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Independent Monophonic Sets in Graphs

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Abstract

In this paper, we obtain two variables for the connected (p,q)-graphs G which is the minimum size of an independent monophonic set and an outer independent monophonic set of G, termed as an independent monophonic number $m_\alpha(G)$ and an outer independent monophonic number $m_\alpha^\perp(G)$ of the connected (p,q)-graphs G, respectively.

Keywords: Chordless Path, Monophonic Number, Independence Number, Geodetic Number, Independent Monophonic Number, Outer Independent Monophonic Number

Mathematics Subject Classification (2010): 05C05, 05C12, 05C69

1. Introduction

All graphs considered in this paper are finite, connected, simple and undirected. For the basic graph theoretical terms, refer to [**? ? ?**]. A graph G is an ordered couplet of V and E, denoted by $G = (V, E)$ where V is called vertex set and E is an edge set of G respectively. A graph G is said to be a (p,q)-graph if $n(V) = p$ and $n(E) = q$. The counts p and q are termed as order and size of a graph G respectively. A path $P\colon [x_1, x_2, \ldots, x_n]$ in a connected (p,q)-graph G is a sequence of adjacent vertices $x_1, x_2, \ldots, x_n$ in G. The distance $d(u,v)$ between two vertices u and v is the length of a shortest $u-v$ path in a connected graph G. A $u-v$ path P is called geodesic if it is a shortest $u-v$ path and $E(P) = d(u,v)$.

A subset $S \subseteq V$ in a connected graph G is called a geodetic set if every vertex in $V-S$ lies on a shortest path between two vertices from

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S.The minimum cardinality among all the geodetic set is termed as the geodetic number $g(G)$ of G. The geodetic number of a graph was introduced in [**? ? ?**].

An edge $e = \{x_s, x_t\} \in E(P)$ in a connected graph G is said to be a chord if $t \geq s+2$. A path P is a monophonic path if it is a chordless path (or written as m-path). The monophonic path is special subgraph of a connected graph G. All the variables related to the distance in a connected (p, q)-graphs G can be defined by the monophonic path, see [**? ?**].
The concept of monophonic path first seemed at [**?**] and was initiated by Ignacio M. Pelayo et al. [**?**]. An interval $J : V \times V \longrightarrow \mathcal{P}(V)$ is a mapping from the vertex set V to the power set $\mathcal{P}(V)$ such that

$$J(x, y) = \{ \text{ all the vertices lying on some x-y m-path } \}.$$

$J(x, y)$ is known as a monophonic interval of x and y. The closed monophonic interval of x and y is $J[x, y] = J(x, y) \cup \{x, y\}$. For any set $M \subseteq V$, monophonic closure of M is denoted by M^c and it is defined as

$$M^c = \bigcup_{x,y \in M} J[x, y].$$

If $M^c = V$, then we say that M is a monophonic set of G. In other words, a set of vertices $M \subseteq V$ is said to be a monophonic set of G if each vertex $v \in V$ lies in an $x - y$-monophonic path in G for some $x, y \in M$. The minimum cardinalities of all monophonic sets in G is the monophonic number of G. It is notated as $m(G)$ and written as

$$m(G) = \min\{n(M) : \text{M, monophonic set of G}\}.$$

A vertex $v \in V$ is said to be a monophonic vertex of G if v belongs to every minimum monophonic sets of G. If G has a unique monophonic set M, then all the vertices of M are monophonic.

The monophonic distance in a connected graph was studied by A. P. Santhakumaran et.al [**?**]. The concept of connected monophonic number and upper monophonic number of a graph was introduced by J.John et.al[**? ? ?**]. But the monophonic number of a graph have not yet been fully explored and discussed in details. For the latest literature work on the topic monophonic number and the combined variable monophonic domination number of a graph, refer to [**? ? ? ? ? ?**].

A vertex $v \in V$ in G is said to be an extreme vertex if induced subgraph $< N(v) >$ is complete. The set of all extreme vertices of G is denoted by $Ext(G)$ and the set of all end vertices of G by $End(G)$.

An independent vertex set $S \subseteq V$ is a set of pairwise non-adjacent vertices in G.The maximum cardinality of an independent set of vertices is called an independence number of G. It is denoted by $\beta = \beta(G)$ [Bollobas 1981],refer [**?** **?**] and it is inscribed as

$$\beta(G) = \max\{n(S) : S,\ \text{independent set of G}\}$$

Independent sets were introduced into the communication theory on noisy channels [**?**]. The independence number of a connected graph is difficult to compute. Finding a maximum independent set is an NP-hard problem. Many upper and lower bounds for the independence number of a graph appear in the available Mathematical literature. The common upper bounds of $\beta(G)$ are $\beta(G) \leq p - \frac{q}{\Delta}$ (known as KWOK bound) and $\beta(G) \leq p - \delta$ (known as Minimum degree bound) where δ and Δ are the minimum and maximum degrees of (p, q)-graphs G.

For any connected graph G, some subsets of vertex set V are monophonic but not an independent set in G and conversely. This clear-cut motivation help us to study the subsets of vertex set V which is both monophonic and independent.

In the rest of the paper, we propose two graph variables, namely an independent monophonic number and outer independent monophonic number of connected (p, q)-graphs G respectively.

2. Independent Monophonic Number of a Graph

Definition 2.1. An independent monophonic set $M \subseteq V$ in a connected (p, q)-graph G is a set of vertices which is both monophonic and independent. The minimum size of an independent monophonic set is called an independent monophonic number of G. It is denoted by $m_\alpha(G)$ and it is inscribed as

$$m_\alpha(G) = \min\{n(M) : M,\ \text{independent monophonic set of G}\}$$

An independent monophonic set of size $m_\alpha(G)$ is known as m_α-set.

An independent monophonic set is always a monophonic set but the converse need not be true. An independence number $\beta(G)$ is the maximum size of an independent set in G, so that the graph variables $m_\alpha(G)$ and $\beta(G)$ are different in the general sense. It follows that $m_\alpha(G) \leq \beta(G) \leq p$.

Definition 2.2. A monophonic vertex set $M \subseteq V$ is said to be an outer independent monophonic set if the vertices of $V - M$ are non adjacent. The minimum size of an outer independent monophonic set is called an outer independent monophonic number of G. It is denoted by $m_\alpha^\perp(G)$ and it is inscribed as

$$m_\alpha^\perp(G) = \min\{n(M) : M, \text{ outer independent monophonic set of G}\}$$

An outer independent monophonic set of size $m_\alpha^\perp(G)$ is known as $m_\alpha^\perp$-set.

Example 2.3. Consider a graph G given in the Figure 1. The sets $M_1 = \{v_1, v_3\}$ and $M_2 = \{v_3, v_5\}$ are the minimum monophonic sets of G, also they are independent. Therefore $m(G) = 2$ and $m_\alpha(G) = 2$.

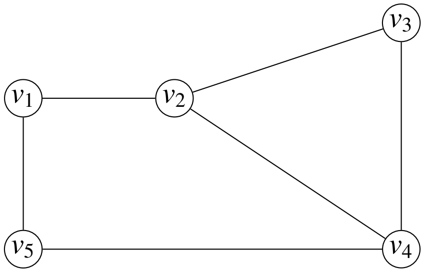

Figure 1: A graph G with $m(G) = m_\alpha(G) = 2$ and $m_\alpha^\perp(G) = 0$

It follows that the variables $m(G)$ and $m_\alpha(G)$ are coincide and v_3 is the only monophonic vertex of G. $V - M_1 = \{v_2, v_4, v_5\}$ or $\{v_1, v_2, v_4\}$. Since the vertices of $V - M_1$ are adjacent, M_1 is not an outer independent monophonic set of G.
Hence graph G has an independent monophonic set, but does not have an outer independent monophonic set, that is $m_\alpha^\perp(G) = 0$

Example 2.4. Consider the graph G given in the Figure 2.

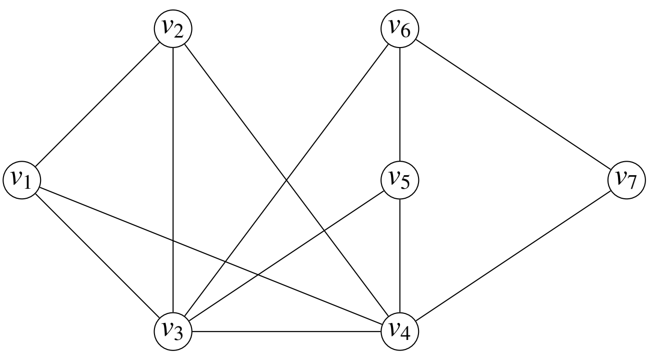

Figure 2: A graph G with $m(G) = 5$, $m_\alpha(G) = 0$ and $m_\alpha^\perp(G) = 2$

The set $M = \{v_1, v_2, v_3, v_5, v_7\}$ is the minimum monophonic set with some adjacent vertices. Clearly see that M is not an independent monophonic set of G and $m_\alpha(G) = 0$. But $V - M = \{v_4, v_6\}$ is independent. It follows that $m_\alpha^\perp(G) = 7 - 5 = 2$.

Note 2.5. In a cycle graph C_6, the two monophonic variables are coincide but an independence number of C_6 is different, that is $m(C_6) = m_\alpha(C_6) = 2$ and $m_\alpha^\perp(C_6) = 0$ but the independence number of C_6 is $\beta(C_6) = 3$.

For the connected (p, q)-graphs G, the trivial lower and upper bounds of $m(G)$ is $2 \leq m(G) \leq p$. Since complete graphs G has no non-adjacent vertices, the trivial lower and upper bounds of an independent monophonic number $m_\alpha(G)$ becomes $0 \leq m_\alpha(G) \leq \beta(G) \leq p$.

Next example shows that all the variables of some graphs G are lying in the strict boarder, that is $0 < m_\alpha(G) < \beta(G) < p$

Example 2.6. Consider a graph G given in the Figure 3. Here $p = 7$ and $M_1 = \{v_3, v_6, v_7\}$ is the minimum monophonic set of G so that $m(G) = 3$. But $M_2 = \{v_1, v_3, v_6, v_7\}$ is the maximum independent set in G so that $\beta(G) = 4$. It follows that M_1 is both monophonic and independent set with minimum cardinality, that is $m_\alpha(G) = 3$. But the set $V - M = \{v_1, v_2, v_4, v_5\}$ is not an independent set of G.
Hence we obtain the strict inequality $0 < m_\alpha(G) < \beta(G) < p$ in the given graph G.

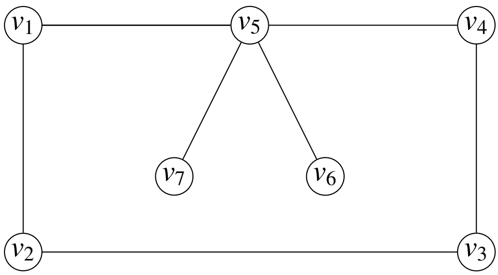

Figure 3: A graph G with strict inequality $0 < m(G) < \beta(G) < p$

For the common terms and symbols, refer to [**? ? ? ?**]. Based on the Definitions 2.1, 2.2, we present some preliminary observations which are to be used wherever required.

1. All extreme vertices of a connected (p, q)- graph G are included in all independent monophonic sets of G.

2. If the set $Ext(G)$ is an independent monophonic set of connected graphs G, then it is unique .

3. An extreme vertex in a connected graph G is also a monophonic vertex of the connected graph G.

4. All the vertices of the complete graph K_p are known as extreme vertices and also known as monophonic vertices of K_p.

5. All end vertices of a connected graph G are included in all independent monophonic sets of G.

6. If both the sets m_α- sets and $m_\alpha^\perp$ -sets are non-empty sets in V, then
$$m_\alpha^\perp(G) = |V| - m_\alpha(G) = p - m_\alpha(G).$$

7. The graph variables $m_\alpha(G)$ and $m_\alpha^\perp(G)$ are may or may not be exists in a connected graph G.
In the case of a complete graph K_p, $m_\alpha(K_p) = 0$ and for the Star graph S_6 on 6 vertices, $m_\alpha^\perp(S_6) = 1$.

3. Basic Results

Proposition 3.1. For a complete graph $G = K_p$, $m_\alpha(G) = 0$.

Proof. Let $V = \{v_1, v_2, \ldots v_p\}$ be the vertex set of G. Then we see that V is the unique monophonic set of G. Since all the vertices are adjacent, graph G does not have an independent monophonic set so that $m_\alpha(G) = 0$. □

Proposition 3.2. For an edge deleted graph $G = K_p - \{e\}$, $m_\alpha(G) = 2$ where K_p is a complete graph on p vertices and $e \in E(K_p)$, $p \geq 3$.

Proof. Consider an edge $e = \{u, v\} \in E(K_p)$. Choose a subset $M = \{u, v\}$ of $V(G)$. For each vertex $w \in V(G) - M$, there exists a $u - v$ monophonic path of length 2 containing w. Since $d(u) = d(v) = p - 2$, M is both monophonic and an independent set of G. It gives that $m_\alpha(G) = 2$. □

Proposition 3.3. For an edge deleted graph $G = K_p - \{e_1, e_2\}$,

$$m_\alpha(G) = \begin{cases} 2 & \text{edges } e_1 \text{ and } e_2 \text{ are non-adjacent} \\ 3 & \text{edges } e_1 \text{ and } e_2 \text{ are adjacent} \end{cases}$$

where K_p is a complete graph and edge set $\{e_1, e_2\} \subseteq E(K_p)$, $p \geq 4$.

Proof. Consider the edge set $\{e_1, e_2\} \subseteq E(K_p)$ where $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$ for the vertices $u_1, u_2, v_1, v_2 \in V(K_p)$.

Case(i) (e_1 and e_2 are non-adjacent edges): Let $M = \{u_1, v_1\}$ be a subset of V. Clearly, $d(u_1) = d(v_1) = p - 2$. By the Proposition 3.2, the set M is a minimum independent monophonic set of G and its cardinality is $m_\alpha(G) = 2$.

Case(ii) (e_1 and e_2 are adjacent edges): In this case e_1 and e_2 have a common vertex say $v_1 = u_2$. Let $M = \{u_1, v_1, v_2\} \subseteq V$. Then every vertex in $V - M$ lies in a $u_1 - v_1$ monophonic path of length 2. Thus M is an independent monophonic set of G and $2 \leq m_\alpha(G) \leq 3$.

Finally, we have to show that $m_\alpha(G) = 3$. Let $M = \{x, y\}$ be an independent monophonic set of G. Since $p \geq 4$, vertices x and y are not adjacent in G. Clearly M is either $\{u_1, u_2\}$ or $\{u_2, v_2\}$. In all cases, there is a vertex in $\{u_1, v_1, v_2\} - M$ which does not lie in a monophonic path of some vertices from M. Therefore, the vertex set M of G is an independent monophonic set of G. Hence $m_\alpha(G) = 3$. □

Proposition 3.4. For a graph $G = K_p - \{e_1, e_2\}$ obtained from K_p by removing non-adjacent edges e_1 and e_2, $m_\alpha^\perp(G) = p - 2$, where $p \geq 5$.

Proof. For the vertices $u, v, x, y \in V$, we choose two non-adjacent vertices $e_1 = \{u, v\}$ and $e_2 = \{x, y\}$. Clearly $M = \{u, v\}$ or $\{x, y\}$ is the monophonic set with minimum size and $V - M$ is not independent. If $M_1 = M \cup N$ where $N \subseteq V - M$ having $p - 4$ elements and $\max deg(v) \leq p - 1$, for $v \in N$. Therefore the closure $M_1^c = V$ and $V - M_1$ is an independent set of G. Hence M_1 is an outer independent monophonic set of G. It gives that $n(M_1) = n(M \cup N) = 2 + p - 4 = p - 2$. Hence $m_\alpha^\perp(G) = p - 2$. □

Proposition 3.5. Let $\xi_\alpha(G) = \{G : m_\alpha(G) \geq 1\}$ be the family of connected graphs. Then $m_\alpha(G) = 2$ if and only if $m(G) = 2$

Proof. Assume that $m_\alpha(G) = 2$. By the definition , it follows that $m(G) = 2$. Conversely assume that $m(G) = 2$, there exists a minimum monophonic set $M = \{x, y\}$ such that $n(M) = 2$. Clearly we have $V - M \neq \phi$ and every $x - y$ monophonic path contains one more vertex and $d_m(x, y) \geq 2$. Hence M is independent and $m_\alpha(G) = 2$. □

Proposition 3.6. For connected (p, q)-graphs G with maximum degree Δ,

$$m_\alpha(G) \geq \frac{\sum_{i=1}^{k}(1 + deg(v_i))}{1 + \Delta}$$

Proof. The vertex set V may or may not be an independent set of G, so we may assume without loss of generality that $m_\alpha(G) = k$ where $k \leq p$. Then there exists a monophonic set $M = \{v_1, v_2, \ldots v_k\}$ such that $n(M) = k$. For $i = 1, 2, 3, \ldots k$, since $deg(v_i) \leq \Delta$, sum of all degrees,

$$deg(v_1) + deg(v_2) + \cdots + deg(v_k) \leq k\Delta.$$

This implies that $(1+deg(v_1))+(1+deg(v_2))+\cdots+(1+deg(v_k)) \leq k+k\Delta$ and $\sum_{i=1}^{k} 1+deg(v_i) \leq k(1+\Delta)$. Hence we obtain the required bound

$$k \geq \frac{\sum_{i=1}^{k} 1+deg(v_i)}{1+\Delta} \Rightarrow m_\alpha(G) \geq \frac{\sum_{i=1}^{k} 1+deg(v_i)}{1+\Delta}$$

Remark 3.7. The trivial upper bound of $m_\alpha(G)$ is $p = n(V)$.

Remark 3.8. If T is a tree of order $p \geq 3$ and $|End(T)| = k$, then $m_\alpha(T) = k$ and independence number $\beta(T) \geq |End(T)|$.

□

4. Some Realizations

Theorem 4.1. For any $k \in Z^+$ with $2 \leq k \leq p-1$, there exists connected graphs G such that $m_\alpha(G) = k$ and $n(V) = p$.

Proof. Choose a positive integer $k \in Z^+$ such that $2 \leq k \leq p-1$. Let $P : [v_1, v_2, \ldots, v_{p-k+1}]$ be a path graph of length at most $p-k$. Adjoin the $k-1$ non-adjacent vertices $\{u_1, u_2, \ldots, u_{k-1}\}$ to an end vertex v_{p-k+1} of the path graph P.

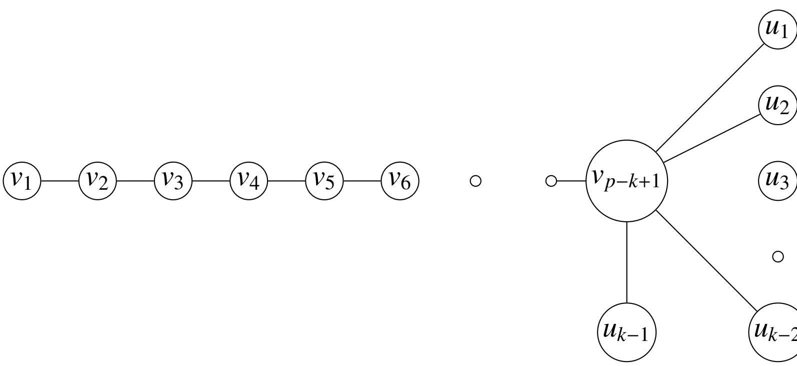

Figure 4: A graph G with $m_\alpha(G) = k$ and $n(V) = p$

We obtain a graph G with $|End(G)| = k$ is given in Figure 4. Clearly $\{u_1, u_2, \ldots, u_{k-1}\}$ is a unique monophonic set of G and all of its vertices are non-adjacent. It follows that $m_\alpha(G) = k$ and total vertex count $n(V) = p-k+1+k-1 = p$. □

Remark 4.2. For $k = p$, there is no connected graph G with the requirements of the Theorem 4.1.

Remark 4.3. For any connected (p,q)-graph, $m_\alpha(G) = 0 \Leftrightarrow G = K_p$

Theorem 4.4. For $\lambda, \mu \in Z^+$ with $2 \le \lambda \le \mu$, there exists a connected graph G such that $m_\alpha(G) = \lambda$, $g(G) = \mu$ and $\beta(G) = \mu + 1$.

Proof. When we consider the case $2 \le \lambda = \mu$, choose any tree with λ-end vertices has the required properties. Next we assume that $2 \le \lambda < \mu$.

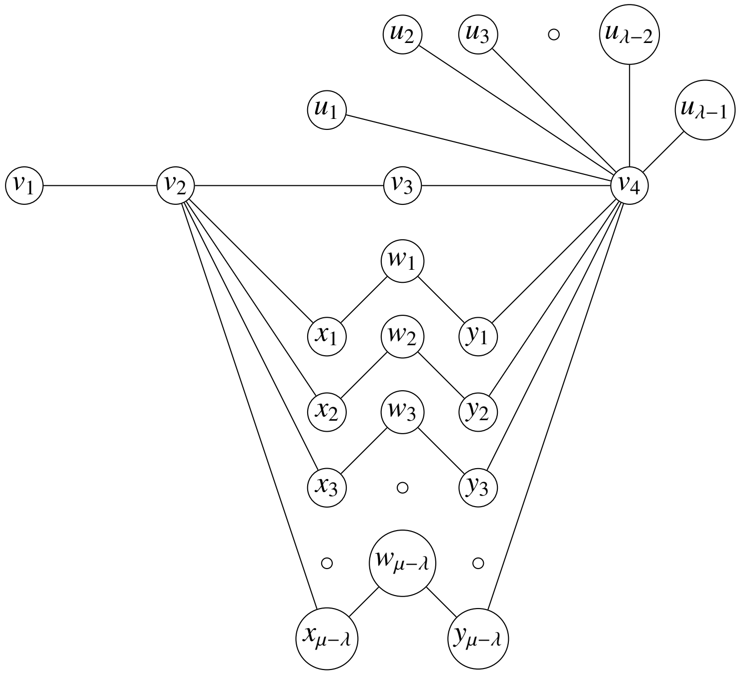

Figure 5: A graph G with $m_\alpha(G) = \lambda$, $g(G) = \mu$ and $\beta(G) = \mu + 1$

Consider $\mu - \lambda$ copies of a path graph P_i ($i = 1, 2, 3 \ldots \mu - \lambda$) of length 2 where $P_1 : [x_1, w_1, y_1]$, $P_2 : [x_2, w_2, y_2]$, $\ldots$, and

$$P_{\mu-\lambda} : [x_{\mu-\lambda}, w_{\mu-\lambda}, y_{\mu-\lambda}].$$

Also choose another path graph on four vertices $P : [v_1, v_2, v_3, v_4]$. We obtain a new graph G by joining each x_i in P_i to v_2 in P and joining each y_i in P_i to v_4 in P. Finally adding a new set of $\lambda - 1$ non-adjacent vertices $\{u_1, u_2, u_3, \ldots u_{\lambda-1}\}$ to end vertex v_4 of P. The new graph G is given in Figure5.

Since $M = \{v_1, u_1, u_2, \ldots u_{\lambda-1}\}$ is a set of monophonic vertices of G, $m(G) = 1 + \lambda - 1 = \lambda$. Also, all the elements of M are independent but not maximal. Therefore M is the minimum independent monophonic set of G so that the variable $m_\alpha(G) = \lambda$.

If we add the set of vertices $N = \{v_3, w_1, w_2, \ldots w_{\mu-\lambda}\}$ to M, then $S = M \cup N$ becomes the maximal independent set of vertices in G so that independence number of G is

$$\beta(G) = |S| = \lambda + 1 + \mu - \lambda = \mu + 1.$$

Finally we have to show that geodetic number $g(G) = \mu$. All the vertices of S except the vertex v_3 are under an extreme category or end vertices. So that $S - \{v_3\}$ is the minimum geodetic set of G. Hence we obtain a minimum geodetic set $S - \{v_3\}$ of the graph G so that geodetic number $g(G) = |S| - 1 = \mu$. □

5. Conclusion

This work can be extended to find the number of non-adjacent vertices in an edge monophonic sets, connected monophonic sets and upper monophonic sets etc in the connected graphs.

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