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Some Numerical Examples on the Stability of Fractional Linear Dynamical Systems

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Abstract

The concept of stability of a class of fractional-order linear system is considered in this paper. Existing sufficient conditions are assumed to guarantee the stability of linear models with the Caputo fractional derivatives. The results have been developed by using the concept of Laplace transform, and approximations of Mittag-Leffler. Furthermore, results concerning asymptotical stability of linear fractional-order models are also achieved. The proposed method is based upon Eigen values and the characteristic polynomials. Numerical illustrations are specified to exhibit effectiveness of the proposed method.

Keywords: Fractional-order, Mittag-Leffler function, stability, linear systems

Mathematics Subject Classification (2010): 05C10

1. Introduction

Fractional Differential Equation (FDE) is responsible for a mathematical model for many systems in different fields such as control systems, population dynamics, physical, biological, chemical kinetics, and so forth. Specific depiction of the real life phenomena can be enriched by FDE. Deficient in the application background and complexity of fractional system fails to attract much consideration. In recent times, it has been established to be valuable tools.

Stability of fractional linear autonomous dynamical systems has been considered and analyzed by different methods such as stability theorem [2] to guarantee stability of the systems through the lo-

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cation in the complex plane of the Eigen values and Linear Matrix Inequality (LMI), Fractional Lyapunov direct method, finite time stability [6] and Mittag-Leffler Stability [5]. The stability problem for fractional nonlinear systems has been studied in prior works but it remains open.

Nonlinear Fractional system can be solved by numerical methods. Several numerical methods are available to analyze the nonlinear system such as Homotopy Perturbation Method [4, 12], variation Iteration Methods [3] , Euler algorithm [7, 9] . In this work linear fractional system has been solved by a numerical method proposed by Momani and Odibat [9].

The organization of this paper is as follows. In the next section, preliminary concepts and basic definitions have been specified. Section 3 summarizes the prevailing concept of analyzing stability theory. In Section 4, some examples were given and solutions plotted to illustrate the considered theory. Also a conclusion is given in Section 6.

2. Preliminaries

Some basic definitions and some results are given in this section.

Definition 2.1. [8]**(Riemann - Liouville Fractional Integral).** *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in L^1(\mathcal{R}^+)$ is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \tag{1}$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. [8]**(Riemann - Liouville Fractional Derivative).** *The Riemann-Liouville fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in N$, is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\mathrm{d}s, \tag{2}$$

where D^n is the ordinary differential operator and the function $f(t)$ has absolutely continuous derivative upto order $(n-1)$.

Definition 2.3. [8]**(Caputo Fractional Derivative).** *The Caputo fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in N$, is defined as*

$${}^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)\mathrm{d}s, \tag{3}$$

where the function $f(t)$ has absolutely continuous derivative upto order $(n-1)$.

Definition 2.4. [8]**Gamma function.** *The Gamma function is defined as*

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\mathrm{dx}, n > 0 \tag{4}$$

Definition 2.5. [8]**Mittag-Leffler function** *The one-parameter Mittag-Leffler function is defined as*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad (\alpha > 0, z \in \mathbb{C}). \tag{5}$$

The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha,\beta > 0, z \in \mathbb{C}). \tag{6}$$

Definition 2.6. *The Laplace transform of Caputo fractional derivative is dfined by*

$$\begin{aligned}\mathcal{L}(^C D^\alpha x)(t) &= s^\alpha(\mathcal{L}x)(s) - x_1 s^{\alpha-1}, 0 < \alpha < 1\\ \mathcal{L}(^C D^\alpha x)(t) &= s^\alpha(\mathcal{L}x)(s) - x_1 s^{\alpha-1} + x_2 s^{\alpha-1}, 1 < \alpha < 2\end{aligned}$$

Here $x_1 = x(0), x_2 = x'(0)$.

Definition 2.7. *The Laplace transform of Mittag-Leffler functions are given by*

$$\mathcal{L}\{E_\alpha(\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda}, \qquad (\mathcal{R}(s) > |\lambda|^{\frac{1}{\alpha}}), \tag{7}$$

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha - \lambda}, \ (\mathcal{R}(s) > |\lambda|^{\frac{1}{\alpha}}), \tag{8}$$

where $t \geq 0, \lambda \in \mathbb{R}$.

Lemma 2.8. *[8] Let $0 < \alpha < 2, \beta$ be a an arbitrary complex number and μ be an arbitrary real number such that $\frac{\pi\alpha}{2} < \mu < min\{\pi, \pi\alpha\}$. Then, for an arbitrary integer $p \geq 1$, we have the following expansions:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} exp(z^{1/\alpha}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}), \tag{9}$$

when $|\arg(z)| \leq \mu$ *and* $|z| \to \infty$;

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}), \tag{10}$$

when $\mu \leq |\arg(z)| \leq \pi$ *and* $|z| \to \infty$.

In particular, if $\beta = 1$, then we have

$$E_{\alpha}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(1 - \alpha k)} + O(|z|^{-1-p}),$$

when $\mu \leq |\arg(z)| \leq \pi$ and $|z| \to \infty$.

Definition 2.9. *Consider the following fractional differential system*

$$^{C}D^{\alpha} x(t) = Ax(t), \tag{11}$$

with initial value $x(0) = x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$, *where* $x = (x_1, x_2, \ldots, x_n)^T$, $\alpha \in (0, 2)$ *and* $A \in \mathbb{R}^{n\times n}$. *The autonomous system* (12) *is said to be*

(i) stable iff for any x_0, *there exists* $\epsilon > 0$ *such that* $\|x(t)\| \leq \epsilon$ *for* $t \geq 0$,

(ii) asymptotically stable iff $\lim_{t\to\infty} \|x(t)\| = 0$.

3. Stability Analysis of Linear Differential System

In this section, we consider the following linear fractional differential system with Caputo fractional derivative

$$^{C}D^{\alpha} x(t) = Ax(t), (0 < \alpha < 1) \tag{12}$$

with initial value $x(0) = x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$, where $x = (x_1, x_2, \ldots, x_n)^T$, and $A \in \mathbb{R}^{n\times n}$. We shall analyze the stability of (12) with non-zero initial conditions.

Theorem 3.1. *[1] If all the eigenvalues of A satisfy*

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}, \tag{13}$$

then the zero solution of the system (12) *is asymptotically stable.*

Proof. Solution of the system (12) can be found by using Laplace Transform technique. Taking Laplace transform on both sides we get,

$$X(s)s^{\alpha} - s^{\alpha-1}x_0 = AX(s), where X(s) = \mathcal{L}x(t) \tag{14}$$

By taking inverse Laplace transform of the abov equation , we obtain

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{\frac{x_0}{s^{\alpha}I - A}\}$$

It immediately follows from the definition 4, the solution of system is $x(t) = E_{\alpha}(At^{\alpha})x_0$

Case 1: First, suppose the matrix A is diagonalizable. Then there exists an invertible matrix T such that $D = T^{-1}AT = diag(\lambda_1, \ldots, \lambda_n)$. Then, $E_{\alpha}(At^{\alpha}) = TE_{\alpha}(Dt^{\alpha})T^{-1} = Tdiag[E_{\alpha}(\lambda_1 t^{\alpha}), \ldots, E_{\alpha}(\lambda_n t^{\alpha})]T^{-1}$. Applying the lemma and definition, we get
$E_{\alpha}(\lambda_i t^{\alpha}) = -\sum_{k=1}^{p} \frac{\lambda_i (t^{\alpha})^{-k}}{\Gamma(1-\alpha k)} + O(|z|^{-1-p}), \to zero as t \to \infty$.
As a result, $\|E_{\alpha}(At^{\alpha})\| = \|E_{\alpha}(\lambda_1 t^{\alpha}), E_{\alpha}(\lambda_2 t^{\alpha}), \cdots, E_{\alpha}(\lambda_n t^{\alpha})\| \to 0$. Hence the conclusion holds.

Case 2 Next, suppose the matrix A is similar to a Jordan canonical form, i.e., there exists an invertible matrix T such that $J = T^{-1}AT = diag(J_1, \ldots, J_r)$, where $J_i,\ i \leq i \leq r$ has the following form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{n_i \times n_i}$$

and $\sum_{i=1}^{r} n_i = n$. Obviously,

$$E_{\alpha}(At^{\alpha}) = Tdiag[E_{\alpha}(J_1 t^{\alpha}), \ldots, E_{\alpha}(J_r t^{\alpha})]T^{-1},$$

where for $1 \leq i \leq r$,

$$E_\alpha(J_i t^\alpha) = \sum_{k=0}^{\infty} \frac{(J_i t^\alpha)^k}{\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+1)} J_i^k$$

$$= \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+1} \begin{pmatrix} \lambda_i^k & C_k^1 \lambda_i^k & \cdots & C_k^{n_i-1} \lambda_i^{k-n_i+1} \\ & \lambda_i^k & \ddots & \vdots \\ & & \ddots & C_k^1 \lambda_i^k \\ & & & \lambda_i^k \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(\lambda_i t^\alpha)^k}{\Gamma(\alpha k+1)} & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+1)} C_k^1 \lambda_i^k & \cdots & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+1)} C_k^{n_i-1} \lambda_i^{k-n_i+1} \\ & \sum_{k=0}^{\infty} \frac{(\lambda_i t^\alpha)^k}{\Gamma(\alpha k+1)} \lambda_i & \ddots & \vdots \\ & & \ddots & \sum_{k=0}^{\infty} \frac{(t^\alpha)^k}{\Gamma(\alpha k+1)} C_k^1 \lambda_i^k \\ & & & \sum_{k=0}^{\infty} \frac{(\lambda_i t^\alpha)^k}{\Gamma(\alpha k+1)} \end{pmatrix}$$

(C_k^j, $1 \leq j \leq n_i - 1$ are the binomial coefficients)

$$= \begin{pmatrix} E_\alpha(\lambda_i t^\alpha) & \frac{1}{(1)!}\left(\frac{d}{d\lambda_i}\right) E_\alpha(\lambda_i t^\alpha) & \cdots & \frac{1}{(n_i-1)!}\left(\frac{d}{d\lambda_i}\right)^{n_i-1} E_\alpha(\lambda_i t^\alpha) \\ & E_\alpha(\lambda_i t^\alpha) & \ddots & \vdots \\ & & \ddots & \frac{1}{(1)!}\left(\frac{d}{d\lambda_i}\right) E_\alpha(\lambda_i t^\alpha) \\ & & & E_\alpha(\lambda_i t^\alpha) \end{pmatrix}.$$

We shall now show that if $|arg(\lambda(A))| \geq \alpha\pi/2$, then we have $|E_\alpha(\lambda_i \theta^\alpha)| \to 0$ and $\left|\frac{1}{(n-1)!}\left(\frac{d}{d\lambda_i}\right)^{n_i-1} E_\alpha(\lambda_i t^\alpha)\right| \to 0 \qquad 1 \leq i \leq n, 1 \leq j \leq n_i - 1.$
These can be seen from the following:

$E_\alpha(\lambda_i t^\alpha) = -\sum_{k=1}^{\infty} \frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(1-\alpha k)} + O(|\lambda_i t^\alpha|^{-1-p}), \to zero$ as $t \to \infty$.

$\left|\frac{1}{(n-1)!}\left(\frac{d}{d\lambda_i}\right)^{n_i-1} E_\alpha(\lambda_i t^\alpha)\right| = \left|\frac{1}{j!}\left(\frac{d}{d\lambda_i}\right)^j\right| \left\{-\sum_{k=2}^{p} \frac{(\lambda_i t^\alpha)^{-k}}{\Gamma(1-\alpha k)} + O(|\lambda_i t^\alpha|)^{-1-p}\right\}$

$= -\sum_{k=2}^{p} \frac{(-1)^j (k+j-1)\cdots(k+1)k \lambda_i^{-k-j} t^{-alphak}}{j!\Gamma(1-\alpha k)} + O(|\lambda_i|^{1-p-j}|t^\alpha|^{-1-p})$

$= -\sum_{k=2}^{p} \frac{(-1)^j (k+j-1)! \lambda_i^{-k-j} t^{-alphak}}{j!\Gamma(1-\alpha k)} + O(|\lambda_i|^{1-p-j}|t^\alpha|^{-1-p})$

This leads to $\left|\frac{1}{(n-1)!}\left(\frac{d}{d\lambda_i}\right)^{n_i-1} E_\alpha(\lambda_i t^\alpha)\right| \to 0$ as $t \to \infty$. □

Consider the following linear system with Caputo fractional derivative

$${}^C D^\alpha x(t) = Ax(t), (1 < \alpha < 2) \tag{15}$$

with initial value $x(0) = x_0 = (x_{10}, x_{20}, \ldots, x_{n0})^T$, $x'(0) = x_1 = (x_{11}, x_{21}, \ldots, x_{n1})^T$, *where* $x = (x_1, x_2, \ldots, x_n)^T$, *and* $A \in \mathbb{R}^{n \times n}$.

We shall analyze the stability of (15) *with non-zero initial conditions. Taking Laplace Transform on* (15)

$$X(s)s^{\alpha} - s^{\alpha-1}x_0 - s^{\alpha-2}x_1 = AX(s), where X(s) = \mathcal{L}x(t)$$

*By taking inverse Laplace transform of the above equation , we obtain It immediately follon*ws from the definition 4, the solution of system is

$$x(t) = E_{\alpha}(At^{\alpha})x_0 + E_{\alpha,2}(At^{\alpha})x_1$$

Theorem 3.2. [*1*] *If all the eigenvalues of A satisfy*

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}, \tag{16}$$

then the zero solution of the system (15) *is asymptotically stable.*

Same as previous theorem, we may analyze the system (15).

4. Analysis of Region of Stability

From the above theorem we may conclude the following results and it may be verified by the examples.

1. If $\alpha \leq 1$, region of instability is smaller than region of stable.
2. If $\alpha > 1$, unstable region is larger than stable region.
3. If $\alpha = 2$, all stability disappear except for negative real axis and the system is oscillatory.
4. If $\alpha > 2$, the given system is unstable. The argument condition is $|arg(\lambda)| > \frac{\alpha\pi}{2}$, we know that $arg(\lambda) = tan^{-1}(y/x)$, here $\lambda = x + iy$. It may be noted that range of tan inverse lies between $-\frac{\pi}{2}$ to $frac\pi 2$. Hence the argument condition $|arg(\lambda)| > \frac{\alpha\pi}{2}$ does not hold for $\alpha > 2$.

5. Examples

Example 5.1. *Consider the following linear system* $^{C}D^{\alpha}x(t) = Ax(t)$, $0 < \alpha \leq 2$, *where* $A = \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix}$.

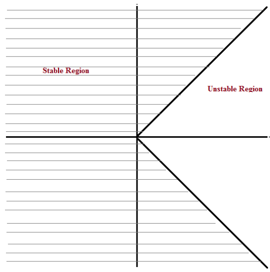

Figure 1: Stability Region for $0 < \alpha < 1$

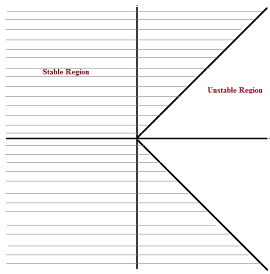

Figure 2: Stability Region for $1 < \alpha < 2$

Characteristic equation is given by $w^2 - w + 1 = 0$, whose roots are given by $w = \frac{1i\sqrt{(3)}}{2}$. Stability of the given system can be found by analyzing eigenvalues of the matrix A. Here, $arg(w_1) = -\frac{\pi}{3}, arg(w_2) = \frac{\pi}{3}$. Hence $|argw| = \frac{\pi}{3}$. Stability of the given system is examined for different values of α.

Case 1: When $\alpha = 1/4$, then $\frac{\pi}{3} > \frac{\pi}{8}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen in Figure 3.

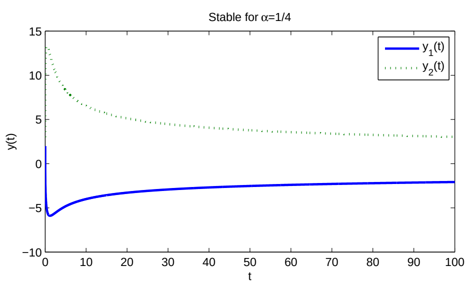

Figure 3: Stable for $\alpha = 1/4$

Case 2: When $\alpha = 1/3$, then $\frac{\pi}{3} > \frac{\pi}{6}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen in Figure

4.

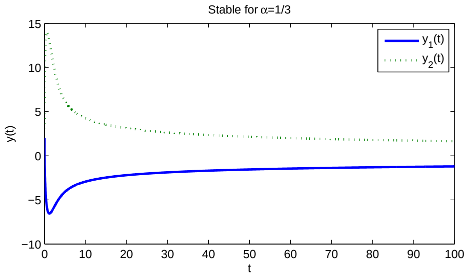

Figure 4: Stable for $\alpha = 1/3$

Case 3: When $\alpha = 1/2$, then $\frac{\pi}{3} > \frac{\pi}{4}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen in Figure 5.

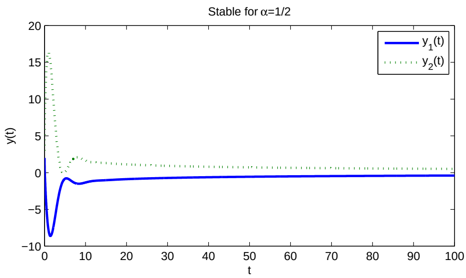

Figure 5: Stable for $\alpha = 1/2$

Case 4: When $\alpha = 2/3$, then $\frac{\pi}{3} = \frac{\alpha\pi}{2}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is oscillatory. This can be seen in Figure 6.

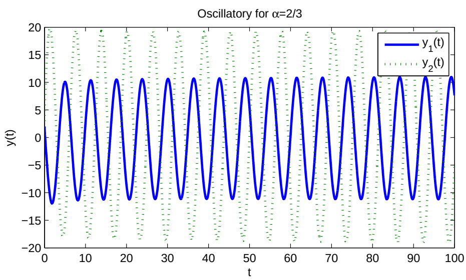

Figure 6: Oscillatory for $\alpha = 2/3$

Case 5: When $\alpha = 3/4$, then $\frac{\pi}{3} < \frac{3\pi}{8}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is not satisfied. Hence the given system is unstable. This can be seen in Figure 7.

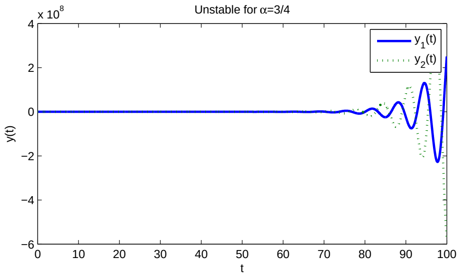

Figure 7: Unstable for $\alpha = 4/4$

Case 6: When $\alpha = 1$, then $\frac{\pi}{3} < \frac{\pi}{2}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is not satisfied. Hence the given system is unstable. This can be seen in Figure 8.

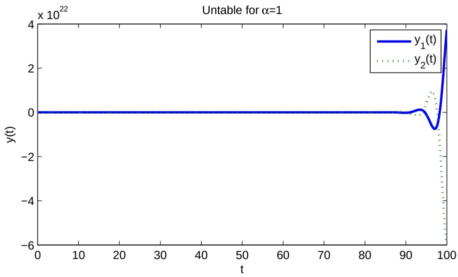

Figure 8: Unstable for $\alpha = 1$

Case 7: When $\alpha = 4/3$, then $\frac{\pi}{3} < \frac{\alpha\pi}{2}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is not satisfied. Hence the given system is oscillatory. This can be seen in Figure 9.

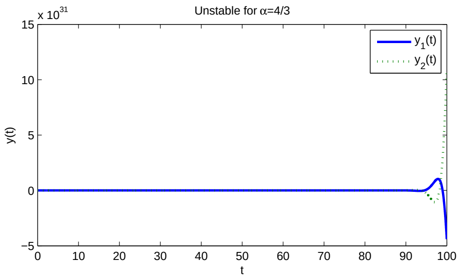

Figure 9: Unstable for $\alpha = 4/3$

Case 8: When $\alpha = 3/2$, then $\frac{\pi}{3} < \frac{3\pi}{4}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is not satisfied. Hence the given system is unstable. This can be seen in Figure 10.

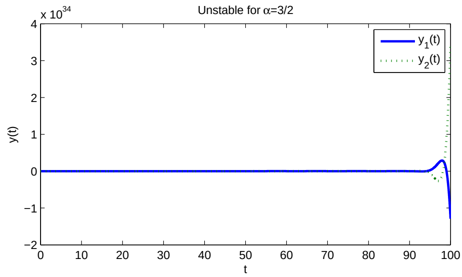

Figure 10: Unstable for $\alpha = 3/2$

Case 9: When $\alpha = 2$, then $\frac{\pi}{3} < \pi$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is not satisfied. Hence the given system is unstable. This can be seen in Figure 11.

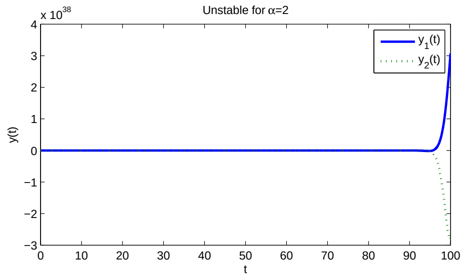

Figure 11: Unstable for $\alpha = 2$

By using the theorem, the following results have been concluded.

Table 1: Stability Results when $0 < \alpha \le 2$

α	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{\alpha\pi}{2}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$
Argument Condition	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) = \frac{\alpha\pi}{2}$	$arg(w) < \frac{\alpha\pi}{2}$
Stability	Stable	Stable	Stable	Oscillatory	Untable
α	1	$\frac{4}{3}$	$\frac{3}{2}$	2	
$\frac{\alpha\pi}{2}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	
Argument Condition	$arg(w) < \frac{\alpha\pi}{2}$	$arg(w) < \frac{\alpha\pi}{2}$	$arg(w) < \frac{\alpha\pi}{2}$	$arg(w) < \frac{\alpha\pi}{2}$	
Stability	Unstable	Unstable	Unstable	Unstable	

Example 5.2. *Consider the following linear system* ${}^{C}D^{\alpha}x(t) = Ax(t)$*,* $0 < \alpha \le 2$*, where* $A = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$.

Characteristic equation is given by $w^2 + w + 1 = 0$, whose roots are given by $w = \frac{-1i\sqrt(3)}{2}$. Stability of the given system can be found by analyzing eigenvalues of the matrix A. Here, $arg(w_1) = \frac{5\pi}{6}, arg(w_2) = \frac{7\pi}{6}$. Hence $|argw| = \frac{5\pi}{6}$. Stability of the given system is examined for different values of α.

Case 1: When $\alpha = 1/4$, then $\frac{5\pi}{6} > \frac{\pi}{8}$ and $\frac{7\pi}{6} > \frac{\pi}{8}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen in Figure 12.

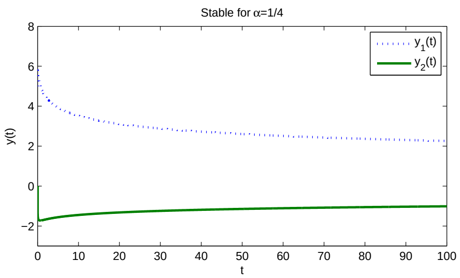

Figure 12: Stable for $\alpha = 1/4$

Case 2: When $\alpha = 1/3$, then $\frac{5\pi}{6} > \frac{\pi}{6}$ and $\frac{7\pi}{6} > \frac{\pi}{6}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen is 13.

Case 3: When $\alpha = 1/2$, then $\frac{5\pi}{6} > \frac{\pi}{4}$ and $\frac{7\pi}{6} > \frac{\pi}{4}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen is 14.

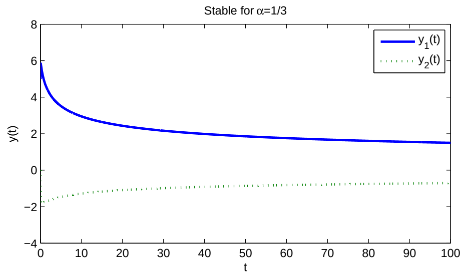

Figure 13: Stable for $\alpha = 1/3$

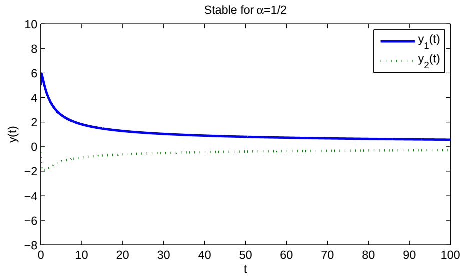

Figure 14: Stable for $\alpha = 1/2$

Case 4: When $\alpha = 2/3$, then $\frac{5\pi}{6} > \frac{\pi}{3}$ and $\frac{7\pi}{6} > \frac{\pi}{3}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen is 15.

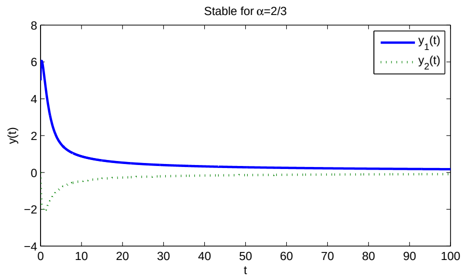

Figure 15: Stable for $\alpha = 2/3$

Case 5: When $\alpha = 3/4$, then $\frac{5\pi}{6} > \frac{3\pi}{8}$ and $\frac{7\pi}{6} > \frac{3\pi}{8}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen is 16.
Case 6: When $\alpha = 1$, then $\frac{5\pi}{6} > \frac{\pi}{2}$ and $\frac{7\pi}{6} > \frac{\pi}{2}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is stable. This can be seen is 17.

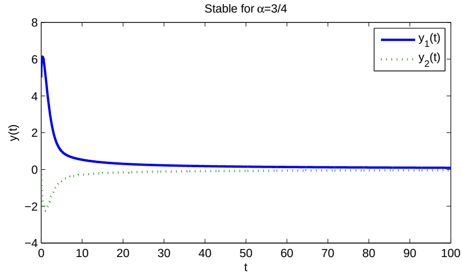

Figure 16: Stable for $\alpha = 3/4$

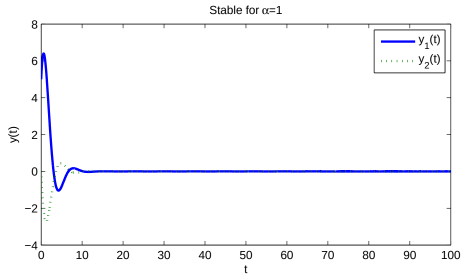

Figure 17: Stable for $\alpha = 1$

Case 7: When $\alpha = 4/3$, then $\frac{5\pi}{6} = \frac{\alpha\pi}{2}$ and $\frac{7\pi}{6} > \frac{\alpha\pi}{2}$. i. e., condition $argw = \frac{\alpha\pi}{2}$ is satisfied. Hence the given system is Oscillatory. This can be seen is 18.

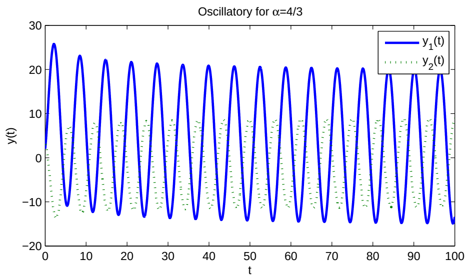

Figure 18: Oscillatory for $\alpha = 4/3$

Case 8: When $\alpha = 3/2$, then $\frac{5\pi}{6} < \frac{3\pi}{4}$ and $\frac{7\pi}{6} < \frac{3\pi}{4}$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is not satisfied. Hence the given system is unstable. This can be seen is 19.
Case 9: When $\alpha = 2$, then $\frac{5\pi}{6} < \pi$ and $\frac{7\pi}{6} < \pi$. i. e., condition $argw > \frac{\alpha\pi}{2}$ is not satisfied. Hence the given system is unstable. This can be seen is 20.

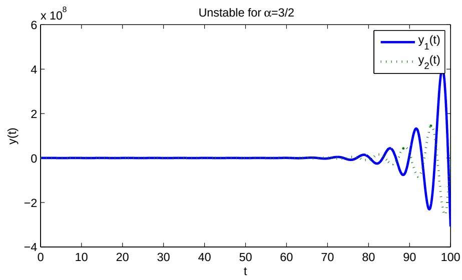

Figure 19: Unstable for $\alpha = 3/2$

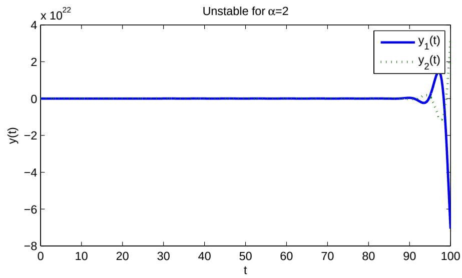

Figure 20: Unstable for $\alpha = 2$

By using the theorem, the following results have been concluded.

6. Conclusion

In this paper, some interesting linear fractional differential equations arising in real life have been solved. It is observed that stability regions not only depend on eigenvalues but also depend on the order of the system. We derived an approximate stability condition for linear fractional system.The properties of the Mittag-Leffler functions are used to deduce the result. Examples were given and its solutions were obtained using the numerical method and Matlab, we can conclude that these solutions are in excellent agreement with the exact solution and show that these approaches can solve the problem effectively.

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Table 2: Stability Results when $0 < \alpha \leq 2$

α	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{\alpha\pi}{2}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$
Argument Condition	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$
Stability	Stable	Stable	Stable	Stable	Stable
α	1	$\frac{4}{3}$	$\frac{3}{2}$	2	
$\frac{\alpha\pi}{2}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	
Argument Condition	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	$arg(w) > \frac{\alpha\pi}{2}$	
Stability	Stable	Oscillatory	Unstable	Unstable	

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