MJS, Vol. 7, No. 2, July - Dec 2008, pp. 25-33
ISSN 0975-3311
https://doi.org/10.12725/mjs.13.3

SASAKIAN HYPERSURFACES OF THE GENERALIZED CONCIRCULAR RECURRENT KAHLERIAN MANIFOLD

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ABSTRACT

In this paper we consider a recurrent sasakian hypersurface of the generalized concircular recurrent Kahlerian manifold and determine some conditions on the vector fields used in the sasakian structure. Further we determine such conditions for φ sasakian hypersurface also.

1. Preliminaries

Let M^{2n+2} be a $2n+2$ dimensional almost Hermitian manifold, with structure tensors (J,G) and the Riemannian connection $\tilde{\nabla}$ such that $J^2=-I$ and $G(JX,JY)=G(X,Y)$. An almost Hermitian manifold with $\tilde{\nabla}J=0$ is known as Kahlerian manifold. Suppose that M^{2n+1} is a C^∞ hypersurface of M^{2n+2} with unit normal N and the induced metric g. Thus if di denotes the differential of the imbedding $i: M^{2n+1} \rightarrow M^{2n+2}$, X a vector field on M^{2n+1}, then $\tilde{X}$ is

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the extension on M^{2n+2} of $\tilde{X}$ and is such that $\tilde{X}$ restricted to M^{2n+1} under the imbedding is diX. Also let $\Theta = \{e_i\}$, $i = 1,2,......,2n+1$ be an orthonormal basis of the tangent space at any point of the manifold M^{2n+1} then $\bar{\Theta} = \{e_i, N\}$ $i = 1,2,......,2n+1$ is an orthonormal basis for the tangent space at any point on the manifold M^{2n+2}. Hence

$$G(\tilde{X}, \tilde{Y}) = g(X, Y), \quad G(N, N) = 1, \quad G(\tilde{X}, N) = 0 \tag{1.1}$$

and its Riemannian connection ∇ is governed by Gauss-Weingarten equations

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \widetilde{(\nabla_X Y)} + h(X, Y)N, \quad \tilde{\nabla}_{\tilde{X}} N = -\widetilde{(H'X)} \tag{1.2}$$

where h denotes the second fundamental form and H' the corresponding Weingarten map. Also the submanifold M^{2n+1} inherits an almost contact metric structure (φ, ξ, η, g) [1] [2] given by

$$J\tilde{X} = \widetilde{(\varphi X)} + \eta(X)N, \qquad JN = -\tilde{\xi} \tag{1.3}$$

Then (1.1), (1.2) and (1.3) lead to the following conditions in M^{2n+1}:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \tag{1.4}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad \eta(X) = g(X, \xi). \tag{1.5}$$

An almost contact metric structure (φ, ξ, η, g) is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{1.6}$$

If K is the curvature tensor of type $(0,4)$ and S is the Ricci tensor of type $(0,2)$ in the sasakian manifold M^{2n+1}, then the following conditions [1] hold in a sasakian manifold

$$S(X,\xi) = 2n\,\eta(X) \tag{1.7}$$

$$g(K(\xi,X)Y,\xi) = g(X,Y) - \eta(X)\eta(Y) \tag{1.8}$$

$$K(\xi,X)\xi = -X + \eta(X)\xi \tag{1.9}$$

$$g(K(X,Y)\xi,Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \tag{1.10}$$

$$(\nabla_X \phi)(Y) = R(\xi,X)Y \tag{1.11}$$

If Ω is the 2-form on M^{2n+1} defined by

$$\Omega(X,Y) = g(X,\varphi Y) = -g(\varphi X,Y) = -\Omega(Y,X) \tag{1.12}$$

then from (1.5), we get

$$(\nabla_X \eta)(Y) = g(X,\varphi Y) \tag{1.13}$$

If $\tilde{K}$ the curvature tensor in the Kahlerian manifold M^{2n+2}, then we have the following well known Gauss-Codazzi equations

$$\begin{aligned} K(X,Y,Z,W) = {} & \tilde{K}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) \\ & + h(X,W)h(Y,Z) - h(X,Z)h(Y,W) \end{aligned} \tag{1.14}$$

$$\tilde{K}(\tilde{X},\tilde{Y},\tilde{Z},N) = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \tag{1.15}$$

where $\tilde{K}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W})$ and $K(X,Y,Z,W)$ are given by $G(\tilde{K}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W})$ and $g(K(X,Y)Z,W)$ respectively.

A Kahlerian manifold M^{2n+2} is said to be a generalized concircular recurrent manifold [6] if there exists a non-zero 1-forms $\tilde{A}$ and $\tilde{B}$ such that

$$(\tilde{\nabla}_{\tilde{U}}\tilde{C})(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) = \tilde{A}(\tilde{U})\tilde{C}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) + \tilde{B}(\tilde{U})\tilde{F}(\tilde{X},\underline{\tilde{Y}},\underline{\tilde{Z}},\overline{\tilde{W}}) \tag{1.16}$$

for arbitrary vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, $\tilde{W}$ and $\tilde{U}$ on M^{2n+2}, where

$$\tilde{C}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) = \tilde{K}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) - \frac{R}{(2n+1)(2n+2)}\tilde{F}(\tilde{X},\overline{\underline{\tilde{Y}}},\underline{\tilde{Z}},\overline{\tilde{W}}) \tag{1.17}$$

and

$$\tilde{F}(\tilde{X},\overline{\underline{\tilde{Y}}},\underline{\tilde{Z}},\overline{\tilde{W}}) = G(\tilde{X},\tilde{W})G(\tilde{Z},\tilde{Y}) - G(\tilde{X},\tilde{Z})G(\tilde{Y},\tilde{W}) \tag{1.18}$$

with $\tilde{A}(\tilde{U}) = G(\tilde{U},\tilde{\rho}')$ and $\tilde{B}(\tilde{U}) = G(\tilde{U},\tilde{\rho}'')$ for some vector fields $\tilde{\rho}'$, $\tilde{\rho}''$. (Note: In (1.18), the bars above $\tilde{Y}$ and $\tilde{W}$ indicate that they are swapped to get the first term from $G(\tilde{X},\tilde{Y})G(\tilde{Z},\tilde{W})$ and the bars below $\tilde{Y}$ and $\tilde{Z}$ indicates that they are swapped to get the second term from $G(\tilde{X},\tilde{Y})G(\tilde{Z},\tilde{W})$. If the second fundamental tensor $h(X,Y)$ satisfies the condition [4]

$$h(X,Y) = g(X,Y) + \mu\eta(X)\eta(Y) \tag{1.19}$$

then M^{2n+1} is called a c-umbilical hypersurface. Seiichi Yamaguchi [4] has proved that an immersed hypersurface in Kahlerian manifold is sasakian if and only if it is cumbilical, with

$$\mu = (2n+1)(H-1) \tag{1.20}$$

in (1.19), where H is the mean curvature. If H is a constant then $\nabla\mu = 0$ and the immersed hypersurface is called as the CMC hypersurface.

2. Recurrent Sasakian Hypersurfaces:

Definition 2.1 A Sasakian manifold is said to be a Recurrent manifold [3] if there exists a non-zero 1-form $A(U)$ such that

$$(\nabla_U K)(X,Y,Z,W) = A(U)K(X,Y,Z,W) \tag{2.1}$$

Where X, Y, Z, W and U are the arbitrary vector fields and $A(U) = g(U,\rho') = G(\tilde{U},\tilde{\rho}') = \tilde{A}(\tilde{U})$, for some vector field ρ'.

Theorem 2.2 Suppose M^{2n+1} is a recurrent sasakian hypersurface of a generalized concircular recurrent Kahlerian manifold M^{2n+2}, then ξ is orthogonal to the vector field $\rho'' - \lambda\rho'$ if and only if $\tilde{\xi}[R] = A(\xi)\,R - 4(n+1)\,\xi[\mu]$, where $\lambda = 1 + (2\mu/(2n+1))$ and ρ' and ρ'' are the vector fields associated with the one forms A and B.

Proof By applying ∇_U to (1.17) and using (1.1), (1.2), (1.4), (1.5), (1.12), (1.13), (1.14), (1.15), (1.16), (1.17), (1.18) and (1.19) we get

$$
\begin{aligned}
(\nabla_U K)(X,Y,Z,W) = {} & A(U)K(X,Y,Z,W) \\
& + \left\{ B(U) - A(U) + \frac{\tilde{U}\left[\tilde{R}\right] - R\,A(U)}{(2n+1)(2n+2)} \right\} F(X,\underline{Y},\underline{Z},\overline{W}) \\
& + \{U[\mu] - A(U)\}\{\eta(X)F(\xi,\underline{Y},\underline{Z},\overline{W}) \\
& \qquad + \eta(Y)F(X,\underline{\bar{\xi}},\underline{Z},\overline{W})\} \\
& + g(U,X)\{(W[\mu]\,\eta(Z) - Z[\mu]\,\eta(W))\eta(Y)\} \\
& + g(U,Y)\{(Z[\mu]\,\eta(W) - W[\mu]\,\eta(Z))\eta(X)\} \\
& + g(U,Z)\{(Y[\mu]\,\eta(X) - X[\mu]\,\eta(Y))\eta(W)\} \\
& + g(U,W)\{(X[\mu]\,\eta(Y) - Y[\mu]\,\eta(X))\eta(Z)\} \\
& + \mu\, g(U,X)\left(2\Omega(W,Z)\eta(Y) - F(\xi,\overline{\underline{\varphi Y}},\underline{Z},\overline{W})\right) \\
& + \mu\, g(U,Y)\left(2\Omega(Z,W)\eta(X) - F(\varphi X,\underline{\bar{\xi}},\underline{Z},\overline{W})\right) \\
& + \mu\, g(U,Z)\left(2\Omega(Y,X)\eta(W) - F(X,\underline{\bar{Y}},\xi,\overline{\varphi W})\right) \\
& + \mu\, g(U,W)\left(2\Omega(X,Y)\eta(Z) - F(X,\underline{\bar{Y}},\underline{\varphi Z},\bar{\xi})\right) \\
& + \mu(\Omega(U,X)F(\xi,\underline{\bar{Y}},\underline{Z},\overline{W}) + \Omega(U,Y)F(X,\underline{\bar{\xi}},\underline{Z},\overline{W})) \\
& + \mu(\Omega(U,Z)F(X,\underline{\bar{Y}},\underline{\xi},\overline{W}) + \Omega(U,W)F(X,\underline{\bar{Y}},\underline{Z},\bar{\xi}))
\end{aligned} \tag{2.2}
$$

Since M^{2n+1} is recurrent, we use (2.1) in (2.2). Then by replacing U by ξ and choosing $X = W = e_i$, $Y = Z = e_j$, and taking summation over i, j $1 \le i, j \le 2n+1$, we get

$$\begin{aligned}\tilde{\xi}[R] = {} & A(\xi)\,R \\ & +2(n+1)[(2(n+\mu)+1)A(\xi)-(2n+1)B(\xi)] \\ & -4(n+1)\xi[\mu]\end{aligned} \qquad (2.3)$$

and hence the theorem.

By virtue of theorem 2.2 and (1.20), we have

Corollary 2.3 Suppose M^{2n+1} is a recurrent sasakian hypersurface of a generalized concircular recurrent Kahlerian manifold M^{2n+2}. Then M^{2n+1} is CMC if and only if $\tilde{\xi}[R] = R\,\tilde{A}(\tilde{\xi})$.

Corollary 2.4 Suppose M^{2n+1} is a recurrent sasakian CMC hypersurface of a generalized concircular recurrent Kahlerian manifold M^{2n+2}, then M^{2n+2} is flat if and only if $R = 0$.

3. φ-recurrent Sasakian Hypersurfaces:

Definition 3.1 A Sasakian manifold is said to be $\varphi-$ recurrent manifold [5] if there exists a non-zero 1-form A such that

$$\varphi^2((\nabla_U K)(X,Y,Z,W) = A(U)K(X,Y,Z,W) \qquad (3.1)$$

for arbitrary vector fields X, Y, Z, W and U where K is the curvature tensor in Sasakian manifold.

If a Sasakian manifold M^{2n+1} is a $\varphi-$ recurrent manifold, then the following relation holds [5]

$$(\nabla_U K)(X,Y,Z,W) = \{g(Y,U)g(\varphi X,Z) - g(X,U)g(\varphi Y,Z) - g(\varphi K(X,Y)U,Z)\}g(\xi,W) - A(U)K(X,Y,Z.W) \tag{3.2}$$

Theorem 3.2 *Suppose* M^{2n+1} *is a* $\varphi-$*recurrent sasakian hypersueneralized concircular recurrent Kahlerian manifold* M^{2n+2}. *Then* ξ *is orthogonal to the vector field* $\rho''-\lambda\rho'$ *if and only if* $\tilde{\xi}[R] = A(\xi)\,R - 4(n+1)\,\xi[\mu]$, *where* $\lambda = 1 + \frac{2\mu n - r}{n(2n+1)}$ *and* ρ', ρ'' *are the vector fields associated with the one forms* A *and* B.

Proof By applying ∇_U to (1.12) and using (1.1), (1.2), (1.4), (1.5), (1.7), (1.8), (1.9), (1.10), (1.11), (1.12), (1.13) and (1.14) we get the expression, same as (2.2). Since M^{2n+1} is φ -recurrent we use (3.2) in (2.2) to obtain

$$\begin{aligned}
&2A(U)K(X,Y,Z,W)\\
&+A(U)\{[g(X,U)g(\varphi Y,U) - g(Y,U)g(\varphi X,Z)] - K(X,Y,U,\varphi Z)\}g(\xi,W)\\
&+\left\{B(U) - A(U) + \frac{\tilde{U}\left[\tilde{R}\right] - R\,A(U)}{(2n+1)(2n+2)}\right\}F(X,\underline{\overline{Y}},\underline{Z},\overline{W})\\
&+\{U[\mu] - A(U)\}\left\{\eta(X)F(\xi,\underline{\overline{Y}},\underline{Z},\overline{W}) + \eta(Y)F(X,\underline{\overline{\xi}},\underline{Z},\overline{W})\right\}\\
&+g(U,X)\{(W[\mu]\,\eta(Z) - Z[\mu]\,\eta(W))\eta(Y)\}\\
&+g(U,Y)\{(Z[\mu]\,\eta(W) - W[\mu]\,\eta(Z))\eta(X)\}\\
&+g(U,Z)\{(Y[\mu]\,\eta(X) - X[\mu]\,\eta(Y))\eta(W)\}\\
&+g(U,W)\{(X[\mu]\,\eta(Y) - Y[\mu]\,\eta(X))\eta(Z)\}\\
&+\mu\, g(U,X)\left(2\Omega(W,Z)\eta(Y) - F(\xi,\underline{\overline{\varphi Y}},\underline{Z},\overline{W})\right)\\
&+\mu\, g(U,Y)\left(2\Omega(Z,W)\eta(X) - F(\varphi X,\underline{\overline{\xi}},\underline{Z},\overline{W})\right)\\
&+\mu\, g(U,Z)\left(2\Omega(Y,X)\eta(W) - F(X,\underline{\overline{Y}},\underline{\xi},\overline{\varphi W})\right)\\
&+\mu\, g(U,W)\left(2\Omega(X,Y)\eta(Z) - F(X,\underline{\overline{Y}},\underline{\varphi Z},\overline{\xi})\right)\\
&+\mu(\Omega(U,X)F(\xi,\underline{\overline{Y}},\underline{Z},\overline{W}) + \Omega(U,Y)F(X,\underline{\overline{\xi}},\underline{Z},\overline{W}))\\
&+\mu(\Omega(U,Z)F(X,\underline{\overline{Y}},\underline{\xi},\overline{W}) + \Omega(U,W)F(X,\underline{\overline{Y}},\underline{\xi},\overline{W})) = 0
\end{aligned} \tag{3.3}$$

Replacing U by ξ, and choosing $X = W = e_i$, $Y = Z = e_j$ and taking the summation over i, j, $1 \le i, j \le 2n+1$, we get

$$\tilde{\xi}[R] = A(\xi)\,R$$
$$+2(n+1)[(2(n+\mu)-\frac{r}{n})A(\xi)-(2n+1)B(\xi)]$$
$$-4(n+1)\xi[\mu] \qquad (3.4)$$

hence the theorem.

In [5] it has been proved that "A $\varphi-$*recurrent sasakian manifold* (M^{2n+1}, g), $n > 1$, is a space of constant curvature, provided that X and Y are orthogonal to ξ".

Using the condition in the above theorem that X and Y are orthogonal to ξ in (3.3) and replacing U by ξ, and choosing $X = W = e_i$, $Y = Z = e_j$, and taking summation over i, j, $1 \le i, j \le 2n+1$, we get

$$\tilde{\xi}[R] = A(\xi)\,R$$
$$+2(n+1)[(2n+1-\frac{r}{n})A(\xi)-(2n+1)B(\xi)] \qquad (3.5)$$

Hence we have the theorem:

Theorem 3.4 Suppose M^{2n+1} is a $\varphi-$ recurrent sasakian space of constant curvature immersed in a generalized concircular recurrent Kahlerian manifold M^{2n+2}, ξ is orthogonal to $\rho''-\lambda\rho'$ if and only if $\tilde{\xi}[R] = R\,A(\xi)$, where $\lambda = 1-\frac{r}{n(2n+1)}$ and ρ', ρ'' are the vector fields associated with the one forms A and B.

By virtue of the theorem 3.2, theorem 3.4 and using (1.20) we have:

Corollary 3.5 Suppose M^{2n+1} is a φ-recurrent sasakian space of constant curvature immersed in a generalized concircular recurrent Kahlerian manifold M^{2n+2}. Then the mean curvature $H = 1$ in M^{2n+1}.

Acknowledgement: The second author would like to acknowledge the support of Christ University administration for deputing him to Ph.D Programme.

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