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ON THE CONTINUOUS MONOTONIC DECOMPOSITION OF SOME COMPLETE TRIPARTITE GRAPHS

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ABSTRACT

Let $G=(v,\mathcal{E})$ be a connected simple graph of order p and size q. If $H_1, H_2, \ldots, H_k$, $k \in \mathbb{N}$ are edge-disjoint subgraphs of $G \ni \mathcal{E}(G) = \mathcal{E}(H_1) \cup \mathcal{E}(H_2) \cup \ldots \cup \mathcal{E}(H_k)$, then $H_1, H_2, \ldots, H_k$ is said to be a decomposition of G. Ascending Subgraph Decomposition (ASD) is a decomposition of G into subgraph H_i (not necessarily connected) $\ni |\mathcal{E}(H_i)| = i$ and is isomorphic to a proper subgraph of H_{i+1}. A decomposition, $\{H_1, H_2, \ldots, H_k, \forall k \in \mathbb{N}\}$, is said to be a Continuous Monotonic Decomposition (CMD) if each H_i is connected and $|\mathcal{E}(H_i)| = i$ for each $i \in \mathbb{N}$. Necessary and Sufficient Conditions for $K_{1,3,m}$, $K_{2,3m}$, $K_{2,5m}$ and $K_{3,5,m}$ to accept CMD.

Key words: Graph Theory, Graph Decomposition, Complete Tripartite Graph, Continuous Monotonic Decomposition, Triangular Numbers.

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I. Introduction

An undirected graph with the property that there is a path between every pair of vertices is known as a *connected graph*. A graph G, referred to here is an undirected connected graph without loops or multiple edges. The degree of a vertex *u* of any graph is the number of edges incident with u and is denoted by d(*u*) and the distance between two vertices *u* and *v* of G is the length of the shortest *u-v* path in G and is denoted by d(*u*,*v*). A graph G is called *n*-regular graph if $deg(v) = n \in \mathbb{N}, \forall v \in V(G)$.

A complete graph with vertices $n \in \mathbb{N}$, denoted by K_n, is a connected simple graph with every vertex is connected with every other vertex by an edge. A graph with n vertices $v_1, v_2, \ldots v_n$, where $n \geq 3$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$ is known as a *cycle*, C_n.

A path of length *t* is denoted by P_t. A complete *m*-partite graph $G = K_{n1, n2, \ldots, nm}$ $\forall n_1, n_2, \ldots n_m \in \mathbb{N}$ is a graph whose vertex set *V* can be partitioned into *m* subsets $V_1, V_2, \ldots V_m$ such that every edge of G joins every vertex of V_i with every vertex of V_j where $i \neq j$ and $|V_i| = i$. When m=2, G is a complete bipartite graph and m=3, G is a complete tripartite graph. Terms not defined here are used in the sense of Harary [1].

II. Graph Decompositions

Let $G = (V, \mathcal{E})$ be a connected simple graph of order *p* and size *q*. If $H_1, H_2, \ldots H_k$ $\forall k \in \mathbb{N}$ are edge-disjoint subgraphs of $G \ni \mathcal{E}(G) = \mathcal{E}(H_1) \cup \mathcal{E}(H_2), \cup \ldots \cup \mathcal{E}(H_k)$, then $H_1, H_2, \ldots H_k$ is said to be a ***decomposition*** of G. Different types of decomposition of G have been studied in the literature by imposing suitable conditions on the subgraphs H_i.

Alavi *et al* [2], introduced Ascending Subgraph Decomposition (ASD) as a decomposition of G into subgraph H_i (not necessarily connected) $\ni |\mathcal{E}(H_i)| = i$ and is isomorphic to a proper subgraph of H_{i+1}. Gnana Dhas and Paulraj Joseph introduced a new concept known as continuous monotonic decomposition of graphs [3]. A decomposition, $\{H_1, H_2, \ldots H_k\} \forall k \in \mathbb{N}$, is said to be a **Continuous Monotonic Decomposition** (CMD) if each H_i is connected and

$|\mathcal{E}(H_i)| = i \ \forall i \in \mathbb{N}$. If G admits a CMD, $\{H_3, H_4, \ldots.. H_k\} \ \forall k \in \mathbb{N}$, where each H_i is a cycle of length i in G, then we say that G admits **Continuous Monotonic Cycle Decomposition** (CMCD) [4]. A CMD in which each H_i is a star is said to be a **Continuous Monotonic Star Decomposition** (CMSD) and a CMD in which each H_i is a path is said to be a **Continuous Monotonic Path Decomposition** (CMPD) [3].

Example 2.1

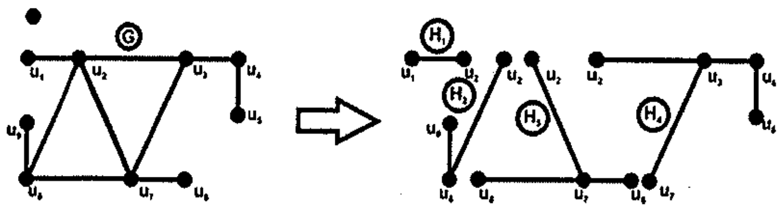

Fig. 2.1 Continuous Monotonic Decomposition of G into H_1, H_2, H_3, and H_4

III. Triangular Numbers

Triangular number is a natural number that is the sum consecutive natural numbers, beginning with 1. Pythagoras found that number is triangular if and only if it is of the form $\frac{n(n+1)}{2}$ for some $n \geq 1$. Plutarch stated that n is a triangular number if and only if $8n+1$ is a perfect square. The square of any integer is either of the form 3k or 3k+1 for some $k \in \mathbb{N}$.

Euler identified that if n is a triangular number, then so are $9n+1$, $25n+3$ and $49n+6$. If t_n denotes the n^{th} triangular number, then $t_n = {}^{(n+1)}C_2$. All these number theory results are used in the sense of David M. Burton [6].

IV. Continuous Monotonic Decomposition of Some Complete Tripartite Graphs

Continuous Monotonic Decomposition of a wide variety of graphs had been studied by Gnana Dhas and Paulraj Joseph, and Navaneetha Krishnan and Nagarajan [3]-[5]. If a graph G admits a CMD $\{H_1, H_2, \ldots.. H_k\} \ \forall k \in \mathbb{N}$ if and only if $q = {}^{(n+1)}C_2$ [3]. But we know that for any positive integer n, ${}^{(n+1)}C_2$ is a triangular number. Hence, if we are able to find out the number of the edges of any connected graph, it is easy for us to conclude whether it admits CMD or not. In this paper, I am presenting the necessary and sufficient condition for a collection of complete tripartite graphs and tensor product of Graphs which admit CMD.

The following four results are about particular classes of complete tripartite graphs which accept CMD.

Theorem 4.1 A complete tripartite graph $K_{1,3,m}$ accepts CMD of $H_1, H_2, \ldots, H_{4n+1}$ if and only if $m=(4n^2+3n-1)/2$ when n is odd and CMD of $H_1, H_2, \ldots, H_{4n+2}$ if and only if $m=(4n^2+5n)/2$ when n is even $\forall n \in \mathbb{N}$.

Proof. Assume that a complete tripartite graph $K_{1,3,m}$ accepts CMD of $H_1, H_2, \ldots, H_{4n+1}$ when n is odd and CMD of $H_1, H_2, \ldots, H_{4n+2}$ when n is even, $\forall n \in \mathbb{N}$.

We have, $q(K_{1,3,m}) = [m(1+3)+1(m+3)+3(m+1)]/2$

$= 4m+3 \ \forall m \in \mathbb{N}$. ..... (1)

We know that G accepts CMD $H_1, H_2, \ldots, H_n$ iff $q(G)= n(n+1)/2$, $\forall n \in \mathbb{N}$.

Case 1: when n is odd

$K_{1,3,m}$ accepts CMD $H_1, H_2, \ldots, H_{4n+1}$ iff $q(K_{1,3,m}) = (4n+1)(4n+2)/2$ where $n \in \mathbb{N}$ and n odd.

i.e., $= (4n+1)(2n+1)$, for $n \in \mathbb{N}$.....and n odd (2)

i.e., $q(K_{1,3,m})$ must be a member of the sequence 1, 3, 6, 10, 15,...$k(k+1)/2 \ \forall k \in \mathbb{N}$. (3)

i.e., $(4n+1)(2n+1)= k(k+1)/2$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}$ and n odd.

i.e., $k = 4n+1$ for $n \in \mathbb{N}$ and n odd.... (4)

Also, $K_{1,3,m}$ accepts CMD iff $q(K_{1,3,m})$ is one among the members of the sequence (3).

i.e., $4m+3$ should be one of these values..... using (1) and (3) i.e., $4m+3 = k(k+1)/2$ for some $k \in \mathbb{N}$.

i.e., $4m+3 = (4n+1)(2n+1)$.....using (4)

i.e., $4m = (4n+1)(2n+1)-3$, for $n \in \mathbb{N}$ and n odd.

i.e., $m= (4n^2+3n-1)/2$ for $n \in \mathbb{N}$ and n odd.

The values of m are 3, 22, 57, 108, 175.......

Example 4.1

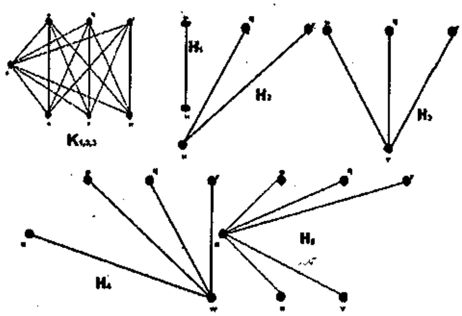

Fig 4.1: Continuous Monotonic Decomposition of $K_{1,3,3}$

Case 2: when n is even

$K_{1,3,m}$ accepts CMD $H_1, H_2, \ldots, H_{4n+2}$ iff $q(K_{1,3,m}) = (4n+3)(4n+2)/2$ where $n \in \mathbb{N}$ and n even.

i.e., $= (4n+3)(2n+1)$, for $n \in \mathbb{N}$ and n even..... (2)

i.e., $q(K_{1,3,m})$ must be a member of the sequence 1, 3, 6, 10, 15,...$k(k+1)/2\ \forall\ k \in \mathbb{N}$. (3)

i.e., $(4n+3)(2n+1)= k(k+1)/2$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}$ and n even.

i.e., $k = 4n+2$ for $n \in \mathbb{N}$ and n even.... (4)

Also, $K_{1,3,m}$ accepts CMD iff $q(K_{1,3,m})$ is one among the members of the sequence (3).

i.e., 4m+3 should be one of these values..... using (1) and (3)

i.e., $4m+3 = k(k+1)/2$ for some $k \in \mathbb{N}$.

i.e., $4m+3 = (4n+3)(2n+1)$....using (4)

i.e., $4m = (4n+3)(2n+1)-3$, for $n \in \mathbb{N}$ and n even.

i.e., $m = (4n^2+5n)/2$ for $n \in \mathbb{N}$ and n even.

The values of m are 13, 42, 87, 148, 225.......

Example 4.2

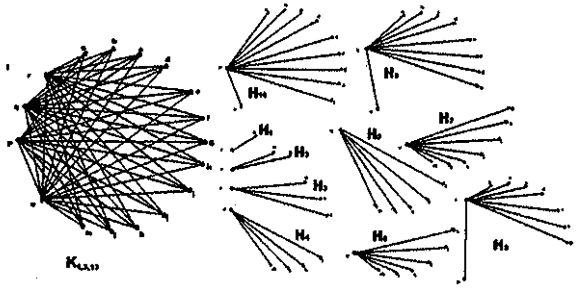

Fig 4.2: Continuous Monotonic Decomposition of $K_{1,3,13}$

Hence, a complete tripartite graph $K_{1,3,m}$ accepts CMD of $H_1, H_2, \ldots, H_{4n+1}$ if $m=(4n^2+3n-1)/2$ when n is odd and CMD of $H_1, H_2, \ldots, H_{4n+2}$ if $m=(4n^2+5n)/2$ when n is even, $\forall n \in \mathbb{N}$

Conversely,

Suppose that $K_{1,3,m}$ is a complete tripartite graph with $m=(4n^2+3n-1)/2$ when n is odd and $m=(4n^2+5n)/2$ when n is even, $\forall n \in \mathbb{N}$

We know that $q(K_{1,3,m}) = 4m+3$

Case 1: When $m=(4n^2+3n-1)/2$

$$
\begin{aligned}
q(K_{1,3,m}) &= 4m+3 \\
&= 4(4n^2+3n-1)/2+3 \\
&= (8n^2+6n-2)+3 \\
&= (8n^2+6n+1) \\
&= 2n(4n+1)+(4n+1) \\
&= (2n+1)(4n+1)\ldots\ldots\ldots \qquad (4)
\end{aligned}
$$

(4) is of the form $k(k+1)/2$ $\forall k \in \mathbb{N}$.

This implies that $K_{1,3,m}$ being a connected simple graph, can be decomposed into $H_1, H_2, \ldots\ldots H_k$ $\forall k \in \mathbb{N}$.

i.e., $K_{1,3,m}$ can be decomposed into $H_1, H_2, \ldots\ldots H_{4n+1}$, for $n \in \mathbb{N}$ and n odd.

Case 2: When $m=(4n^2+5n)/2$

$$
\begin{aligned}
q(K_{1,3,m}) &= 4m+3 \\
&= 4(4n^2+5n)/2+3 \\
&= (8n^2+10n)+3 \\
&= 2n(4n+3)+4n+3 \\
&= (2n+1)(4n+3)\ldots\ldots\ldots \qquad (5)
\end{aligned}
$$

(5) is of the form $k(k+1)/2$ $\forall k \in \mathbb{N}$.

This implies that $K_{1,3,m}$ being a connected simple graph, can be decomposed into $H_1, H_2, \ldots\ldots H_k$ $\forall k \in \mathbb{N}$.

i.e., $K_{1,3,m}$ can be decomposed into $H_1, H_2, \ldots\ldots H_{4n+2}$, for $n \in \mathbb{N}$ and n even.

Table 4.1 First 25 $K_{1,3,m}$'s which admit CMD and their CMDs

m	$q(K_{1,3,m})$	CMD
3	15	$H_1, H_2, \ldots\ldots H_5$
13	55	$H_1, H_2, \ldots\ldots H_{10}$
22	91	$H_1, H_2, \ldots\ldots H_{13}$
42	171	$H_1, H_2, \ldots\ldots H_{18}$
57	231	$H_1, H_2, \ldots\ldots H_{21}$
87	351	$H_1, H_2, \ldots\ldots H_{26}$
108	435	$H_1, H_2, \ldots\ldots H_{29}$
148	595	$H_1, H_2, \ldots\ldots H_{34}$
175	703	$H_1, H_2, \ldots\ldots H_{37}$
225	903	$H_1, H_2, \ldots\ldots H_{42}$
258	1035	$H_1, H_2, \ldots\ldots H_{45}$
318	1275	$H_1, H_2, \ldots\ldots H_{50}$
357	1431	$H_1, H_2, \ldots\ldots H_{53}$
427	1711	$H_1, H_2, \ldots\ldots H_{58}$
472	1891	$H_1, H_2, \ldots\ldots H_{61}$
552	2211	$H_1, H_2, \ldots\ldots H_{66}$
603	2415	$H_1, H_2, \ldots\ldots H_{69}$
693	2775	$H_1, H_2, \ldots\ldots H_{74}$
750	3003	$H_1, H_2, \ldots\ldots H_{77}$
850	3403	$H_1, H_2, \ldots\ldots H_{82}$
913	3655	$H_1, H_2, \ldots\ldots H_{85}$
1023	4095	$H_1, H_2, \ldots\ldots H_{90}$
1092	4371	$H_1, H_2, \ldots\ldots H_{93}$
1212	4851	$H_1, H_2, \ldots\ldots H_{98}$
1287	5151	$H_1, H_2, \ldots\ldots H_{101}$

Proofs of the following three theorems follow the same arguments of Theorem 4.1.

Theorem 4.2 A complete tripartite graph $K_{2,3,m}$ accepts CMD of $H_1, H_2, \ldots, H_{(5n+7)/2}$ if and only if $m=(5n^2+16n+3)/8$ when n is odd and CMD of $H_1, H_2, \ldots, H_{(5n+6)/2}$ if and only if $m=(5n^2+14n)/8$ when n is even $\forall n \in \mathbb{N}$.

Example 4.3

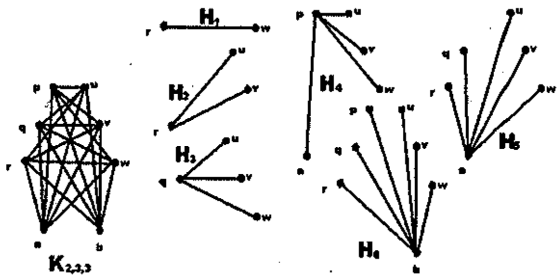

Fig 4.3: Continuous Monotonic Decomposition of $K_{2,3,3}$

Table 4.2 First 25 $K_{2,3,m}$'s which admit CMD and their CMDs

m	$q(K_{2,3,m})$	CMD
3	21	$H_1, H_2, \ldots H_6$
6	36	$H_1, H_2, \ldots H_8$
12	66	$H_1, H_2, \ldots H_{11}$
17	91	$H_1, H_2, \ldots H_{13}$
26	136	$H_1, H_2, \ldots H_{16}$
33	171	$H_1, H_2, \ldots H_{18}$
45	231	$H_1, H_2, \ldots H_{21}$
54	276	$H_1, H_2, \ldots H_{23}$
69	351	$H_1, H_2, \ldots H_{26}$
80	406	$H_1, H_2, \ldots H_{28}$
98	496	$H_1, H_2, \ldots H_{31}$
111	561	$H_1, H_2, \ldots H_{33}$
132	666	$H_1, H_2, \ldots H_{36}$
147	741	$H_1, H_2, \ldots H_{38}$
171	861	$H_1, H_2, \ldots H_{41}$
188	946	$H_1, H_2, \ldots H_{43}$
215	1081	$H_1, H_2, \ldots H_{46}$
234	1176	$H_1, H_2, \ldots H_{48}$
264	1326	$H_1, H_2, \ldots H_{51}$
285	1431	$H_1, H_2, \ldots H_{53}$
318	1596	$H_1, H_2, \ldots H_{56}$
341	1711	$H_1, H_2, \ldots H_{58}$
377	1891	$H_1, H_2, \ldots H_{61}$
402	2016	$H_1, H_2, \ldots H_{63}$
441	2211	$H_1, H_2, \ldots H_{66}$

Theorem 4.3 A complete tripartite graph $K_{2,5,m}$ accepts CMD of $H_1, H_2, \ldots, H_{7n+2}$ and $H_1, H_2, \ldots, H_{7n+4}$ if and only if $m=(7n^2+5n-2)/2$ and $m=(7n^2+9n)/2$ respectively, $\forall n \in \mathbb{N}$.

Example 4.4

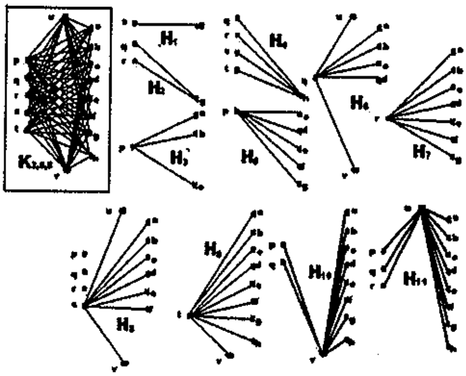

Fig 4.4: Continuous Monotonic Decomposition of $K_{2,5,8}$

Table 4.3 First 25 $K_{2,5,m}$'s which admit CMD and their CMDs

m	$q(K_{2,3,m})$	CMD
5	45	$H_1, H_2, \ldots H_9$
8	66	$H_1, H_2, \ldots H_{11}$
18	136	$H_1, H_2, \ldots H_{16}$
23	171	$H_1, H_2, \ldots H_{18}$
38	276	$H_1, H_2, \ldots H_{23}$
45	325	$H_1, H_2, \ldots H_{25}$
65	465	$H_1, H_2, \ldots H_{30}$
74	528	$H_1, H_2, \ldots H_{32}$
99	703	$H_1, H_2, \ldots H_{37}$
110	780	$H_1, H_2, \ldots H_{39}$
140	990	$H_1, H_2, \ldots H_{44}$
153	1081	$H_1, H_2, \ldots H_{46}$
188	1326	$H_1, H_2, \ldots H_{51}$
203	1431	$H_1, H_2, \ldots H_{53}$
243	1711	$H_1, H_2, \ldots H_{58}$
260	1830	$H_1, H_2, \ldots H_{60}$
305	2145	$H_1, H_2, \ldots H_{65}$
324	2278	$H_1, H_2, \ldots H_{67}$
374	2628	$H_1, H_2, \ldots H_{72}$
395	2775	$H_1, H_2, \ldots H_{74}$
450	3160	$H_1, H_2, \ldots H_{79}$
473	3321	$H_1, H_2, \ldots H_{81}$
533	3741	$H_1, H_2, \ldots H_{86}$
558	3916	$H_1, H_2, \ldots H_{88}$
623	4371	$H_1, H_2, \ldots H_{93}$

Theorem 4.4 A complete tripartite graph $K_{3,5,m}$ accepts CMD of $H_1, H_2, \ldots, H_{16n-6}$ and $H_1, H_2, \ldots, H_{16n+5}$ if and only if $m=16n^2-11n$ and $m=16n^2+11n$ respectively, $\forall n \in \mathbb{N}$.

Example 4.5

Let us consider the graph $K_{3,5,5}$.

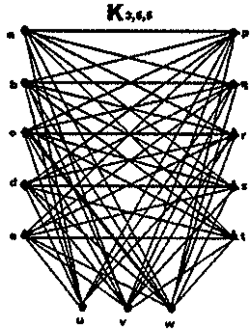

Fig 4.5: $K_{3,5,5}$

Let the three sets of vertices be $V_1=\{u,v,w\}$ $V_2=\{a,b,c,d,e\}$ and $V_3=\{p,q,r,s,t\}$.

Continuous Monotonic Decomposition of $K_{3,5,5}$ is as follows:

H_1 = {(q,e)}

H_2 = {(p,a), (p,b)}

H_3 = {(p,c), (p,d), (p,e)}

H_4 = {(q,a), (q,b), (q,c), (q,d)}

H_5 = {(r,a), (r,b), (r,c), (r,d), (r,e),}

H_6 = {(s,a), (s,b), (s,c), (s,d), (s,e), (s,w)}

H_7 = {(t,a), (t,b), (t,c), (t,d), (t,e), (t,v), (t,w)}

H_8 = {(w,a), (w,b), (w,c), (w,d), (w,e), (w,p), (w,q), (w,r)}

H_9 = {(v,a), (v,b), (v,c), (v,d), (v,e), (v,p), (v,q), (v,r), (v,s),}

H_{10} = {(u,a), (u,b), (u,c), (u,d), (u,e), (u,p), (u,q), (u,r), (u,s), (u,t)}

Table 4.4 First 25 $K_{3,5,m}$'s which admit CMD and their CMDs

m	$q(K_{2,3,m})$	CMD	m	$q(K_{2,3,m})$	CMD
5	55	$H_1, H_2, \ldots H_{10}$	707	5671	$H_1, H_2, \ldots H_{106}$
27	231	$H_1, H_2, \ldots H_{21}$	861	6903	$H_1, H_2, \ldots H_{117}$
42	351	$H_1, H_2, \ldots H_{26}$	936	7503	$H_1, H_2, \ldots H_{122}$
86	703	$H_1, H_2, \ldots H_{37}$	1112	8911	$H_1, H_2, \ldots H_{133}$
111	903	$H_1, H_2, \ldots H_{42}$	1197	9591	$H_1, H_2, \ldots H_{138}$
177	1431	$H_1, H_2, \ldots H_{53}$	1395	11175	$H_1, H_2, \ldots H_{149}$
212	1711	$H_1, H_2, \ldots H_{58}$	1490	11935	$H_1, H_2, \ldots H_{154}$
300	2415	$H_1, H_2, \ldots H_{69}$	1710	13695	$H_1, H_2, \ldots H_{165}$
345	2775	$H_1, H_2, \ldots H_{74}$	1815	14535	$H_1, H_2, \ldots H_{170}$
455	3655	$H_1, H_2, \ldots H_{85}$	2057	16471	$H_1, H_2, \ldots H_{181}$
510	4095	$H_1, H_2, \ldots H_{90}$	2172	17391	$H_1, H_2, \ldots H_{186}$
642	5151	$H_1, H_2, \ldots H_{101}$	2436	19503	$H_1, H_2, \ldots H_{197}$
			2561	20503	$H_1, H_2, \ldots H_{202}$

V. Conclusion

The results described above are about four complete tripartite graphs that accept CMD. There are many other classes of complete tripartite graphs that accept CMD. Study can be extended to find the algorithms for the above graphs to accept Continuous Monotonic Star Decomposition (CMSD) and Continuous Monotonic Path Decomposition (CMPD). Finding the size of the graph is the major task in the process. The study can also be extended to complete m-partite graphs for greater values of m.

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