Mapana Journal of Sciences
2018, Vol. 17, No. 3, 17–28
ISSN 0975-3303 | https://doi.org/10.12723/mjs.46.2

Odd-Even Sum Labeling in the Context of Duplication of Graph Elements

K. Monika* and K. Murugan†

Abstract

In this paper, odd-even sum labeling of the graphs obtained by duplication of graph elements of star graphs and path graphs are studied.

Keywords: odd-even sum graph, odd-even sum labeling

Mathematics Subject Classification (2010): 05C78

1. Introduction

We study finite, undirected and non-trivial graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The elements $V(G)$ and $E(G)$ are commonly termed as graph elements. Throughout this paper, P_n denotes the path on n vertices. Star $K_{1,n}$ is a graph with a vertex of degree n called apex and n vertices of degree one called pendent vertices $|V(G)|$ and $|E(G)|$ denote the cardinality of the vertex set and edge set respectively. For various graph theoretic notation and terminology, [2] is followed.

If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling. Harary [3] introduced the notion of a sum graph. A graph $G = (V, E)$ is called a sum graph if there is an bijection f from V to a set of $+ve$ integers S such that $xy \in E$ if and only if $(f(x) + f(y)) \in S$. In 1991, Harary [3] defined a real sum graph. S. Arockiaraj *et al.* introduced odd sum graph [1]. Monika and Murugan [7] introduced odd-even sum graph. Odd-even sum graph Labeling of some other graphs are in [6].

The following definitions are used in the subsequent discussion.

*Department of Mathematics, M. S. U. College, Govindaperi, Tirunelveli; kmonikalyani@gmail.com

†Department of Mathematics, The M. D. T. Hindu College (Manonmaniam Sundaranar University), Tirunelveli; muruganmdt@gmail.com

Received: March 2017. Reviewed: April 2017

Definition 1.1. [7] *A (p, q) graph $G = (V, E)$ is said to be an odd–even sum graph if there exists an injective function $f : V(G) \to \{\pm 1, \pm 3 \pm 5, ..., \pm(2p-1)\}$ such that the induced mapping $f^* : E(G) \to \{2, 4, 6, ..., 2q\}$ defined by $f^*(uv) = f(u) + f(v)\ \forall\ uv \in E(G)$ is bijective.*

The function f is called an odd-even sum labeling of G. A graph which admits odd-even sum labeling is called an odd-even sum graph.

Definition 1.2. [8] *Duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G.*

Definition 1.3. [8] *Duplication of a vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produces a new graph G' such that $N(v'_k) = \left\{v_k, v''_k\right\}$ and $N(v''_k) = \left\{v_k, v'_k\right\}$.*

Definition 1.4. [8] *Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.*

Definition 1.5. [8] *Duplication of an edge $e = uv$ of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \left\{v'\right\} - \{v\}$ and $N(v') = N(v) \cup \left\{u'\right\} - \{u\}$*

2. Labelings Associated with Star

Theorem 2.1. *The graph obtained by duplicating the apex vertex of the star $K_{1,n} (n \geq 1)$ is an odd-even sum graph.*

Proof. Let $v, v_1, v_2, ..., v_n$ be the vertices of $K_{1,n}$.

Let G be the graph obtained by duplicating the apex vertex v by v'

Let $V(G) = \{v, v_i, v' / 1 \leq i \leq n\}$; and $E(G) = \{vv_i, v'v_i / 1 \leq i \leq n\}$

Then $|V(G)| = n + 2, |E(G)| = 2n$

Let $f : V(G) \to \{\pm 1, \pm 3 \pm 5, ..., \pm(2n + 3)\}$be defined as follows

$f(v) = 2n + 3$

$f(v') = 2n + 1$

$f(v_i) = 2n + 1 - 4i, 1 \leq i \leq n$

Let f^* be the induced edge labeling of f.

Then $f^*(vv_{n-i}) = 4 + 4i, 0 \leq i < n$

$f^*(v'v_{n-i}) = 2 + 4i, 0 \leq i < n$

The induced edge labels are $2, 4, 6, \ldots, 4n$ which are all distinct.Hence G is an odd-even sum graph. □

Example 2.2. *Odd-even sum labeling of the graph obtained by duplicating the apex vertex of $K_{1,5}$ is shown in Figure 1.*

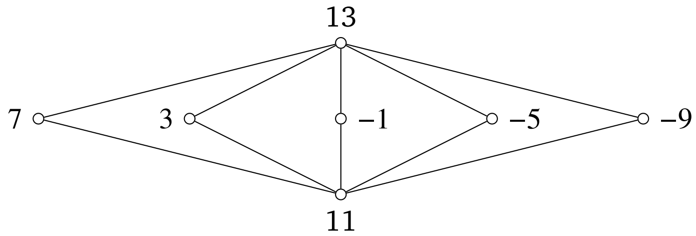

Figure 1: Duplicating the apex vertex of $K_{1,5}$

Theorem 2.3. *The graph obtained by duplicating all the pendent vertices of the star $K_{1,n}(n \geq 1)$ is an odd-even sum graph.*

Proof. Let u be the apex and $v_1, v_2, ..., v_n$ be the pendent vertices of the star $K_{1,n}$
Let G be the graph obtained by duplicating all the pendent vertices $v_1, v_2, ..., v_n$ by $v_1^{'}, v_2^{'}, ..., v_n^{'}$.
Let $V(G) = \{u, v_i, v_i^{'}/1 \leq i \leq n\}$ and $E(G) = \{uv_i, uv_i^{'}/1 \leq i \leq n\}$
Then $|V(G)| = 2n + 1$ *and* $|E(G)| = 2n$
Let $f : V(G) \rightarrow \{\pm 1, \pm 3, ..., \pm(4n + 1)\}$ be defined as follows
$f(u) = 4n + 1$
$f(v_{i+1}) = -1 - 2i; 0 \leq i < n$
$f(v_i^{'}) = f(v_n) - 2i; 1 \leq i \leq n$
Let f^* be the induced edge labeling of f
Then, $f^*(uv_i) = 4n - 2i; 0 \leq i < n$
$f^*(uv_i^{'}) = f^*(uv_n) - 2i; 1 \leq i \leq n$
The induced edge labels are $2, 4, 6, ..., 4n$
Hence the graph G is an odd-even sum graph.

□

Example 2.4. *See Figure 2.*

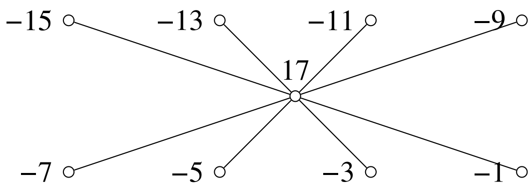

Figure 2: Duplicating all the pendent vertices of the star $K_{1,4}$

Theorem 2.5. *The graph obtained by duplicating the apex vertex of the star $K_{1,n}(n \geq 1)$ by an edge is an odd-even sum graph.*

Proof. Let $v, v_1, v_2, ..., v_n$ be the vertices of $K_{1,n}$.
Let G be the graph obtained by duplicating v by edge uw
Let $V(G) = \{v, v_i, u, w/1 \leq i \leq n\}$;
Let $E(G) = \{vv_i, uw, uv, vw/1 \leq i \leq n\}$
Then $|V(G)| = n + 3, |E(G)| = n + 3$
Let $f : V(G) \to \{\pm 1, \pm 3 \pm 5, ..., \pm 2n + 5\}$be defined as follows:
$f(u) = 3$;
$f(w) = 1$;
$f(v) = 2n + 3$;
$f(v_{i+1}) = -1 - 2i, 0 \leq i \leq n - 2$;
$f(v_n) = f(v_{n-1}) + 4$.
Let f^* be the induced edge labeling of f.
Then $f^*(uw) = 4$
$f^*(uv) = 2n + 6$
$f^*(vw) = 2n + 4$
$f^*(vv_{i+1}) = 2n + 2 - 2i; 0 \leq i \leq n - 2$
$f^*(vv_n) = 2$
The induced edge labels are $2, 4, 6, ..., 2n + 6$ which are all distinct
Hence G is an odd-even sum graph.

□

Example 2.6. *See Figure 3.*

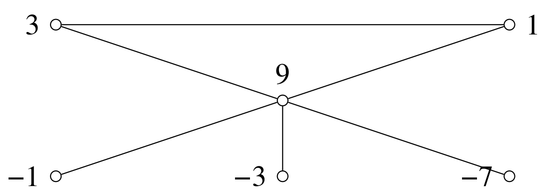

Figure 3: Duplicating the apex vertex of the star $K_{1,3}$

Corollary 2.7. *The graph obtained by duplicating any one of the edges of $K_{1,n}$ by a vertex is $K_{1,n+1}$ which is obviously odd-even sum.*

3. Labelings Associated with Path

Theorem 3.1. *The graph obtained by duplicating of a vertex of the path $P_n(n \geq 2)$ is an odd-even sum graph.*

Proof. Let $v_1, v_2, ..., v_n$ be the consecutive vertices of the path P_n and G be the graph obtained by duplication of the vertex v_i by a new vertex v_i'.
Depending upon the $deg(v_i)$ we have the following cases:

Case(i) $n = 2$
The graph obtained by duplicating a vertex is $K_{1,2}$
Which is obviously odd-even sum.
Case(ii)
If $deg(v_i) = 1$ then v_i is either v_1 or v_n
Without loss of generality, let $v_i = v_1$
Then $|V(G)| = n + 1$ and $|E(G)| = n$
Let $f : V(G) \rightarrow \{\pm 1, \pm 3 \pm 5, ..., \pm(2n + 1)\}$be defined as follows
Let $f(v_1^{'}) = 3$
Subcase(a): Let 'n' be odd

$$f(v_{2i+1}) = 2n + 1 - 2i, 0 \leq i < \frac{n+1}{2}$$

$$f(v_{2(i+1)}) = -1 - 2i, 0 \leq i < \frac{n-1}{2}$$

Subcase(b): Let 'n' be even

$$f(v_{2i+1}) = 2n + 1 - 2i, 0 \leq i < \frac{n}{2}$$

$$f(v_{2(i+1)}) = -1 - 2i, 0 \leq i < \frac{n}{2}$$

Let f^* be the induced edge labeling of f.
Then $f^*(v_1^{'}v_2) = 2$
$f^*(v_{i+1}v_{i+2}) = 2n - 2i, 0 \leq i < n - 1$
Case(iii)
If $deg(v_i) \neq 1$ then $i = \{2, 3, ..., n - 2, n - 1\}$
Then $|V(G)| = n + 1$ and $|E(G)| = n$
Let $f : V(G) \rightarrow \{\pm 1, \pm 3 \pm 5, ..., \pm(2n + 1)\}$be defined as follows
Subcase(a):Let 'n' be odd
subsubcase(i): when i is odd

$v_1 = -n + 2$

$v_2 = n + 4$

$$v_{2i+1} = v_1 + 2i, 1 \leq i \leq \frac{n-1}{2}$$

$$v_{2i+2} = v_2 + 2i, 1 \leq i < \frac{n-1}{2}$$

$v_i^{'} = -n - (i - 1)$

Subsubcase(ii): when i is even

$v_1 = -n + 2$

$v_2 = n + 4$

$$v_{2i+1} = v_1 + 2i, 1 \leq i \leq \frac{n-1}{2}$$

$$v_{2i+2} = v_2 + 2i, 1 \leq i < \frac{n-1}{2}$$

$v_i^{'} = v_2 - (i + 2)$

Subcase(b):Let 'n' be even
Subsubcase(i): when i is odd

$v_1 = n + 3$

$v_2 = -n + 3$

$v_{2i+1} = v_1 + 2i, 1 \leq i < \frac{n}{2}$
$v_{2i+2} = v_2 + 2i, 1 \leq i < \frac{n}{2}$
$v_i^{'} = v_i - (i+1)$
Subsubcase(ii): when i is even
$v_1 = n + 3$
$v_2 = -n + 3$
$v_{2i+1} = v_1 + 2i, 1 \leq i < \frac{n}{2}$
$v_{2i+2} = v_2 + 2i, 1 \leq i < \frac{n}{2}$
$v_i^{'} = -n - (i-1)$
Let f^* be the induced edge labeling of f.
Then $f^*(v_i^{'} v_{i-1}) = 2$
$f^*(v_i^{'} v_{i+1}) = 4$
$f^*(v_i v_{i+1}) = 4 + 2i, 1 \leq i \leq n-1$
The induced edge labels are $2, 4, 6, ..., 2n+2$ which are all distinct
Hence G is an odd-even sum graph.

□

Example 3.2. *See Figure 4.*

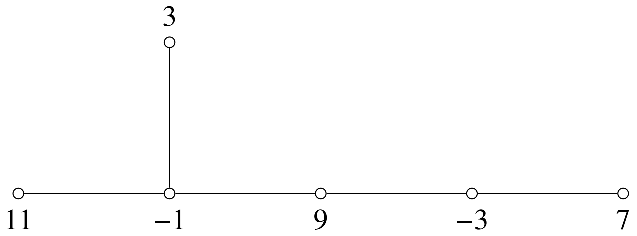

Figure 4: Duplicating of a vertex of the path P_5

Example 3.3. *See Figure 5.*

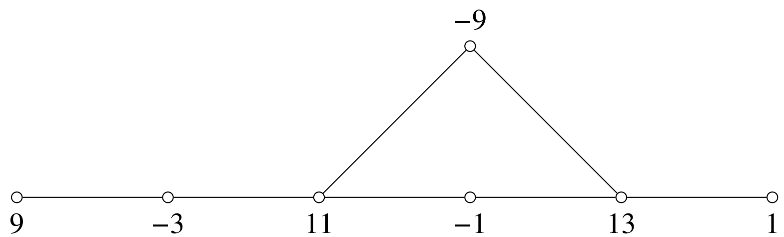

Figure 5: Duplicating of a vertex of the path P_6

Proposition 3.4. *The graph obtained by duplication of an edge by a vertex of the path $P_n(n \geq 3)$ is an odd-even sum graph.*

Proof. Let $v_1, v_2, ..., v_n$ be the consecutive vertices of the path P_n and G be the graph obtained by duplication of the edge $v_i v_{i+1}$ by a new vertex $v_i^{'}$

Case(i)

Without loss of generality duplicating the pendent edge $v_1 v_2$ by $v_i^{'}$

Then $|V(G)| = n + 1$ and $|E(G)| = n + 1$

Let $f : V(G) \rightarrow \{\pm 1, \pm 3 \pm 5, ..., \pm(2n + 1)\}$be defined as follows

Let $f(v_1^{'}) = 3$

Subcase(a): Let 'n' be odd

$f(v_{2i+1}) = 1 - 2i, 0 \leq i < \dfrac{n-1}{2}$

$f(v_{2i+2}) = 2n - 1 - 2i, 0 \leq i < \dfrac{n-1}{2}$

$f(v_n) = f(v_{n-2}) - 4$

Subcase(b): Let 'n' be even

$f(v_{2i+1}) = 1 - 2i, 0 \leq i < \dfrac{n}{2}$

$f(v_{2i+2}) = 2n - 1 - 2i, 0 \leq i < \dfrac{n-2}{2}$

$f(v_n) = f(v_{n-2}) - 4$

Let f^* be the induced edge labeling of f.

Then $f^*(v_1^{'} v_1) = 4$

$f^*(v_1^{'} v_2) = 2n + 2$

$f^*(v_i v_{i+1}) = 2n + 2 - 2i, 1 \leq i < n - 1$

$f^*(v_{n-1} v_n) = 2$

Case(ii)

If duplicating internal edge of the graph Then $|V(G)| = n + 1$ and $|E(G)| = n + 1$

Let $f : V(G) \rightarrow \{\pm 1, \pm 3 \pm 5, ..., \pm(2n + 1)\}$be defined as follows

Subcase(a):Let 'n' be odd

Subsubcase(i): when i is odd

$v_i^{'} = i + 2$

$v_1 = -(2n - 3)$

$v_{2i+1} = 3 - 2i, 1 \leq i < \dfrac{n-1}{2}$

$v_{2i+2} = (2n - 1) - 2i, 0 \leq i < \dfrac{n-1}{2}$

$v_n = -(n - 2)$

Subsubcase(ii): when i is even

$v_i^{'} = i + 1$

$v_1 = -(2n - 3)$

$v_{2i+1} = 3 - 2i, 1 \leq i \leq \dfrac{n-1}{2}$

$v_{2i+2} = (2n - 1) - 2i, 0 \leq i < \dfrac{n-1}{2}$

Subcase(b):Let 'n' be even
Subsubcase(i): when i is odd
$v_i^{'} = i$
$v_1 = -(2n-3)$
$v_{2i+1} = 3 - 2i, 1 \leq i < \frac{n}{2}$
$v_{2i+2} = (2n-1) - 2i, 0 \leq i < \frac{n}{2}$
Subsubcase(ii):when i is even
$v_i^{'} = i + 1$
$v_1 = -(2n-3)$
$v_{2i+1} = 3 - 2i, 1 \leq i < \frac{n}{2}$
$v_{2i+2} = 2n - 1 - 2i, 0 \leq i < \frac{n}{2}$
Let f^* be the induced edge labeling of f.
Then when n is odd and i is odd $f^*(v_1v_2) = 2$
$f^*(v_{n-1}v_n) = 4$
$f^*(v_i^{'}v_i) = 6$
$f^*(v_i^{'}v_{i+1}) = 2n + 2$
$f^*(v_{i+1}v_{i+2}) = 2n + 2 - 2i, 1 \leq i < n - 2$
when i is even $f^*(v_1v_2) = 2$
$f^*(v_i^{'}v_i) = 2n + 2$
$f^*(v_i^{'}v_{i+1}) = 4$
$f^*(v_{i+1}v_{i+2}) = 2n + 2 - 2i, 1 \leq i < n - 2$
When n is odd and i is even
$f^*(v_1v_2) = 4$
$f^*(v_i^{'}v_i) = 2$
$f^*(v_i^{'}v_{i+1}) = 2n + 2$
$f^*(v_{i+1}v_{i+2}) = 2n + 2 - 2i, 1 \leq i < n - 2$

The induced edge labels are $2, 4, 6, ..., 2n + 2$ which are all distinct Hence G is an odd-even sum graph.

□

Example 3.5. *See Figure 6.*

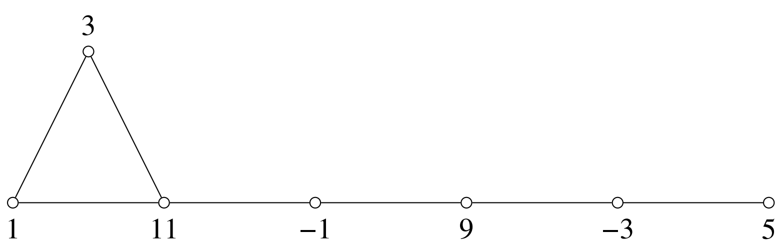

Figure 6: Duplicating of an edge by a vertex of the path P_6

Theorem 3.6. *The graph obtained by duplicating of a pendant vertex by an edge of the path $P_n (n \geq 2)$ is an odd-even sum graph.*

Proof. Let $v_1, v_2, ..., v_n$ be the consecutive vertices of the path P_n
Without loss of generality,G be the graph obtained by duplication of the vertex v_1 by an edge $v_1^{'} v_1^{''}$
Then G contains a cycle C_3 whose vertices are $v_1, v_1^{'}, v_1^{''}$
Then $|V(G)| = n + 2$ and $|E(G)| = n + 2$
Let $f : V(G) \to \{\pm 1, \pm 3 \pm 5, ..., \pm(2n + 3)\}$be defined as follows
Case(i): When $n = 2$

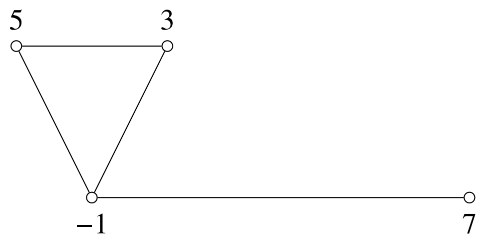

Figure 7: Duplicating of a pendant vertex by an edge of the path P_2

Case(ii): Let 'n' be odd
$f(v_1^{'}) = 2n + 1$
$f(v_1^{''}) = 3$
$f(v_1) = 1$
$f(v_{2i+1}) = f(v_1) - 2i, 1 \leq i < \dfrac{n-1}{2}$
$f(v_{2i}) = f(v_1^{'}) - 2i, 1 \leq i \leq \dfrac{n-1}{2}$
$f(v_n) = f(v_{n-2}) - 4$
Case(iii): Let 'n' be even
$f(v_1^{'}) = 2n + 1$
$f(v_1^{''}) = 3$
$f(v_1) = 1$
$f(v_{2i+1}) = f(v_1) - 2i, 1 \leq i < \dfrac{n}{2}$
$f(v_{2i}) = f(v_1^{'}) - 2i, 1 \leq i < \dfrac{n}{2}$
$f(v_n) = f(v_{n-2}) - 4$
Let f^* be the induced edge labeling of f.
Then $f^*(v_1^{'} v_1^{''}) = 2n + 4$
$f^*(v_1^{'} v_1) = 2n + 2$
$f^*(v_1 v_1^{''}) = 4$
$f^*(v_i v_{i+1}) = 2n + 2 - 2i, 1 \leq i < n - 1$
$f^*(v_{n-1} v_n) = 2$
The induced edge labels are $2, 4, 6, ..., 2n + 4$ which are all distinct
Hence G is an odd-even sum graph. □

Example 3.7. *See Figure 8.*

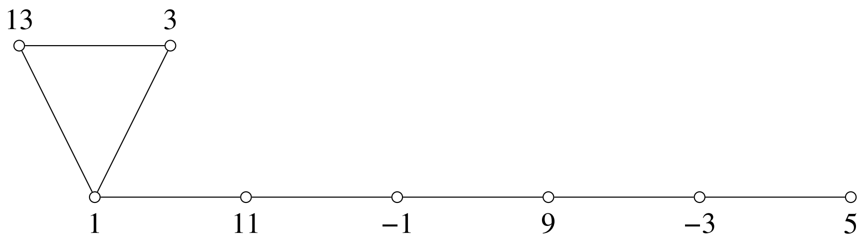

Figure 8: Duplicating of a pendant vertex by an edge of the path P_6

Theorem 3.8. *The graph obtained by duplicating a pendent edge of path $P_n (n \geq 2)$ by edge is an odd-even sum graph.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of the path P_n
Without loss of generality, let G be the graph obtained by duplicating a pendent edge v_1v_2 by $v_1^{'}v_2^{'}$
Let$V(G) = \{v_i, v_1^{'}, v_2^{'}/1 \leq i \leq n\}$ and
$E(G) = \{v_iv_{i+1}, v_1^{'}v_2^{'}, v_2^{'}v_3/1 \leq i < n\}$
Then $|V(G)| = n + 2 \quad and \quad |E(G)| = n + 1$
Let $f : V(G) \rightarrow \{\pm1, \pm3, ..., \pm(2n + 3)\}$ be defined as follows
Case(i): When $n = 2$ The graph obtained is $2P_2$ which is odd-even

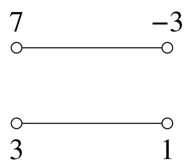

Figure 9: Duplicating a pendent edge of a path P_2

sum
Case(ii): When $n \geq 3$ is odd
For n=3 $f(v_2) = 7$
$f(v_1^{'}) = -7$
$f(v_2^{'}) = 9$
$f(v_1) = -3$
$f(v_3) = -5$
For $n > 3$ $f(v_2) = n + 4$
$f(v_1^{'}) = -(n + 4)$
$f(v_2^{'}) = n + 6$
$f(v_{2i+1}) = -(n - 2i); 0 \leq i < \frac{n+1}{2}$

$f(v_{2i+2}) = n + 6 + 2i; 1 \leq i < \dfrac{n-1}{2}$
Case(iii): If $n \geq 4$ is even
$f(v_2) = -(n+1);$
$f(v_1') = n + 1;$
$f(v_2') = -(n-1);$
$f(v_{2i+1}) = n + 5 + 2i; 0 \leq i < \dfrac{n}{2};$
$f(v_{2i+2}) = f(v_2') - 2i; 1 \leq i < \dfrac{n}{2};$
Let f^* be the induced edge labeling of f.
Then, $f^*(v_1'v_2') = 2;$
$f^*(v_1v_2) = 4;$
$f^*(v_2v_3) = 6;$
$f^*(v_3v_2') = 8;$
$f^*(v_iv_{i+1}) = 2i + 4; 3 \leq i \leq n-1.$
The induced edge labels are $2, 4, 6, ..., 2n+2$
Hence the graph G is an odd-even sum graph.

□

Example 3.9. *See Figure 10.*

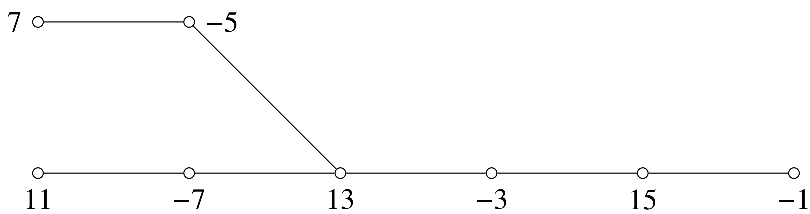

Figure 10: Duplicating a pendent edge of path P_6

Theorem 3.10. *C_3 is not odd-even sum graph.*

Proof. Suppose, C_3 is an odd-even sum.
To get an odd-even sum labeling for C_3 the following are impossible.
All the three labels are positive.
All the three labels are negative.
Any two of the labels are negative.
So, the only possiblity is one negative label and two positive labels.
The two maximum positive labels(ie,5 and 3) can't be assigned.
similarly, two minimum positive labels (ie, 3 and 1) can't be assigned.
So the possible labels are 1 and 5.
Case(i) if $x = -1$ (or) -5
Then we get an edge label 0.
Case(ii) if $x = -3$
Then we get an edge label -2.
In both the cases we get a contradiction. □

4. Conclusion

Similar works can be done on this topic. Authors can attempt on characterzing the graphs which are not Odd-Even sum.

Acknowledgments

The authors are thankful to the anonymous referee for the valuable comments and suggestions which improved the quality of the paper.

References

[1] S. Arockiaraj, P. Mahalakshmi and P. Namasivayam, “Odd Sum Labeling Of Some Subdivision Graphs,” *Kragujevac Journal of Mathematics*, vol. 38, no. 1, pp. 203-222, 2014.

[2] F. Harary, *Graph Theory*. Narosa Publishing House, New Delhi, 2001.

[3] F. Harary, ”Sum Graphs and Difference graphs,” *Congr. Numer.*, vol.72, pp.101-108, 1990.

[4] F. Harary, ”Sum Graphs over all the integers,” *Discrete Math.*, vol. 124, pp.99-105, 1994.

[5] J. A. Gallian, “A Dynamic Survey of Graph Labeling,” *The Electronic Journal of Combinatorics*, vol. 15, DS6, 2008

[6] K. Monika and K. Murugan, “Further odd-even sum labeling Graphs,” *International Journal of Mathematics and its applications*, vol. 5, no. 3-A, pp.33-37, 2017.

[7] K. Monika and K. Murugan, “Odd-even sum labeling of some Graphs,” *International Journal of Mathematics and Soft Computing*, vol. 7, no.1, 2017.

[8] S. K. Vaidya and U. M. Prajapati, “Prime labeling in the context of duplication of graph elements,” *International Journal of Mathematics and Soft Computing*, vol. 3, no. 1, pp.13-20, 2013.