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Unique Metro Domination of a Ladder

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Abstract

A dominating set D of a graph G which is also a resolving set of G is called a metro dominating set. A metro dominating set D of a graph $G(V, E)$ is a unique metro dominating set (in short an UMD-set) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$ and the minimum cardinality of an UMD-set of G is the unique metro domination number of G. In this paper, we determine unique metro domination number of $P_n \times P_2$.

Keywords: domination, metric dimension, metro domination, uni-metro domination

Mathematics Subject Classification (2010): 05C20, 05C26

1. Introduction

All the graphs considered in this paper are simple, connected and undirected. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d(u, v)$. For a vertex $v \in V(G)$, the closed neighborhood of v is given by $N[v] = \{u \in V(G) : d(u, v) \leq 1\}$.

Let $G(V, E)$ be a graph. For each ordered subset $S = \{v_1, v_2, \ldots, v_k\}$ of V, each vertex $v \in V$ can be associated with a vector of distances denoted by $\Gamma(v/S) = (d(v_1, v), d(v_2, v), \ldots, d(v_k, v))$. The set S is said to be a *resolving set* of G, if $\Gamma(v/S) \neq \Gamma(u/S)$, for every $u, v \in V - S$. A resolving set of minimum cardinality is a *metric basis* and cardinality of a metric basis is the *metric dimension* of G. The k-tuple, $\Gamma(v/S)$ associated to the vertex $v \in V$ with respect to a Metric basis S, is referred as a *code generated by S* for that vertex v. If $\Gamma(v/S) = \{c_1, c_2, \ldots, c_k\}$, then $c_1, c_2, \ldots, c_k$ are called components of the code of v generated by

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S and in particular c_i, $1 \leq i \leq k$, is called i^{th}-component of the code of v generated by S.

A *dominating set* D of a graph $G(V, E)$ is the subset of V having the property that for each vertex $v \in V - D$ there exists a vertex u in D such that $uv \in E$. A dominating set D of G which is also a resolving set of G is called a *metro dominating set* or in short an *$MD-set$*. A metro dominating set D of a graph $G(V, E)$ is a *unique metro dominating set* (in short an UMD-set) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$ and the minimum of cardinalities of UMD-sets of G is the *unique metro domination number* of G, denoted by $\gamma_{u\beta}(G)$.

The *Cartesian product* of the graphs G_1 and G_2 denoted by $G_1 \times G_2$, is the graph G such that $V(G) = V(G_1) \times V(G_2)$ and $E(G) = \{\{(u_1, v_1), (u_2, v_2)\} :$ either $[u_1 = u_2$ and $v_1v_2 \in E(G_2)]$ or $[v_1 = v_2$ and $u_1u_2 \in E(G_1)]$

Metric dimensions and locating dominating sets of certain classes of graphs were studied in [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14]. In this paper we determine unique metro domination number of a ladder $P_n \times P_2$.

2. Dominance in Ladder

For convenience, we represent the vertex (g_i, h_k) of a Cartesian product $G \times H$ as $v_{i,k}$. The graph $P_n \times P_2$ is called a ladder. Let D be a minimal dominating set for $P_n \times P_2$.

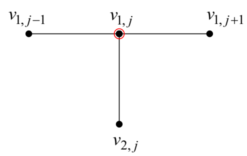


Figure 1: $v_{1,j}$ dominates at most three other vertices

Let $v_{1,j} \in D$, $2 \leq j < n$. Then $v_{1,j}$ can dominate $v_{1,j-1}, v_{1,j+1}$ and $v_{2,j}$. Further $P_n \times P_2$ contains $2n$ vertices. Hence $|D| + 3|D| \geq 2n \Rightarrow |D| \geq \frac{n}{2}$. Thus we have the following lemma:

Lemma 2.1. *If D is a minimal dominating set for $P_n \times P_2$, then $|D| \geq \frac{n}{2}$.*

Let P and P' be two distinct uv-paths between two vertices u, v in $P_n \times P_2$. The vertices u and v are said to be neighboring vertices if u and v are the only vertices of D contained in one of the paths P, P'. If P (or P') is the path containing only u, v from D, then the set of all vertices of $P - \{u, v\}$ is called a gap of D determined by u and v and is denoted by γ. The number of vertices in the gap is called order of the gap and is denoted by $o(\gamma)$.

In order to reduce $|D|$, we have to increase the order of the gaps of D. The most suitable gaps are of order 3. Consider $v_{j,1}$ and $v_{j+4,1}$, the neighboring vertices on first horizontal projection H_1, then $v_{j+1,1}$ is dominated by $v_{j,1}$ and $v_{j+3,1}$ is dominated by $v_{j+4,1}$. The vertex $v_{j+2,1}$ in the first horizontal projection H_1 is dominated by $v_{j+2,2}$ in the second horizontal projection H_2. Thus we have obtained a gap of order 3 on first horizontal projection H_1.

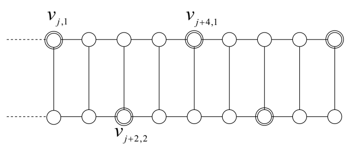


Figure 2: Illustration of an UM-Dominating vertices.

Further, if $v_{j+2,2}$ and $v_{j+6,2}$ are neighboring vertices of a gap of order 3 on second horizontal projection H_2, then $v_{j+4,1}$ will dominate $v_{j+4,2}, v_{j+3,2}$ is dominated by $v_{j+2,2}$ and $v_{j+5,2}$ is dominated by $v_{j+6,2}$. This gives a gap of order 3 on second horizontal projection H_2.

Suppose $v_{j,1}$ and $v_{j+5,1}$ are neighboring vertices of a gap of order 4. Then $v_{j+1,1}$ and $v_{j+4,1}$ are dominated by $v_{j,1}$ and $v_{j+5,1}$ respectively.

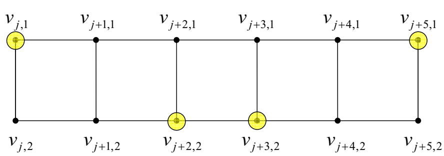


Figure 3: A UMD-set of the graph $P_6 \times P_2$

As $v_{j+2,1}$ and $v_{j+3,1} \in V - D$, it is essential to include $v_{j+2,2}$ and $v_{j+3,2}$ in D. This creates a gap of order 0 on second horizontal projection H_2; which in turn increases $|D|$. Thus we have the following lemma;

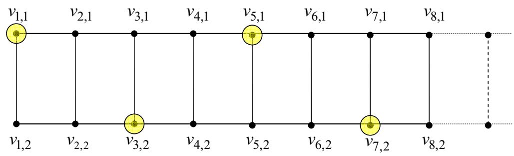
Lemma 2.2. *In order to minimize $|D|$, gaps in each of the Horizontal Projections of $P_n \times P_2$ of order 3 are suitable.*

If $\{v_{1,1}, v_{2,1}, v_{1,2}, v_{2,2}\} \bigcap D = \emptyset$, then $v_{1,1}$ and $v_{1,2}$ are not dominated by any vertex of D, a contradiction that D is a minimal dominating set.

Hence we have;

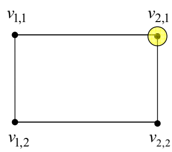
Lemma 2.3. *Let D be a minimal dominating set for $P_n \times P_2$. Then at least one of the vertices in $\{v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}\}$ must be in D.*

Suppose that $v_{1,1} \in D_1$ for some minimal dominating set D_1, then D_1 contains $v_{3,2}, v_{5,1}, v_{7,2}, \ldots$.



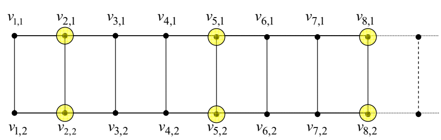
Similarly by symmetry, if $v_{1,2} \in D_2$ for some minimal dominating set, then D_2 contains $v_{3,1}, v_{5,2}, v_{7,1}, \ldots$. So, if $v_{1,1} \in D_1$ and $v_{1,2} \in D_2$, for some minimal dominating set D_1 and D_2, then $|D_1| = |D_2|$. Hence with out loss of generality we assume $v_{1,2}$ will not lie in any minimal dominating set.

Suppose $v_{1,1}$ and $v_{1,2}$ both are not in D, then both $v_{2,1}$ and $v_{2,2}$ are in D; for if $v_{2,1} \in D$ and $v_{2,2} \notin D$, then $v_{1,2}$ is not dominated by any vertex in D.This leads to ;



Lemma 2.4. *If D is any minimal dominating set of $P_n \times P_2$ such that $v_{1,1}, v_{1,2} \notin D$, then D contains both $v_{2,1}$ and $v_{2,2}$.*

Now, if both $v_{2,1}$ and $v_{2,2}$ are in a minimal dominating set D of minimum cardinality, then $D = \{v_{2+3k,1}, v_{2+3k,2} | \ k = 0, 1, \ldots, \} \subseteq V(P_n \times P_2)$.



Lemma 2.5. *If D_1 and D_2 are minimal dominating sets of $P_n \times P_2$ such $v_{1,1} \in D_1$ and $v_{2,1} \in D_2$, then $|D_1| \leq |D_2|$.*

Proof. Even though $v_{1,1} \in D_1$ dominates only two vertices $v_{1,2}$ and $v_{2,1}$ in $V - D_1$, other vertices $v_{2,3}, v_{1,5}, v_{2,7}, \ldots$ of D_1 dominates 3 distinct vertices of $V - D_1$. But each vertex of D_2 dominates two vertices of $V - D_2$. Hence, $|D_2| \geq |D_1|$. □

In view of Lemma 2.4 and Lemma 2.5, here onwards we consider only such D of $P_n \times P_2$ with $v_{1,1} \in D$ and for such a set D, we get the following result;

Lemma 2.6. *The set D is $\{v_{1+4i,1},\ v_{3+4j,2} : 0 \leq 1 + 4i \leq n, 0 \leq 3 + 4j \leq n\}$. If $n = 1 + 4k$, then $|D| = \frac{n+1}{2}$.*

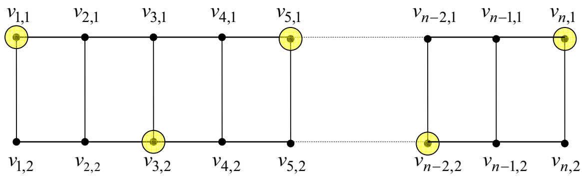


Figure 4: When $n = 4k + 1$.

Proof. On the first horizontal projection H_1, when $i = 0, 1, 2, \ldots, \frac{n-1}{4}$; $k + 1$ vertices $v_{1,1}, v_{5,1}, \ldots, v_{n,1}$ are in D. On the second horizontal projection H_2 when $j = 0, 1, \ldots, \frac{n-1}{4} - 1$; k vertices $v_{3,2}, v_{7,2}, \ldots, v_{n-2,2}$ are in D. Thus, D has $k+1+k$ vertices. Therefore, $|D| = 2k+1 = 2\left(\frac{n-1}{4}\right)+1 = \frac{n+1}{2}$. □

Lemma 2.7. *In Lemma 2.6, if $n = 4k + 3$, then $|D| = \frac{n+1}{2}$.*

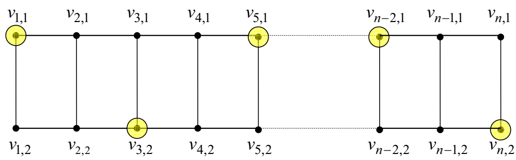


Figure 5: When $n = 4k + 3$.

Proof. On the first horizontal projection H_1, $v_{1,1}, v_{5,1}, \ldots, v_{4k+1,1}$ are in D and on the second horizontal projection H_2, $v_{3,2}, v_{7,2}, \ldots, v_{4k+3,2}$ are in D. Thus, $k + 1$ vertices on the first horizontal projection H_1 and $k + 1$ vertices on the second horizontal projection H_2 are in D. Hence $|D| = 2k + 2 = \frac{n-3}{2} + 2 = \frac{n+1}{2}$. □

Lemma 2.8. *In Lemma 2.6, if $n = 4k + 2$, then $|D| = \frac{n}{2} + 1$.*

Proof. By Lemma 2.6, D contains $v_{1+4i,1}, 0 \leq i \leq \frac{n-2}{4}$ and $v_{3+4j,2}$, $0 \leq j \leq \frac{n-2}{4}$. The vertex $v_{n-1,1}$ on the first horizontal projection H_1 is in D as $n-1 = 1+4k$ and the vertex $v_{n-3,2}$ on the second horizontal projection H_2 belongs to D as $n - 3 = 4j + 3$.

Observe that $v_{n,1}, v_{n-2,1}$ and $v_{n-1,2}$ are dominated by $v_{n-1,1}$. The vertex $v_{n-2,2}$ is dominated by $v_{n-3,2}$. But $v_{n,2}$ is not dominated by any vertex in D. Hence it is required to include one more vertex in D. We include $v_{1,n}$ in D. Thus D contains $k + 1 = \frac{n-2}{4} + 1$ vertices from first horizontal projection H_1 and $k + 1$ vertices from second horizontal projection H_2. Thus, $|D| = 2k + 2 = 2\frac{n-2}{4} + 2 = \frac{n}{2} + 1$. Hence the lemma □

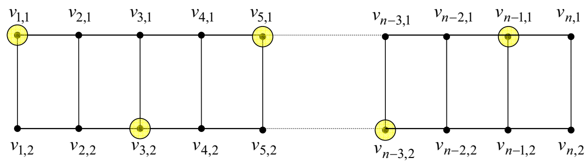


Figure 6: When $n = 4k + 2$.

When $n = 4k$, D contains $v_{1+4i,1}$, $0 \le i \le \frac{n-4}{4}$ and $v_{3+4j,2}$, $0 \le j \le \frac{n-4}{4}$. Hence $v_{n-3,1}$ and $v_{n-1,2}$ are in D. But then $v_{n,1}$ is not dominated by any vertex in D. Hence we include $v_{n,2}$ in D. Therefore, the set D will have k vertices from first horizontal projection and $k + 1$ vertices from the second horizontal projection. Thus, $|D| = k + k + 1 = 2k + 1 = \frac{n}{2} + 1$. This leads to the lemma

Lemma 2.9. *In Lemma 2.6, if $n = 4k$, then $|D| = \frac{n}{2} + 1$.*

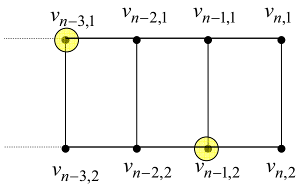


Figure 7: When $n = 4k$.

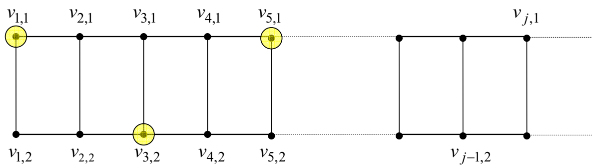
When $n = 4k + 1$, from Lemma 2.6, $|D| = \frac{n+1}{2} = \lfloor \frac{n+2}{2} \rfloor$. When $n = 4k + 3$, from Lemma 2.7, $|D| = \lfloor \frac{n+2}{2} \rfloor$. From lemma 2.8, when $n = 4k + 2$, $|D| = \frac{n}{2} + 1 = \lfloor \frac{n+2}{2} \rfloor$ and from Lemma 2.9 when $n = 4k$, $|D| = \frac{n}{2} + 1 = \lfloor \frac{n+2}{2} \rfloor$. Thus in all the cases we conclude

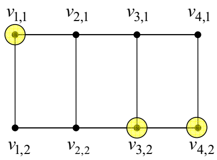
$$\gamma(P_n \times P_2) = \left\lfloor \frac{n+2}{2} \right\rfloor$$

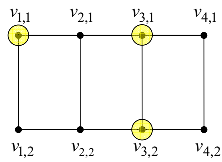
Lemma 2.10. *For an integer $n \ge 5$, the vertices $v_{1,1}, v_{3,2}$ and $v_{5,1}$ of $P_n \times P_2$ resolves all the vertices of $V - D$.*

Proof. Observe that $d(v_{1,1}, v_{j,1}) = d(v_{1,1}, v_{j-1,2}) = j - 1$. If $j \ge 4$, then $d(v_{3,2}, v_{j,1}) = j - 2$ and $d(v_{3,2}, v_{j-1,2}) = j - 4$. Hence $d(v_{3,2}, v_{j,1}) \ne d(v_{3,2}, v_{j-1,2})$. Thus $v_{3,2}$ resolves all vertices with $j \ge 4$.

If $j \le 3$, then $v_{5,1}$ resolves these vertices, for; $d(v_{5,1}, v_{j,1}) = 5 - j$ and $d(v_{5,1}, v_{j-1,2}) = 6 - j$ and hence $d(v_{5,1}, v_{j,1}) \ne d(v_{5,1}, v_{j-1,2})$. Thus, $v_{5,1}$ resolves the vertices with $j \le 3$. □





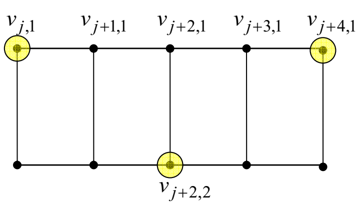


When $n = 4$, by Lemma 2.9, $D = \{v_{1,1}, v_{3,2}, v_{4,2}\}$. But then, D does not resolve $V - D$. Code of $v_{2,1}$ is (1, 2, 3) and code of $v_{1,2}$ is also (1, 2, 3). Further code of $v_{3,1}$ is (2, 1, 2) and code of $v_{2,2}$ is also (2, 1, 2). Therefore, we delete $v_{4,2}$ from D. Then $v_{4,1}$ is not dominated by D. If $v_{4,1}$ is included in D, then $v_{4,2}$ is not uniquely dominated by D. If we take $D = \{v_{1,1}, v_{3,2}, v_{3,1}\}$, then D is a dominating set. Codes of $v_{2,1}, v_{1,2}, v_{2,2}, v_{4,1}$ and $v_{4,2}$ are respectively (1, 2, 1), (1, 2, 3), (2, 1, 2), (3, 2, 1) and (4, 1, 2), which are all distinct.

Lemma 2.11. *When $n = 4$, the vertices $v_{1,1}$ and $v_{4,1}$ resolves all vertices of $V - D$.*

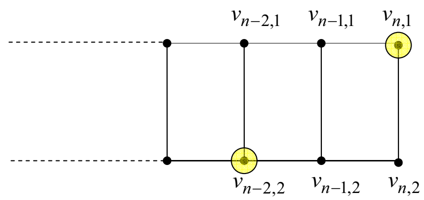
Remark 2.12. *We note that when $n = 4$, $D = \{v_{1,1}, v_{3,2}, v_{3,1}\}$ is a UMD-set with $|D| = 3$ and $\left\lfloor \frac{n+2}{2} \right\rfloor = 3$.*

Now, consider the minimal dominating sets used in Lemma 2.6 to Lemma 2.9. Take any gap of order 3, say, with neighboring vertices $v_{1,j}$ and $v_{1,j+4}$. Then $v_{1,j}, v_{1,j+4}$ and $v_{2,j+2}$ are in D.

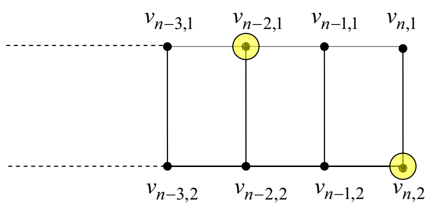


The vertex in the gap $v_{j+1,1}$, $v_{j+2,1}$ and $v_{j+3,1}$ are uniquely dominated. Further $v_{1,2}$ and $v_{2,2}$ are not in any gap of order 3 (in all cases, lemma 2.6 to 2.9). However, $v_{1,2}$ is dominated uniquely by $v_{1,1}$ and $v_{2,2}$ is dominated uniquely by $v_{3,2}$.

When $n = 4k + 1$ (as in lemma 2.6), $v_{n-1,2}$ and $v_{n,2}$ are not in a gap of order 3. There are uniquely dominated by the vertices $v_{n-2,2}$ and $v_{n,1}$ respectively.

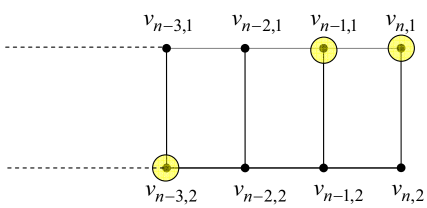


When $n = 4k + 3$ (as in lemma 2.7), $v_{n-1,1}$ and $v_{n,1}$ are the vertices which are not in a gap of order 3. The vertices $v_{n-2,1}$ and $v_{n,2}$ in D (respectively) uniquely dominate them.



When $n = 4k + 2$ (as in lemma 2.8), $v_{2,1}$ and $v_{2,2}$ are uniquely dominated.

Note that $v_{n-1,1}, v_{n,1}$ and $v_{n-3,2}$ are in D. The vertex $v_{n-2,2}$ is uniquely dominated by $v_{n-3,2}$, the vertex $v_{n-1,2}$ is uniquely dominated by $v_{n-1,1}$ and the vertex $v_{n,2}$ is uniquely dominated by $v_{n,1}$.



When $n = 4k$, the vertices $v_{n-3,1}, v_{n-1,2}$ and $v_{n,2}$ are in D and they uniquely dominate the vertices $v_{n-2,1}$, $v_{n-1,1}$ and $v_{n,1}$ respectively. Hence D is a UMD-set in all the four cases. Finally, when $n = 3$, the set $D = \{v_{1,1}, v_{3,2}\}$ is a unique dominating set but does not resolve $V - D$. Similarly, by Symmetry, $D = \{v_{1,2}, v_{3,1}\}$ is also not an UMD-set. The set $D = \{v_{2,1}, v_{2,2}\}$ is a unique dominating set (UD-set) but does not resolve $V - D$.

If D consists of any two adjacent vertices, then it is not a dominating set. As gaps of order 1 are not allowed, no set with 2 vertices can be a UMD-set. Therefore, $|D| > 2$ for a UMD-set. We now observe that $D = \{v_{1,1}, v_{2,1}, v_{3,1}\}$ is a UMD-set. Therefore, $\gamma_{\mu\beta}(P_3 \times P_2) = 3$. Lastly, when $n = 2$, the graph is isomorphic to the cycle C_4, hence it follows that its Unique metro domination number is 2.

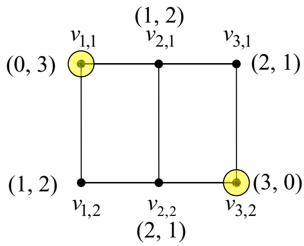


Figure 8: An UD-set but not a resolving set of the case $n = 3$.

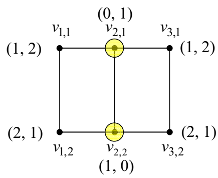


Figure 9. An UD-set but not a resolving set of the case $n = 3$.

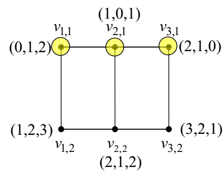


Figure 10. An UD-set but not a resolving set of the case $n = 3$.

The fact that $\gamma_{\mu\beta}(P_m \times P_2) \geq \gamma(P_m \times P_2)$ and the discussions we had so far leads to the theorem,

Theorem 2.13. *For any integer $n \geq 2$,*

$$\gamma_{\mu\beta}(P_n \times P_2) = \begin{cases} 3\,, & \text{if } n = 3 \\ \left\lfloor \frac{n+2}{2} \right\rfloor\,, & \text{otherwise} \end{cases}$$

3. Conclusion

We intend to find unique metrodomination number of $C_n \times P_2$. Finding unique upper metrodomination number also is a big task.

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