EFFECT OF SECOND SOUND ON THE ONSET OF RAYLEIGH-BENARD CONVECTION IN A MICROPOLAR FLUID

S. Pranesh*

ABSTRACT

The effects resulting from the substitution of the classical Fourier law by the non-classical Maxwell–Cattaneo law in Rayleigh–Benard convection in micropolar fluid is studied. The classical approach predicts an infinite speed for the propagation of heat. The present non-classical theory involves a wave type heat transport (SECOND SOUND) and does not suffer from the physically unacceptable drawback of infinite heat propagation speed. It is found that the results are noteworthy at short times and the critical eigenvalues are less than the classical ones.

1. Introduction

Convection in fluid suspensions has been the subject of intensive study because of the remarkable physical properties of the fluid as well as its practical applications (see Power 1995, Lukaszewicz 1998, Eringen 1999). Rayleigh–Benard convection

* Head, P.G. : Department of Mathematics, Christ College (Autonomous), Hosur Road, Bangalore–560 029, INDIA.

in fluids with suspended particles have been studied by a number of authors [1-20]. The reported works on convection concern fluid suspensions with classical Fourier heat flux law.

The drawback of the classical law motivated Lindsay and Stranghan (1978), Stranghan and Franchi (1984), Lebon and Cloot (1984) and Siddheshwar (1993, 1999) to adopt a non-classical heat flux law in studying Rayleigh – Benard / Marangoni convection. The hyperbolic heat equation (SECOND SOUND) model adopted by these authors does not suffer from the physically unacceptable drawback of infinite heat propagation speed. The principle of exchange of stability is also not guaranteed in these problems.

The objective of this chapter is to replace the classical parabolic heat equations by non-classical Maxwell-Cattaneo heat flux law and study Rayleigh-Benard convection in micropolar fluids.

2. Mathematical Formulation

Consider an infinite horizontal layer of a Boussinesquian, Micropolar fluid of depth d. A cartesian co-ordinate system is taken with origin in the lower boundary and z-axis vertically upwards. Let ΔT be the temperature difference between the upper and lower boundaries. The body force acting on the fluid is buoyancy.
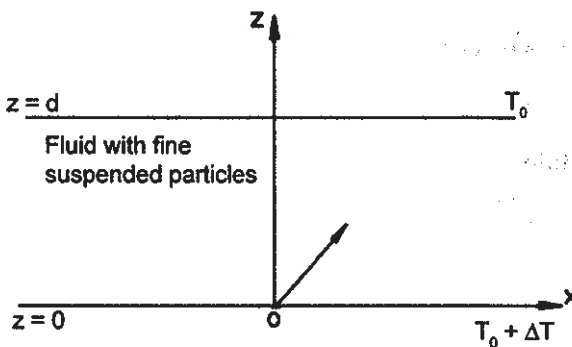

Fig 1 : Schematic Diagram of the Rayleigh-Benard Situation for a Fluid with Suspended Particles

The governing equations for the Rayleigh-Benard situation in a Boussinesquian Micropolar fluid are:

Continuity equation

$$\nabla.\vec{q} = 0, \tag{1}$$

Conservation of linear momentum

$$\rho_0 \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} \right] = -\nabla P - \rho g \hat{k} + (2\zeta + \eta)\nabla^2 \vec{q} + \zeta \nabla \times \vec{\omega}, \tag{2}$$

Conservation of angular momentum

$$\rho_0 I \left[\frac{\partial \vec{\omega}}{\partial t} + (\vec{q}.\nabla)\vec{\omega} \right] = (\lambda' + \eta')\nabla (\nabla.\vec{\omega}) + \eta'\nabla^2 \vec{\omega} + \zeta(\nabla \times \vec{q} - 2\vec{\omega}), \tag{3}$$

Conservation of energy

$$\frac{\partial T}{\partial t} + \left(\vec{q} - \frac{\beta}{\rho_0 C_V} \nabla \times \vec{\omega} \right).\nabla T = -\frac{\nabla.\vec{Q}}{\rho_0 C_V}, \tag{4}$$

Maxwell – Cattaneo heat flux law

$$\tau \left[\dot{\vec{Q}} + \vec{\omega}_1 \times \vec{Q} \right] = -\vec{Q} - \kappa \nabla T, \tag{5}$$

Equation of state

$$\rho = \rho_0 \left[1 - \alpha(T - T_0) \right], \tag{6}$$

where

$\vec{\omega}_1 = \frac{1}{2}\nabla \times \vec{q}$, $\vec{Q}$ is the heat flux vector and τ is the constant relaxation time.

The aim of this paper is to investigate the stability of a quiescent state to infinitesimal perturbations superposed on the basic state. The basic state of the fluid being quiescent is described by

$$\vec{q}_b = 0, \ \vec{\omega}_b = 0, \ p = p_b(z), \ \rho = \rho_b(z), \ T = T_b(z), \vec{Q} = (0,0,Q_b(z)). \tag{7}$$

Equations (2), (4) and (6) in the basic state specified by equation (7) respectively become

$$\left.\begin{aligned}
&\frac{dP_b}{dz} = -\rho_0 g, \\
&\frac{d^2 T_b}{dz^2} = 0, \\
&\rho_b = \rho_0\left[1-\alpha(T_b - T_0)\right], \\
&Q_b = -\kappa\frac{dT_b}{dz}
\end{aligned}\right\}. \tag{8}$$

Equations (1), (3) and (5) are identically satisfied by the concerned basic state variables. We now superpose infinitesimal perturbations on the quiescent basic state and study the stability.

2.1 Linear Stability Analysis

Let the basic state be disturbed by an infinitesimal thermal perturbation. We now have

$$\left.\begin{aligned}
&\vec{q} = \vec{q}_b + \vec{q}', \quad \vec{\omega} = \vec{\omega}_b + \vec{\omega}', \quad p = p_b + p', \\
&\rho = \rho_b + \rho', \quad T = T_b + T', \quad \vec{Q} = \vec{Q}_b + \vec{Q}'
\end{aligned}\right\} \tag{9}$$

The primes indicate that the quantities are infinitesimal perturbations and subscript b indicates basic state value.

Substituting equation (9) into equations (1)-(6) and using the basic state (8), we get the linearised equations governing the infinitesimal perturbations in the form:

$$\nabla \cdot \vec{q}' = 0 , \tag{10}$$

$$\rho_0 \frac{\partial \vec{q}'}{\partial t} = -\nabla p' - \rho' g \hat{k} + (2\zeta + \eta)\nabla^2 \vec{q}' + \zeta \nabla \times \vec{\omega}' , \tag{11}$$

$$\rho_0 I \frac{\partial \vec{\omega}}{\partial t} = (\lambda' + \eta')\nabla(\nabla \cdot \vec{\omega}') + \eta' \nabla^2 \vec{\omega}' + \zeta(\nabla \times \vec{q}' - 2\vec{\omega}') , \tag{12}$$

$$\frac{\partial T'}{\partial t} = \frac{\Delta T}{d}\left(\frac{\beta}{\rho_0 C_v}\nabla \times \vec{\omega}'.\hat{k} - W\right) - \frac{\nabla \cdot \vec{Q}'}{\rho_0 C_v} , \tag{13}$$

$$\left(1 + \tau \frac{\partial}{\partial t}\right)\vec{Q}' = -\frac{1}{2}\chi_1 \frac{\Delta T}{d}\left(\frac{\partial \vec{q}'}{\partial z} - \nabla W'\right) - \chi \nabla T , \tag{14}$$

$$\rho' = -\alpha \rho_0 T' , \tag{15}$$

Operating divergence on the equation (14) and substituting in equation (13), on using equation (10), we get

$$\left(1 + \tau \frac{\partial}{\partial t}\right)\frac{\partial T'}{\partial t} = \left(1 + \tau \frac{\partial}{\partial t}\right)\frac{\Delta T}{d}\left[\frac{\beta}{\rho_0 C_v}\nabla \times \vec{\omega}'\hat{k} - W'\right] \\ -\frac{1}{2}\chi_1 \frac{\Delta T}{d}\nabla^2 W' + \chi \nabla^2 T' \Bigg\} , \tag{16}$$

The perturbation equations (11)-(12) and equation (16) are non-dimensionalised using the following definition:

$$\left(x^*, y^*, z^*\right) = \frac{(x, y, z)}{d}, \quad \vec{q}^* = \frac{\vec{q}'}{\chi/d}, \quad \vec{\omega}^* = \frac{\vec{\omega}'}{\chi/d^2}, \quad t^* = \frac{t}{d^2/\chi}, \quad T^* = \frac{T'}{\Delta T} \tag{17}$$

Using equation (15) in equation (11), operating curl twice on the resulting equation, operating curl on equation (12) and non-dimensionalising the two resulting equations and also equation (16), we get

$$\frac{1}{Pr}\frac{\partial}{\partial t}\left(\nabla^2 W\right) = (1+N_1)\nabla^4 W + N_1\nabla^2\Omega_z + R\nabla_1^2 T \tag{18}$$

$$\frac{N_2}{Pr}\frac{\partial\Omega_z}{\partial t} = N_3\nabla^2\Omega_z - 2N_1\Omega_z - N_1\nabla^2 W \tag{19}$$

$$\left(1+C\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial t} = \left(1+C\frac{\partial}{\partial t}\right)W - \left(1+C\frac{\partial}{\partial t}\right)N_5\Omega_z + \nabla^2 T - C\nabla^2 W, \tag{20}$$

where the asterisks have been dropped for simplicity and the non-dimensional parameters N_1, N_3, N_5, R and C are as defined as

$$N_1 = \frac{\zeta}{\zeta+\eta} \qquad \text{(Coupling Parameter),}$$

$$N_3 = \frac{\eta'}{(\zeta+\eta)d^2} \qquad \text{(Couple Stress Parameter),}$$

$$N_5 = \frac{\beta}{\rho_0 C_v d^2} \qquad \text{(Micropolar Heat Conduction Parameter),}$$

$$Pr = \frac{\zeta+\eta}{\chi} \qquad \text{(Prandtl number),}$$

$$Pm = \frac{\zeta+\eta}{\gamma_m} \qquad \text{(Magnetic Prandtl number),}$$

$$R = \frac{\alpha g \Delta T d^3 \rho_0}{(\zeta+\eta)\chi} \qquad \text{(Rayleigh number) and}$$

$$C = \frac{\tau \chi}{2d^2} \qquad \text{(Cattaneo number).}$$

Equations (18) – (20) are solved for velocity stress free, isothermal and vanishing spin boundaries and hence the assumed boundary conditions are

$$W = \frac{\partial^2 W}{\partial z^2} = T = \Omega_z = 0 \quad \text{at} \quad z = 0, 1. \tag{21}$$

The infinitesimal perturbations W, Ω_z and T are assumed to be periodic waves (see Chandrasekhar 1961) and hence these permit a normal mode solution in the form

$$\begin{bmatrix} W \\ \Omega_z \\ T \end{bmatrix} = \begin{bmatrix} W(z) \\ G(z) \\ T(z) \end{bmatrix} \exp\left[\sigma t + i(lx + my)\right], \tag{22}$$

where l and m are horizontal components of the wave number $\vec{a}$.

Substituting equation (22) into equations (18) – (20), we get

$$\frac{\sigma}{Pr}\left(D^2 - a^2\right)W = (1 + N_1)\left(D^2 - a^2\right)^2 W - Ra^2 T + N_1\left(D^2 - a^2\right)G, \tag{23}$$

$$\frac{N_2 \sigma G}{Pr} = -N_1\left(D^2 - a^2\right)W + N_3\left(D^2 - a^2\right)G - 2N_1 G, \tag{24}$$

$$(1 + 2C\sigma)\sigma T = (1 + 2C\sigma)W - (1 + 2C\sigma)N_5 G + \left(D^2 - a^2\right)T - C\left(D^2 - a^2\right)W, \tag{25}$$

where $D = \dfrac{d}{dz}$.

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Eliminating G and T between the equations (23) – (25), we get a single equation involving W in the form

$$\left[Y_2Y_3 + N_1^2\left(D^2 - a^2\right)\right]Y_1\left(D^2 - a^2\right)W = -Ra^2\left[Y_2Y_4 + (1+2C\sigma)\left(D^2 - a^2\right)^2\right]W \quad (26)$$

where

$$Y_1 = \sigma + 2C\sigma^2 - \left(D^2 - a^2\right),$$

$$Y_2 = \frac{N_2\sigma}{Pr} - N_3\left(D^2 - a^2\right) + 2N_1,$$

$$Y_3 = \frac{\sigma}{Pr} - (1+N_1)\left(D^2 - a^2\right)$$

and

$$Y_4 = 1 + 2C\sigma - C\left(D^2 - a^2\right).$$

The solution for W for the lowest mode is, as per Chandrasekhar (1961), taken in the form

$$W = A_0 \sin\left(\pi z\right) \quad (27)$$

where A_o is a constant. Substituting equation (27) in equation (26), we get

$$\left.\begin{array}{l}\left(k^2 + 2C\sigma^2 + \sigma\right)\left[\left(\frac{N_2\sigma}{Pr} + N_3k^2 + 2N_1\right)\left(\frac{\sigma}{Pr} + (1+N_1)k^2\right) - N_1^2k^2\right]k^2 \\[2mm] = Ra^2\left[\left(N_3k^2 + 2N_1 + \frac{N_2\sigma}{Pr}\right)\left(1 + 2C\sigma + Ck^2\right) - (1+2C\sigma)N_1N_5k^2\right]\end{array}\right\} \quad (28)$$

where $k^2 = \pi^2 + a^2$.

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2.2 Marginal State

If σ is real, then marginal stability occurs when $\sigma = 0$. This gives the stationary Rayleigh number R_s in the form

$$R_S = \frac{(1+N_1)N_3 k^8 + N_1(2+N_1)k^6}{a^2\left[(1+Ck^2)(N_3 k^2 + 2N_1) - N_1 N_5 k^2\right]} \tag{29}$$

If $C = 0$, the equation (29) reduces to

$$R_S = \frac{k^6}{a^2}\left[\frac{(1+N_1)N_3 k^2 + 2N_1 + N_1^2}{(N_3 - N_1 N_5)k^2 + 2N_1}\right] \tag{30}$$

which is the expression for the Rayleigh number discussed by Datta and Sastry (1976) and Bhattacharya and Jena (1983).

2.3 Oscillatory Motions

To study the oscillatory motions we put $\sigma = i\omega$, $\omega \in \Re$, in equation (28). By equating the real and imaginary parts, the following equations are obtained

$$R_o = \frac{\left\{T_3\left[T_5 T_1 - \omega^2 T_2\right] + \omega^2 k^2 T_4\left[T_1 + T_5 T_2\right]\right\}k^2}{a^2\left\{T_3^2 + \omega^2 T_4^2\right\}}, \tag{31}$$

$$A\omega^4 + B\omega^2 + E = 0, \tag{32}$$

where

$$T_1 = X_1 - \frac{\omega^2 N_2}{Pr^2},$$

$$T_2 = \frac{N_3 k^2 + 2N_1 + (1+N_1)N_2 k^2}{Pr},$$

$$T_3 = X_2 - \frac{2CN_2\omega^2}{Pr},$$

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$$T_4 = 2C\left(N_3 k^2 + 2N_1\right) + \frac{N_2\left(1 + Ck^2\right)}{Pr} - 2CN_1 N_5 k^2 ,$$

$$T_5 = k^2 - 2C\omega^2 ,$$

$$X_1 = (1+N_1)\left(N_3 k^2 + 2N_1\right)k^2 - N_1^2 k^2 ,$$

$$X_2 = \left(1 + Ck^2\right)\left(N_3 k^2 + 2N_1\right) - N_1 N_5 k^2 ,$$

$$X_3 = \frac{N_2}{Pr^2} + 2CX_2 ,$$

$$A = \frac{2CN_2}{Pr} X_3 - \frac{2CN_2 X_4}{Pr^2} ,$$

$$B = -\frac{2CN_2}{Pr}\left(X_1 + k^2 T_2\right) - X_2 X_3 + \left(\frac{N_2 k^2}{Pr^2} + 2CX_1 + X_2\right) T_4 ,$$

and $E = X_1 X_2 - k^2 X_1 T_4$

3. Results and Discussions

In this paper, we study the Rayleigh-Benard convection in Micropolar fluid by replacing the classical parabolic heat equation by a non-classical heat flux law. Figure (2) is the plot of critical R_c versus Cattaneo number C for different values of N_1. It is observed that as N_1 increases, R_c also increases. Increase in N_1 indicates the increase in the concentration of microelements. These elements consume the greater part of the energy of the system in developing the gyrational velocities of the fluid and as a result onset of convection is delayed.   From the figure (2) it is observed that C which represents second sound has a destabilising influence. The effects of N_3, N_5 and C on R_c are given in table (1). It is found that an increase in N_3 is to destabilise the system while an increase in N_5 is to stabilise the system. Increase in N_3, decreases the couple stress of the fluid which causes a decrease in microrotation and hence makes the system more unstable. When N_5 increases,

the heat induced into the fluid due to these microelements also increases, thus reducing the heat transfer from bottom to top. It is also found that C has no influence on the critical wave number. For small values of C, oscillatory convection sets in for heating from above and this is clearly shown in table (2).

Table 1 : Values of critical Rayleigh number R_c, critical wave number a_c^2 and critical frequency ω_c^2 for different values of N_1, N_2, N_3, N_5, Pr and C for a Micropolar fluid

C	N_1	N_2	N_3	N_5	Pr	R_c	a_c^2	ω_c^2
0.1	0.1	0.1	2.0	1.0	10.0	224.98	10.08	23.82
0.3						73.58	9.91	12.60
0.5						43.90	9.94	8.87
0.7						31.27	10.02	6.58
0.9						24.28	10.14	6.58
0.1	0.5	0.1	2.0	1.0	10.0	350.94	9.86	10.62
0.3						127.44	9.90	8.90
0.5						76.21	9.90	6.04
0.7						54.36	9.74	4.48
0.9						42.24	9.75	3.60
0.1	0.1	0.5	2.0	1.0	10.0	220.10	9.95	24.44
0.3						72.11	10.01	14.11
0.5						43.00	10.27	9.28
0.7						30.61	10.17	6.80
0.9						23.76	9.98	5.32
0.1	0.1	0.1	6.0	1.0	10.0	217.53	10.01	25.17
0.3						71.27	9.89	14.10
0.5						42.51	9.72	9.08
0.7						30.28	9.64	6.66
0.9						23.50	9.73	4.82
0.1	0.1	0.1	2.0	1.5	10.0	230.65	10.07	22.35
0.3						75.61	9.99	13.19
0.5						45.12	9.86	8.55
0.7						32.14	9.89	6.33
0.9						24.96	9.88	5.01
0.1	0.1	0.1	2.0	1.0	25.0	226.61	10.02	22.88
0.3						74.90	10.19	13.45
0.5						44.81	9.92	8.62
0.7						31.96	10.44	6.55
0.9						24.84	9.67	4.97

Table (2) : Values of critical Rayleigh number R_c, critical wave number a_c^2 and critical frequency ω_c^2 for different values of C and N_1 and for $N_2 = 0.1$, $N_3 = 2.0$, $N_5 = 1.0$ and Pr = 10.0 for a Micropolar fluid.

| N_1	C	R_c	a_c^2	ω_c^2
0.1	0.002	-6991127.00	512.22	64054990.00
	0.005	-3137664.00	198.96	10251850.00
	0.007	-1152220.00	139.28	5230719.00
	0.01	-599794.40	94.53	2563493.00
	0.03	-303159.60	24.92	285262.80
	0.05	-20899.91	11.24	102869.90
0.5	0.002	-24516650.00	727.48	181326400.00
	0.005	-4001038.00	284.87	28997100.00
	0.007	-2068769.00	200.55	14787800.00
	0.01	-1034842.00	137.33	7242847.00
	0.03	-133721.10	38.97	801970.00
	0.05	-57748.59	19.31	287757.20
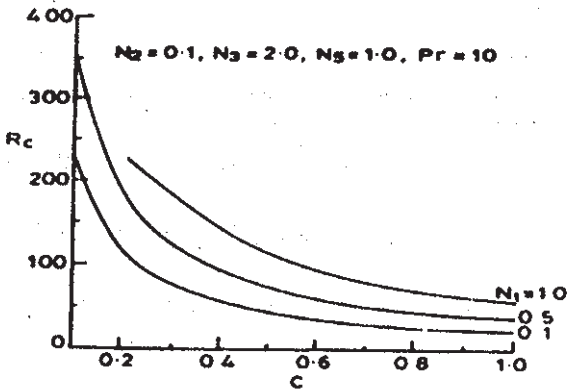
Fig 2 : Plot of Critical Rayleigh Number R_c Vs Cattaned Number C for Different Values of Coupling Parameter N_1.

Analysing equations (29) and (30) we can conclude the following for stationary convection in micropolar fluids:

$$R_c^{HHE} < R_c^{PHE}$$

where HHE – Hyperbolic heat equation and PHE – Parabolic heat equation.

4. Conclusion

The non-classical Maxwell-Cattaneo heat flux law involves a wave type heat transport and does not suffer from the physically unacceptable drawback of infinite heat propagation speed. The classical Fourier flux law overpredicts the critical Rayleigh number compared to that predicted by the non-classical law. Overstability is the preferred mode of convection.

Acknowledgment

References

1. Eringen A. C.: Simple microfluids with stretch. Int. J. Engg. Sci., 7, 115-127, (1969).

2. Eringen A. C.: Micropolar fluids with stretch. Int. J. Engg. Sci., 7, 115-127, (1986).

3. Eringen A. C.: Micropolar theory of liquid crystals. In: Johnson, J. F., Porter R. S. (Eds), Liquid crystals and ordered fluids, 3, Plenum Publishing Co., 1978.

4. Power H.: Bio-Fluid Mechanics, Advances in fluid mechanics, W.I.T. Press, UK, 3, 336, (1995).

5. Lukaszewicz: Micropolar fluid theory and applications, Birkhauser Boston, M. A., USA, 1998.

6. Eringen A. C.: Microcontinuum fluid theories, Springer Verlag, 1999.

7. Chandra K.: Instability of fluids heated from below. Proc. Roy. Soc. London A., 164, 231-242, (1938).

8.   Datta A. B., Sastry V. U. D.: Thermal instability of a horizontal layer of micropolar fluid heated from below. Int. J. Engg. Sci., 14, 631-637, (1976).

9.   Bhattacharya S. P., Jena S. K.: Thermal instability of a horizontal layer of micropolar fluid with heat source. Int. J. Engg. Sci., 23, 13-26, (1984).

10. Payne L. E.:, Straughan B.: Critical Rayleigh number for oscillatory and non-linear convection in an isotropic thermomicropolar fluid. Int. J. Engg. Sci., 27, 827-836, (1989).

11. Siddheshwar P. G., Pranesh S.: Magnetoconvection in a micropolar fluid. Int. J. Engg. Sci., 36, 1173-1181, (1998).

12. Siddheshwar P. G., Pranesh S.: Effect of a non-uniform basic temperature gradient on Rayleigh-Benard convection in a micropolar fluid. Int. J. Engg. Sci., 36, 1183-1196. (1998).

13. Siddheshwar P. G., Pranesh S.: Effect of temperature / gravity modulation on the onset of magneto-convection in a weak electrically conducting fluids with intenal angular momentum. JMMM, 192, 159-176, (1999).

14. Siddheshwar P. G., Pranesh S.: Effect of temperature / gravity modulation on the onset of magneto-convection in an electrically conducting fluids with intenal angular momentum. JMMM, 219, L153 – L162. (2000).

15. Siddheshwar P. G., Pranesh S.: Effects of non-uniform temperature gradients and magnetic field on the onset of convection in fluids with suspended particles under microgravity conditions, Indian Journal of Engineering and Mateials Sciences, 8, 77-83 (2001).

16. Siddheshwar P. G., Pranesh S.: Suction-Injection effects on the onset of Rayleigh-Benard-Marangoni convection in a fluid with suspended particles, Acta Mechanica, 152, 241-252 (2001).

17. Siddheshwar P. G., Pranesh S.: Magnetoconvection in fluids with suspended particles under 1g and mg, International Journal of Aerospace Science and Technology, 6, 105-114 (2002).

18. Pranesh S.: Effects of Suction-Injection-Combination (SIC) on the onset of Rayleigh-Benard magnetoconvection in a fluid with suspended particles, International Journal of Engineering Sciences, 41, 1741-1766 (2003).

19. Siddheshwar P. G., Pranesh S.: Linear and weakly non-linear analyses of convection in a micropolar fluid, Hydrodynamics VI-Theory and applications Cheng & Yeow (eds) 487 – 493 (2005).

20. Pranesh S.: Effects of magnetic field and non-uniform basic temperature gradient on the onset of Rayleigh-Benard convection in a micropolar fluid, MAPANA, 6, 1-33 (2007).

21. Straughan B, Franchi F.: Benard convection and the Cattaneo law of heat conduction. Proc. of Roy. Soc. of Edi. 96, 175 (1984).

22. Lebon G., Cloot A.: Benard-Marangoni instability in Maxwell-Cattaneo fluid, Phy. Let. 105A 361 (1984).

23. Siddheshwar P. G.: Rayleigh-Benard convection in a magnetic field with second sound. Proc. Japan Soc. Mag. Fluids. 32, (1993).

24. Siddheshwar P. G., Rayleigh-Benard convection in a second order ferromagnetic fluid with second sound. Proc. VIII Asian cong., Fluid Mech. China 631, (1999).

25. Chandrasekhar S.: Hydrodynamic, Hydromagnetic Stability, Dover, New York, 1961.