the* i^{th} *copy of* G_2*.*

Definition 1.6. [10] *The tree obtained by joining a new vertex v to one pendant vertex of each of the k disjoint stars* $K_{1,n_1}, K_{1,n_2}, K_{1,n_3}, \cdots, K_{1,n_k}$ *is called a banana tree. The class of all such trees is denoted by* $BT(n_1, n_2, n_3, \cdots, n_k)$*.*

Definition 1.7. [7] *A caterpillar graph is a tree in which all the vertices are within distance 1 of a central path* P_n *for* $n \geq 1$*. A caterpillar graph of order greater than 1 is a star graph when* $n = 1$*, which is* $K(1, r)$ *for some* $r \geq 1$*. When* $n \geq 2$*, a caterpillar graph is obtained from a path* $P_n = u_1 u_2 \cdots u_n$ *by attaching* $m_i \geq 0$ *pendant vertices* $v_{i,j}$ $(1 \leq j \leq m_i)$ *to each* u_i*. We shall denote this caterpillar graph by* $P_n(m_1, m_2, \cdots, m_n)$*.*

Definition 1.8. [9] *A sparkler, denoted as P_m^{+n}, is a graph obtained from the path P_m and appending n edges to an endpoint. This is a special case of a caterpillar. We refer to the hub of P_m^{+n}, the sparkler, as the vertex of degree $n+1$.*

Definition 1.9. [7] *Given $t \geq 3$ paths of length $n_j \geq 1$ with an end vertex $v_{j,1}(1 \leq j \leq t)$. A spider graph $SP(n_1, n_2, n_3, \cdots, n_t)$ is the one-point union of the t paths at vertex $v_{j,1}$.*

Definition 1.10. *The subdivision graph $S(G)$ is obtained from G by subdividing each edge of G with a vertex.*

Definition 1.11. *Consider t copies of stars namely $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \cdots, K_{1,n}^{(t)}$. Then $G = \left\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, \cdots, K_{1,n}^{(t)} \right\rangle$ is the graph obtained by joining apex vertices of each $K_{1,n}^{(j-1)}$ and $K_{1,n}^{(j)}$ to a new vertex x_{j-1} where $2 \leq j \leq t$. Note that G has $t(n+2)-1$ vertices and $t(n+2)-2$ edges.*

Definition 1.12. [8] *The graph $B(m, n, k)$ is a graph obtained from a path of length k by attaching the star $K_{1,m}$ and $K_{1,n}$ with its pendent vertices.*

Definition 1.13. [9] *Given two subgraphs G_1 and G_2 of G, the union $G_1 \cup G_2$ is the subgraph of G with vertex set consisting of all vertices which are in either G_1 or G_2 (or both) and with edge set consisting of all those edges which are in either G_1 or G_2 (or both).*

2. Path and Tree Related Graphs

In this section, we prove that P_n, $P_m@P_n$, banana tree $BT(n, n, n, \cdots, n)$, full binary tree, $P_n(m_1, m_2, \cdots, m_n)$, star, bistar, $P_n \odot \bar{K}_m$, P_m^{+n} and $SP(n_1, n_2, \cdots, n_t)$ admit SD-harmonious labeling. Also we prove that cycle C_n is not SD-harmonious graph.

Theorem 2.1. *The path P_n admits SD-harmonious labeling.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the path P_n. The labeling $f : V(P_n) \to \{0, 1, 2, \cdots, n-1\}$ is defined as $f(v_i) = i - 1$ for $1 \leq i \leq n$. It is easy to verify that f admits SD-harmonious labeling of P_n. □

Theorem 2.2. *The graph $P_m@P_n$ admits SD-harmonious labeling.*

Proof. Let $V(P_m@P_n) = \{u_i, v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n \text{ with } u_i = v_{i,1}\}$ and $E(P_m@P_n) = \{u_i u_{i+1} : 1 \leq i \leq m-1\} \cup \{v_{i,j} v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n-1\}$. Therefore, $P_m@P_n$ is of order mn and size $mn - 1$.
Define $f : V(P_m@P_n) \to \{0, 1, 2, \cdots, mn-1\}$ as follows:

$$f(u_i) = n(i-1), \quad 1 \leq i \leq m;$$
$$f(v_{i,j+1}) = n(i-1) + j, \ 1 \leq i \leq m, \ 1 \leq j \leq n-1.$$

The induced edge labels are

$$f^*(u_iu_{i+1}) = 2ni, \quad 1 \leq i \leq m-1;$$
$$f^*(v_{i,j}v_{i,j+1}) = 2n(i-1) + 2j, \ 1 \leq i \leq m, \ 1 \leq j \leq n-1.$$

Hence f admits SD-harmonious labeling for $P_m@P_n$. □

Corollary 2.3. *The comb $P_m \odot K_1$ admits SD-harmonious labeling.*

Theorem 2.4. *The banana tree $BT(n, n, n, \cdots, n)$ admits SD-harmonious labeling.*

Proof. Let $V(BT(n, n, n, \cdots, n)) = \{v\} \cup \{v_j, v_{j,r} : 1 \leq j \leq k, \ 1 \leq r \leq n\}$ where $d(v_j) = n$ and $E(BT(n, n, n, \cdots, n)) = \{vv_{j,n} : 1 \leq j \leq k\} \cup \{v_jv_{j,r} : 1 \leq j \leq k, \ 1 \leq r \leq n\}$. Therefore, $BT(n, n, \cdots, n)$ is of order $(n+1)k+1$ and size $(n+1)k$.
Define $f : V(BT(n, n, \cdots, n)) \to \{0, 1, 2, \cdots, (n+1)k\}$ as follows:

$$f(v) = 0;$$
$$f(v_j) = (n+1)(j-1) + 2, \ 1 \leq j \leq k;$$
$$f(v_{j,r}) = (n+1)(j-1) + r + 2, \ 1 \leq j \leq k, \ 1 \leq r \leq n-1;$$
$$f(v_{j,n}) = (n+1)(j-1) + 1, \ 1 \leq j \leq k.$$

The induced edge labels are

$$f^*(vv_{j,n}) = 2(n+1)(j-1) + 2, \ 1 \leq j \leq k;$$
$$f^*(v_jv_{j,n}) = 2(n+1)(j-1) + 4, \ 1 \leq j \leq k;$$
$$f^*(v_jv_{j,r}) = 2(n+1)(j-1)+2r+4, \ 1 \leq j \leq k-1, \ 1 \leq r \leq n-1;$$
$$f^*(v_kv_{k,r}) = 2(n+1)(k-1) + 2r + 4, \ 1 \leq r \leq n-2;$$
$$f^*(v_kv_{k,n-1}) = 0.$$

Hence f admits SD-harmonious labeling for $BT(n, n, n, \cdots, n)$. □

Example 2.5. *A SD-harmonious labeling of banana tree $BT(6, 6, 6)$ is shown in Figure 1.*

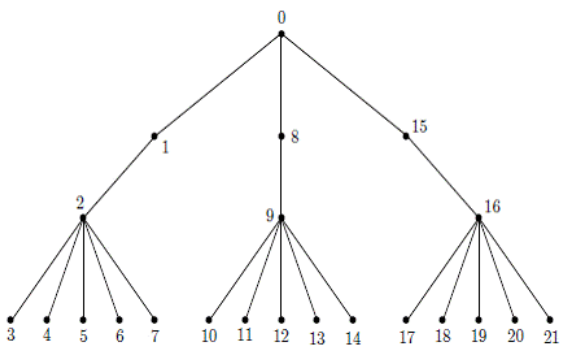

Figure 1: SD-harmonious labeling of banana tree $BT(6, 6, 6)$

Theorem 2.6. *Every full binary tree admits SD-harmonious labeling.*

Proof. We note that every full binary tree has odd number of vertices and hence has even number of edges. Let T be a full binary tree and let v_0 be a root of T which is called zero level vertex. Clearly, the i^{th} level of T has 2^i vertices. If T has m levels, then the number of vertices of T is $2^{m+1}-1$ and the number of edges is $2^{m+1}-2$. Now, assign the label 0 to the root v_0 and assign the labels 1 and 2 to the first level vertices. Next, we assign the labels $2^i-1, 2^i, \cdots, 2^{i+1}$ to the i^{th} level vertices for $2 \leq i \leq m$. It can be easily verified that f admits SD-harmonious labeling for full binary tree. □

Theorem 2.7. *The graph $P_n(m_1, m_2, \cdots, m_n)$ admits SD-harmonious labeling.*

Proof. Let $V(P_n(m_1, m_2, \cdots, m_n)) = \{u_i : 1 \leq i \leq n\} \cup \{v_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$ and $E(P_n(m_1, m_2, \cdots, m_n)) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$. Therefore, $P_n(m_1, m_2, \cdots, m_n)$ is of order $n + m_1 + m_2 + \cdots + m_n$ and size $n + m_1 + m_2 + \cdots + m_n - 1$.
Define $f : V(P_n(m_1, m_2, \cdots, m_n)) \to \{0, 1, 2, \cdots, n + m_1 + m_2 + \cdots + m_n - 1\}$ as follows:

$$f(u_1) = 0;$$
$$f(u_i) = \sum_{l=1}^{i-1} m_l + i - 1,\ 2 \leq i \leq n;$$
$$f(v_{1,j}) = j,\ 1 \leq j \leq m_1;$$
$$f(v_{i,j}) = \sum_{l=1}^{i-1} m_l + i + j - 1,\ 2 \leq i \leq n,\ 1 \leq j \leq m_i.$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = 2\left(\sum_{l=1}^{i} m_l + i\right),\ 1 \leq i \leq n-1;$$
$$f^*(u_1 v_{1,j}) = 2j,\ 1 \leq j \leq m_1;$$
$$f^*(u_i v_{i,j}) = 2\left(\sum_{l=1}^{i-1} m_l + i\right) + 2j - 2,\ 2 \leq i \leq n-1,\ 1 \leq j \leq m_i;$$
$$f^*(u_n v_{n,j}) = 2\left(\sum_{l=1}^{n-1} m_l + n\right) + 2j - 2,\ 1 \leq j \leq m_n - 1;$$
$$f^*(u_n v_{n,m_n}) = 0.$$

Hence f admits SD-harmonious labeling for $P_n(m_1, m_2, \cdots, m_n)$. □

Example 2.8. *A SD-harmonious labeling of $P_5(2, 3, 4, 1, 3)$ is shown in Figure 2.*

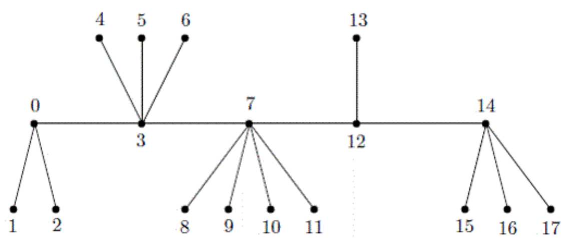

Figure 2: SD-harmonious labeling of $P_5(2, 3, 4, 1, 3)$

Remark 2.9. *In the above Theorem 2.7, if $n = 1$ or 2, we see that both the star graph and bistar graph admit SD-harmonious labeling.*

Corollary 2.10. *The corona $P_n \odot mK_1$ admits SD-harmonious labeling.*

Theorem 2.11. *The graph P_m^{+n} admits SD-harmonious labeling.*

Proof. Let $u_1, u_2, \cdots, u_m$ be the vertices of path P_m. Then $V(P_m^{+n}) = V(P_m) \cup \{v_j : 1 \leq j \leq n\}$ and $E(P_m^{+n}) = E(P_m) \cup \{u_m v_j : 1 \leq j \leq n\}$. Therefore, P_m^{+n} is of order $m+n$ and size $m+n-1$. Define $f : V(P_m^{+n}) \to \{0, 1, 2, \cdots, m+n-1\}$ as follows:

$$f(u_i) = i-1, \ 1 \leq i \leq m;$$
$$f(v_j) = m+j-1, \ 1 \leq j \leq n.$$

The induced edge labels are

$$f^*(u_i u_{i+1}) = 2i, \ 1 \leq i \leq m-1;$$
$$f^*(u_m v_j) = 2(m-1)+2j, \ 1 \leq j \leq n-1;$$
$$f^*(u_m v_n) = 0.$$

Hence f admits SD-harmonious labeling for P_m^{+n}. □

Example 2.12. *A SD-harmonious labeling of P_6^{+5} is shown in Figure 3.*

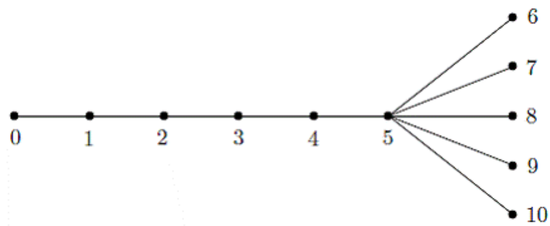

Figure 3: SD-harmonious labeling of P_6^{+5}

Theorem 2.13. *The graph $SP(n_1, n_2, \cdots, n_t)$ admits SD-harmonious labeling.*

Proof. Consider the graph $SP(n_1, n_2, \cdots, n_t)$ which is the one-point union of the t paths. Let the vertices of $SP(n_1, n_2, \cdots, n_t)$ be $v_{j,i}$ for $1 \leq j \leq t$, $1 \leq i \leq n_j$. Therefore, $SP(n_1, n_2, \cdots, n_t)$ is of order $n_1 + n_2 + \cdots + n_t + 1$ and size $n_1 + n_2 + \cdots + n_t$.

Define $f : V(SP(n_1, n_2, \cdots, n_t)) \to \{0, 1, 2, \cdots, n_1 + n_2 + \cdots + n_t\}$ as follows:

$$f(v) = 0 \ ;$$
$$f(v_{1,i}) = i-1, \ 2 \leq i \leq n_1 + 1;$$
$$f(v_{j,i}) = \sum_{l=1}^{j-1} n_l + i - 1, \ 2 \leq i \leq n_j + 1, \ 2 \leq j \leq t.$$

The induced edge labels are

$$f^*(vv_{1,2}) = 2;$$
$$f^*(vv_{j,2}) = 2\left(\sum_{l=1}^{j-1} n_l + 1\right), \ 2 \leq j \leq t;$$

$f^*(v_{1,i}v_{1,i+1}) = 2i,\ 2 \leq i \leq n_1 - 1;$
$f^*(v_{j,i}v_{j,i+1}) = 2\left(\sum_{l=1}^{j-1} n_l + i\right),\ 2 \leq i \leq n_j - 1,\ 2 \leq j \leq t;$
$f^*(v_{t,n_t}v_{t,n_t+1}) = 0.$
Hence f admits SD-harmonious labeling for $SP(n_1, n_2, \cdots, n_t)$. □

Example 2.14. *A SD-harmonious labeling of $SP(4, 3, 3, 2, 5)$ is shown in Figure 4.*

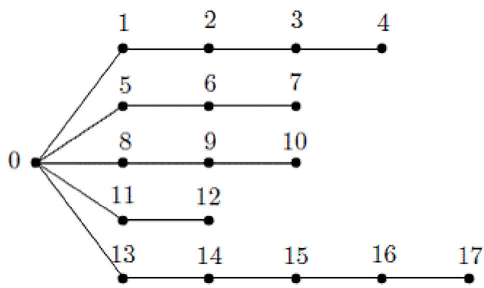

Figure 4: SD-harmonious labeling of $SP(4, 3, 3, 2, 5)$

Theorem 2.15. *The cycle C_n is not SD-harmonious graph.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n. Let $q = n$. Suppose f is SD-harmonious labeling of C_n. To get the edge label 2, there must be the two adjacent vertices v_1 and v_2 with labels 0 and 1 respectively. To get the edge label 0, we must have two adjacent vertices v_1 and v_n with labels 0 and n respectively. We observe that the edges v_1v_n and v_nv_{n-1} both receive the same label 0. This is a contradiction. Hence, the cycle C_n is not SD-harmonious graph. □

3. Star Related Graphs

In this section, we prove that $S(K_{1,n})$, $S(B_{m,n})$, $S(P_n \odot K_1)$, $B(m, n, k)$ and $\left\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, \cdots, K_{1,n}^{(t)}\right\rangle$ admit SD-harmonious labeling.

Theorem 3.1. *The graph $S(K_{1,n})$ admits SD-harmonious labeling.*

Proof. Let $V(S(K_{1,n})) = \{v\} \cup \{v_i, v_i' : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{vv_i', v_i'v_i : 1 \leq i \leq n\}$. Therefore, $S(K_{1,n})$ is of order $2n + 1$ and size $2n$.
Define $f : V(S(K_{1,n})) \rightarrow \{0, 1, 2, \cdots, 2n\}$ as follows:
$f(v) = 2;$
$f(v_i') = 2i - 1,\ 1 \leq i \leq n;$
$f(v_i) = 2i - 2,\ 1 \leq i \leq n.$

The induced edge labels are

$f^*(v_1'v_1) = 2;$
$f^*(vv_1') = 4;$
$f^*(vv_i') = 4i - 2,\ 2 \leq i \leq n;$
$f^*(v_i'v_i) = 4i,\ 2 \leq i \leq n - 1;$
$f^*(v_n'v_n) = 0.$

Hence f admits SD-harmonious labeling for $S(K_{1,n})$. □

Theorem 3.2. *The graph $S(B_{m,n})$ admits SD-harmonious labeling.*

Proof. Let $V(S(B_{m,n})) = \{v, u, w\} \cup \{v_i, v_i' : 1 \leq i \leq m\} \cup \{u_j, u_j' : 1 \leq j \leq n\}$ and $E(S(B_{n,n})) = \{vw, wu\} \cup \{vv_i', v_i'v_i : 1 \leq i \leq m\} \cup \{uu_j', u_j'u_j : 1 \leq j \leq n\}$. Therefore, $S(B_{m,n})$ is of order $2m + 2n + 3$ and size $2m + 2n + 2$.
Define $f : V(S(B_{m,n})) \to \{0, 1, 2, \cdots, 2(m + n + 1)\}$ as follows:

$f(v) = 0;$
$f(w) = 2m + 1;$
$f(u) = 2m + 2;$
$f(v_i') = i,\ 1 \leq i \leq m;$
$f(v_i) = 2m - i + 1,\ 1 \leq i \leq m;$
$f(u_j') = 2m + j + 2,\ 1 \leq j \leq n;$
$f(u_j) = 2m + 2n - j + 3,\ 1 \leq j \leq n.$

The induced edge labels are

$f^*(vw) = 4m + 2;$
$f^*(wu) = 4m + 4;$
$f^*(vv_i') = 2i,\ 1 \leq i \leq m;$
$f^*(v_i'v_i) = 4m + 2 - 2i,\ 1 \leq i \leq m;$
$f^*(uu_j') = 4m + 4 + 2j,\ 1 \leq j \leq n;$
$f^*(u_j'u_j) = 4m + 4n - 2j + 6,\ 2 \leq j \leq n;$
$f^*(u_1'u_1) = 0.$

Hence f admits SD-harmonious labeling for $S(B_{m,n})$. □

Theorem 3.3. *The graph $S(P_n \odot K_1)$ admits SD-harmonious labeling.*

Proof. Let $V(S(P_n \odot K_1)) = \{u_i, v_i, v_i' : 1 \leq i \leq n\} \cup \{u_i' : 1 \leq i \leq n - 1\}$ and $E(S(P_n \odot K_1)) = \{u_iu_i', u_i'u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_iv_i', v_i'v_i : 1 \leq i \leq n\}$. Therefore, $S(P_n \odot K_1)$ is of order $4n - 1$ and size $4n - 2$.
Define $f : V(S(P_n \odot K_1)) \to \{0, 1, 2, \cdots, 4n - 2\}$ as follows:

$f(u_i) = 4(i - 1),\ 1 \leq i \leq n;$
$f(u_i') = 4i - 1,\ 1 \leq i \leq n - 1;$
$f(v_i) = 4i - 2,\ 1 \leq i \leq n;$
$f(v_i') = 4i - 3,\ 1 \leq i \leq n.$

The induced edge labels are

$f^*(u_iu_i') = 8i - 2,\ 1 \leq i \leq n - 1;$
$f^*(u_i'u_{i+1}) = 8i,\ 1 \leq i \leq n - 1;$
$f^*(u_iv_i') = 8i - 6,\ 1 \leq i \leq n;$

$f^*(v_i'v_i) = 8i - 4,\ 1 \le i \le n-1;$
$f^*(v_n'v_n) = 0.$

Hence f admits SD-harmonious labeling for $S(P_n \odot K_1)$. □

Theorem 3.4. *The graph $G = \left\langle K_{1,n}^{(1)}, K_{1,n}^{(2)}, \cdots, K_{1,n}^{(t)} \right\rangle$ admits SD-harmonious labeling.*

Proof. Let $v_1^{(i)}, v_2^{(i)}, \cdots, v_n^{(i)}$ be the pendant vertices of $K_{1,n}^{(i)}$ and let x_i be the apex vertex of $K_{1,n}^{(i)}$ for $i = 1, 2, \cdots, t$. Now w_i is adjacent to x_i and x_i is adjacent to w_{i+1} for $1 \le i \le t-1$. Therefore, G is of order $t(n+2)-1$ and size $t(n+2)-2$.
Define $f : V(G) \to \{0, 1, 2, \cdots, t(n+2)-2\}$ as follows:

$f(w_i) = (i-1)(n+2),\ 1 \le i \le t;$
$f(x_i) = i(n+2) - 1,\ 1 \le i \le t-1;$
$f(v_j^{(i)}) = (i-1)(n+2) + j,\ 1 \le i \le t,\ 1 \le j \le n.$

The induced edge labels are

$f^*(x_iw_i) = 2i(n+2) - 2,\ 1 \le i \le t-1;$
$f^*(x_iw_{i+1}) = 2i(n+2),\ 1 \le i \le t-1;$
$f^*(w_iv_j^{(i)}) = 2(i-1)(n+2) + 2j,\ 1 \le i \le t,\ 1 \le j \le n-1;$
$f^*(w_tv_n^{(t)}) = 0.$

Hence f admits SD-harmonious labeling for G. □

Example 3.5. *A SD-harmonious labeling of $\left\langle K_{1,4}^{(1)}, K_{1,4}^{(2)}, K_{1,4}^{(3)} \right\rangle$ is shown in Figure 5.*

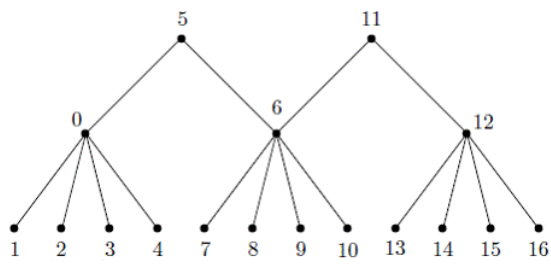

Figure 5: SD-harmonious labeling of $\left\langle K_{1,4}^{(1)}, K_{1,4}^{(2)}, K_{1,4}^{(3)} \right\rangle$

Theorem 3.6. *The graph $B(m, n, k)$ admits SD-harmonious labeling.*

Proof. Let $u_1, u_2, \cdots, u_m$ be the vertices adjacent to v_0 and $w_1, w_2, \cdots, w_n$ be another set of vertices adjacent to v_k. Let v_0 and v_k be the end vertices of a path P_k. Therefore, $B(m, n, k)$ is of order $m + n + k + 1$ and size $m + n + k$.
Define $f : V(B(m, n, k)) \to \{0, 1, 2, \cdots, m + n + k\}$ as follows:

$f(u_i) = i,\ 1 \le i \le m;$
$f(v_0) = 0;$
$f(v_j) = m + j,\ 1 \le j \le k;$
$f(w_r) = m + k + r,\ 1 \le r \le n.$

The induced edge labels are

$f^*(v_0 u_i) = 2i,\ 1 \le i \le m;$
$f^*(v_{j-1} v_j) = 2m + 2j,\ 1 \le j \le k;$
$f^*(v_k w_r) = 2m + 2k + 2r,\ 1 \le r \le n - 1;$
$f^*(v_k w_n) = 0.$

Hence f admits SD-harmonious labeling for $B(m, n, k)$. □

Example 3.7. *A SD-harmonious labeling of $B(6, 5, 4)$ is shown in Figure 6.*

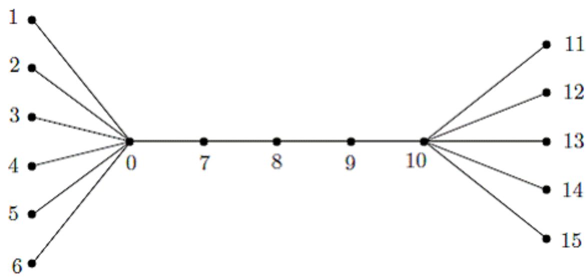

Figure 6: SD-harmonious labeling of $B(6, 5, 4)$

4. Disjoint Union of Graphs

In this section, we prove that $K(1, n_1) \cup K(1, n_2) \cup \cdots \cup K(1, n_t)$, $P_{m_1} \cup P_{m_2} \cup \cdots \cup P_{m_t}$, $G \cup P_n$, $G \cup P_m^{+t}$ and $G \cup K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_t}$ admit SD-harmonious labeling.

Theorem 4.1. *The graph $K(1, n_1) \cup K(1, n_2) \cup \cdots \cup K(1, n_t)$ admits SD-harmonious labeling.*

Proof. Let $G = K(1, n_1) \cup K(1, n_2) \cup \cdots \cup K(1, n_t)$ with $V(G) = \{u_i : 1 \le i \le t\} \cup \{v_{i,j} : 1 \le i \le t,\ 1 \le j \le n_i\}$ and $E(G) = \{u_i v_{i,j} : 1 \le i \le t,\ 1 \le j \le n_i\}$. Therefore, G is of order $t + n_1 + n_2 + \cdots + n_t$ and size $n_1 + n_2 + \cdots + n_t$. Define $f : V(G) \to \{0, 1, 2, \cdots, n_1 + n_2 + \cdots + n_t\}$ as follows:

$f(u_i) = i - 1,\ 1 \le i \le t;$
$f(v_{1,j}) = j,\ 1 \le j \le n_1;$
$f(v_{i+1,j}) = \sum_{l=1}^{i} n_l + j,\ 1 \le i \le t - 1,\ 1 \le j \le n_{i+1}.$

It can be verified that the induced edge labels of G are $0, 2, 4, \cdots, 2(n_1 + n_2 + \cdots + n_t) - 2$. Hence f admits SD-harmonious labeling for $K(1, n_1) \cup K(1, n_2) \cup \cdots \cup K(1, n_t)$. □

Theorem 4.2. *The graph $P_{m_1} \cup P_{m_2} \cup \cdots \cup P_{m_t}$ admits SD-harmonious labeling.*

Proof. Let $G = P_{m_1} \cup P_{m_2} \cup \cdots \cup P_{m_t}$ with $V(G) = \{u_{i,j} : 1 \le i \le t,\ 1 \le j \le m_i\}$ and $E(G) = \{u_{i,j}u_{i,j+1} : 1 \le i \le t,\ 1 \le j \le m_i - 1\}$. Therefore, G is of order $m_1 + m_2 + \cdots + m_t$ and size $m_1 + m_2 + \cdots + m_t - t$.
Define $f : V(G) \to \{0, 1, 2, \cdots, m_1 + m_2 + \cdots + m_t - t\}$ as follows:

$$f(u_{1,j}) = j - 1,\ 1 \le j \le m_1;$$

$$f(u_{i+1,j}) = \sum_{l=1}^{i}(m_l - l) + (j - 1),\ 1 \le i \le t - 1,\ 1 \le j \le m_{i+1}.$$

It can be verified that the induced edge labels of G are $0, 2, 4, \cdots, 2(m_1 + m_2 + \cdots + m_t - t) - 2$. Hence f admits SD-harmonious labeling for $P_{m_1} \cup P_{m_2} \cup \cdots \cup P_{m_t}$. □

Example 4.3. *A SD-harmonious labeling of $P_5 \cup P_4 \cup P_6$ is shown in Figure 7.*

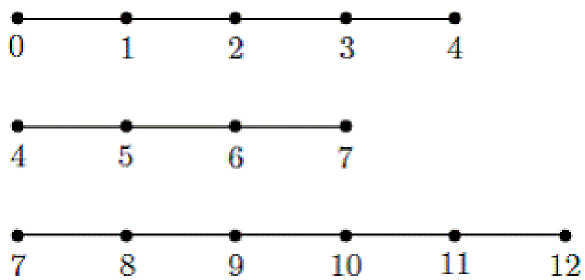

Figure 7: SD-harmonious labeling of $P_5 \cup P_4 \cup P_6$

Theorem 4.4. *Let f be SD-harmonious labeling of graph G of order p and size q. Then $G \cup P_n$ admits SD-harmonious labeling.*

Proof. Let f be SD-harmonious labeling of graph G of order p and size q. Let the labeling of the vertices of G be $0, 1, 2, \cdots, q$. Then the induced edge labels are $0, 2, 4, \cdots, 2q - 2$. Let $u_1, u_2, \cdots, u_n$ be the vertices of path P_n. Let us denote by H the graph obtained by $G \cup P_n$. The labeling $g : V(H) \to \{0, 1, 2, \cdots, q + n - 1\}$ is defined by $g(u_i) = q + i - 1$ for $1 \le i \le n$. It can be verified that the induced edge labels of H are $0, 2, 4, \cdots, 2q + 2n - 4$. Hence g admits SD-harmonious labeling for $G \cup P_n$. □

Theorem 4.5. *Let f be SD-harmonious labeling of graph G of order p and size q. Then $G \cup P_m^{+t}$ admits SD-harmonious labeling.*

Proof. Let f be SD-harmonious labeling of graph G of order p and size q. Let the labeling of the vertices of G be $0, 1, 2, \cdots, q$. Then the induced edge labels are $0, 2, 4, \cdots, 2q - 2$. Let $V(P_m^{+t}) = \cup\{v_i : 1 \le i \le$

$m\} \cup \{w_j : 1 \le j \le t\}$ and $E(P_m^{+t}) = \{v_i v_{i+1} : 1 \le i \le m-1\} \cup \{v_m w_j : 1 \le j \le t\}$. Let us denote by H the graph obtained by $G \cup P_m^{+t}$. We define a labeling $g : V(H) \to \{0, 1, 2, \cdots, q+m+t-1\}$ as follows:

$g(u) = f(u),\ u \in G;$
$g(v_i) = q + i - 1,\ 1 \le i \le m;$
$g(w_j) = q + m - 1 + j,\ 1 \le j \le t.$

It can be verified that the induced edge labels of H are $0, 2, 4, \cdots, 2q + 2m + 2t - 4$. Hence g admits SD-harmonious labeling for $G \cup P_m^{+t}$. □

Theorem 4.6. *Let f be SD-harmonious labeling of graph G of order p and size q. Then $G \cup K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_t}$ admits SD-harmonious labeling.*

Proof. Let f be SD-harmonious labeling of graph G of order p and size q. Let the labeling of the vertices of G be $0, 1, 2, \cdots, q$. Then the induced edge labels are $0, 2, 4, \cdots, 2q-2$. Let $V(K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_t}) = \{v_i, v_{i,j} : 1 \le i \le t,\ 1 \le j \le n_i\}$ and $E(K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_t}) = \{v_i v_{i,j} : 1 \le i \le t,\ 1 \le j \le n_i\}$. Let us denote by H the graph obtained by $G \cup K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_t}$. We define a labeling $g : V(H) \to \{0, 1, 2, \cdots, q + n_1 + \cdots + n_t\}$ as follows:

$g(u) = f(u),\ u \in G;$
$g(v_1) = q;$
$g(v_{i+1}) = q + \sum_{l=1}^{i} n_l,\ 1 \le i \le t-1;$
$g(v_{1,j}) = q + j,\ 1 \le j \le n_1;$
$g(v_{i+1,j}) = q + \sum_{l=1}^{i} n_l + j,\ 1 \le i \le t-1,\ 1 \le j \le n_{i+1}.$

It can be verified that the induced edge labels of H are $0, 2, 4, \cdots, 2(q + n_1 + \cdots + n_t) - 2$. Hence g admits SD-harmonious labeling for $G \cup K_{1,n_1} \cup K_{1,n_2} \cup \cdots \cup K_{1,n_t}$. □

Corollary 4.7. *Let f be SD-harmonious labeling of graph G_1 of order p_1 and size q_1. Let g be SD-harmonious labeling of graph G_2 of order p_2 and size q_2. Then $G_1 \cup G_2$ admits SD-harmonious labeling.*

Proof. Let f be SD-harmonious labeling of graph G_1 of order p_1 and size q_1. Let the labeling of the vertices of G_1 be $0, 1, 2, \cdots, q_1$. Then the induced edge labels are $0, 2, 4, \cdots, 2q_1 - 2$.

Let g be SD-harmonious labeling of graph G_2 of order p_2 and size q_2. Let the labeling of the vertices of G_2 be $0, 1, 2, \cdots, q_2$. Then the induced edge labels are $0, 2, 4, \cdots, 2q_2 - 2$.

We define a labeling $h : V(G_1 \cup G_2) \to \{0, 1, 2, \cdots, q_1 + q_2\}$ by $h(u) = f(u),\ u \in G_1$ and $h(v) = q_1 + g(v),\ v \in G_2$. It can be verified that the induced edge labels of $G_1 \cup G_2$ are $0, 2, 4, \cdots, 2q_1 - 2, 2q_1, 2q_1 + 2, \cdots, 2q_1 + 2q_2 - 2$. Hence h admits SD-harmonious labeling for $G_1 \cup G_2$. □

5. Identifying Vertex of Graphs

In this section, we prove that graphs obtained by identifying a vertex in G_1 and a vertex in G_2 admit SD-harmonious labeling.

Theorem 5.1. *Let f be a SD-harmonious labeling of a graph G of order p and size q. Let $w \in V(G)$ be such that $f(w) = 0$. The graph obtained by identifying a vertex w in G and a vertex of degree n in $K_{1,n}$ admits a SD-harmonious labeling.*

Proof. Let f be a SD-harmonious labeling of a graph G of order p and size q. That is the vertices of G are labeled with numbers $\{0, 1, 2, \cdots, q\}$ and the induced edge labeling is bijective.

Let $w \in V(G)$ be such that $f(w) = 0$. Let us denote by H the graph obtained by identifying a vertex w in G and a vertex of degree n in $K_{1,n}$.

We define a vertex labeling g of H such that

$$g(v) = f(v), \quad v \in V(G);$$
$$g(x_i) = q + i, \quad i = 1, 2, \cdots, n.$$

Thus for the induced edge labeling we get

$$g^*(uv) = f^*(vu), \quad v \in V(G);$$
$$g^*(wx_i) = 2q + 2i, \quad 1 \le i \le n - 1;$$
$$g^*(wx_n) = 0.$$

Then the induced edge labels of $K_{1,n}$ are $2q + 2, 2q + 4, 2q + 6, \cdots, 2q + 2i - 2, 0$. Since the graph G admits a SD-harmonious labeling, the corresponding induced edge labels of G are $2, 4, 6, \cdots, 2q$. Therefore, g is bijective and the induced edge set of H is $\{0, 2, 4, \cdots, 2q - 2i - 2\}$. Hence H admits a SD-harmonious labeling. □

Theorem 5.2. *Let f be a SD-harmonious labeling of a graph G of order p and size q. Let $w \in V(G)$ be such that $f(w) = 0$. The graph obtained by identifying a vertex w in G and a vertex of degree 1 in P_n admits a SD-harmonious labeling.*

Proof. Let f be a SD-harmonious labeling of a graph G of order p and size q. That is the vertices of G are labeled with numbers $\{0, 1, 2, \cdots, q\}$ and the induced edge labeling is bijective.

Let $w \in V(G)$ be such that $f(w) = 0$. Let us denote by H the graph obtained by identifying a vertex w in G and a vertex of degree 1 in P_n.

We define a vertex labeling g of H such that

$$g(v) = f(v), \quad v \in V(G);$$
$$g(x_{i+1}) = q + i, \quad i = 1, 2, \cdots, n - 1.$$

Thus for the induced edge labeling we get

$$g^*(uv) = f^*(vu), \quad v \in V(G);$$
$$g^*(wx_2) = 2q + 2;$$
$$g^*(x_i x_{i+1}) = 2q + 2i, \quad 2 \le i \le n - 2;$$
$$g^*(x_{n-1}x_n) = 0;$$

Then the induced edge labels of P_n are $2q+2, 2q+4, 2q+6, \cdots, 2q+2i-2, 0$. Since the graph G is a SD-harmonious labeling, the corresponding induced edge labels of G are $2, 4, 6, \cdots, 2q$. Therefore, g is bijective and the induced edge set of H is $\{0, 2, 4, \cdots, 2q-2i-2\}$. Hence H admits a SD-harmonious labeling. □

Theorem 5.3. *Let f be a SD-harmonious labeling of a graph G_1 of order p_1 and size q_1. Let g be a SD-harmonious labeling of a graph G_2 of order p_2 and size q_2. Let $x \in V(G_1)$, $y \in V(G_2)$ be such that $f(x) = 0$ and $f(y) = 0$. The graph obtained by identifying a vertex x in G_1 and a vertex y in G_2 admits a SD-harmonious labeling.*

Proof. Let f be a SD-harmonious labeling of a graph G_1 of order p_1 and size q_1. That is the vertices of G_1 are labeled with numbers $\{0, 1, 2, \cdots, q_1\}$ and the induced edge labeling is bijective.

Let g be a SD-harmonious labeling of a graph G_2 of order p_2 and size q_2. That is the vertices of G_1 are labeled with numbers $\{0, 1, 2, \cdots, q_2\}$ and the induced edge labeling is bijective.

Let $x \in V(G_1)$, $y \in V(G_2)$ be such that $f(x) = 0$ and $f(y) = 0$. Let us denote by H the graph obtained by identifying a vertex x in G_1 and a vertex y in G_2.

We define a vertex labeling h of H such that

$$h(x) = 0;$$
$$h(u) = f(u), \quad u \in V(G_1);$$
$$h(w) = g(w) + q_1, \quad w \in V(G_2).$$

Thus for the induced edge labeling we get

$$h^*(uv) = f^*(uv), \quad uv \in V(G_1);$$
$$h^*(xw) = 2q_1 + 2;$$
$$h^*(wz) = g^*(wz) + 2q_1, \quad wz \in V(G_2);$$
$$h^*(wz) = 0 \ (mod\ 2q);$$

Then the induced edge labels of G_1 are $2, 4, 6, \cdots, 2q_1$ and G_2 are $2q_1 + 2, 2q_1 + 4, 2q_1 + 6, \cdots, 2q_1 + 2q_2 - 2, 0$. Therefore, h is bijective and the induced edge set of H is $\{0, 2, 4, \cdots, 2q_1 + 2q_2 - 2\}$. Hence H admits a SD-harmonious labeling. □

6. Conclusion

We investigated SD-harmonious labeling of path related graphs, tree related graphs, star related graphs and disjoint union of graphs. We conclude this paper with the following conjecture.

Conjecture 6.1. *All trees admit SD-harmonious labeling.*

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, MacMillan, New York, 1976.

[2] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs International Symposium, Rome, 1966.

[3] J. A. Gallian, "A Dyamic Survey of Graph Labeling", *The Electronic J. Combin.*, vol. 20, # DS6, 2017.

[4] R. L. Graham and N. J. A. Sloane, "On additive bases and harmonious graphs", *SIAM J. Algebr. Discrete Methods*, vol. 1, pp. 382-404, 1980.

[5] G. C. Lau and W. C. Shiu, "On SD-prime labeling of graphs", *Utilitas Math., (accepted)*.

[6] G. C. Lau, W. C. Shiu, H. K. Ng, C. D. Ng and P. Jeyanthi, "Further results on SD-prime labeling", *JCMCC*, vol. 98, pp. 151-170, 2016.

[7] G. C. Lau, W. C. Shiu and H. K. Ng, "Further results on super graceful labeling of graphs", *AKCE International Journal of Graphs and Combinatorics*, vol. 13, pp. 200-209, 2016.

[8] M. A. Perumal, S. Navaneethakrishnan, S. Arockiaraj and A. Nagarajan, "Super graceful labeling for some special graphs", *Int. J. Res. Rev. Appl. Sci.*, vol. 9, no. 3, pp. 382-404, 2011.

[9] D. Stewart, *Even harmonious labelings of disconnected graphs* (Master's thesis), University of Minnesota Duluth, 2015.

[10] V. Swaminathan and P. Jeyanthi, "Super edge-magic strength of fire cracker, banana trees and unicycle graphs", *Discrete Mathematics*, vol. 306 pp. 1624-1636, 2006.