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EFFECTS OF UNSTEADY FREE CONVECTIVE MHD NON-NEWTONIAN FLOW THROUGH A POROUS MEDIUM BOUNDED BY AN INFINITE INCLINED POROUS PLATE

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ABSTRACT

The problem of unsteady free convective MHD incompressible electrically conducting non-Newtonian fluid through porous medium bounded by an infinite inclined porous plate in the presence of constant suction and absorbing sinks is presented. Uniform magnetic field is applied normal to the plate. The equations governing the fluid flow have been solved using multi-parameter perturbation technique, subject to the relevant boundary conditions. It is noted that the velocity of the fluid and skin friction are increased as permeability parameter and angle of inclination increases, whereas reverse phenomenon is observed in case of magnetic field strength and sink strength. Velocity and

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temperature are greater for mercury than that of electrolytic solution. Rate of heat transfer decreases with increase in the sink strength. The results are discussed through graphs and tables.

Keywords: Free convection; MHD; heat sink; Permeability; inclined porous plate.

NOMENCLATURE

B_0	magnetic field of uniform strength	B_1	kinematical visco-elasticity
C_p	specific heat at constant pressure	T	fluid temperature
T_w	temperature at the plate	T_∞	temperature far away from the plate
A	sink strength	t	time
Ec	Eckert number	M	Hartmann number
Pr	Prandtl number	R_m	Magnetic Reynolds number
Gr	Grashoff number	K	permeability of the porous medium
k'	thermal conductivity	g	acceleration due to gravity
k	permeability parameter	u	fluid velocity component along x-axis
x,y	cartesian coordinates along the plate and normal to it	ν	kinematic viscosity
v_o	constant velocity	β	coefficient of volumetric expansion
σ	electrical conductivity	ρ	density of the fluid
ω	frequency parameter	Φ	angle made by the plate with horizontal

Super script ′ denotes differentiation with y

1. Introduction

Free convection flow occurs frequently in nature. Flows of fluid through Porous media are of principal interest these days and have attracted the attention of a number of scholars due to their applications in the fast growing fields of Science and Technology, viz. in the fields of agricultural engineering to study the underground water resources, seepage of water in riverbeds, in petroleum technology to study the movement of natural gas, oil, and water through the oil reservoirs. In view of these applications, a series of specific investigations have been made. Jha and Prasad [1] studied MHD free-convection and mass transfer flow through a porous medium with heat source. Acharya *et al.*, [2] analyzed the steady two-dimensional free convection and mass transfer flow of a viscous incompressible electrically conducting fluid through a porous medium bounded by a vertical infinite surface with constant suction velocity and constant heat flux in the presence of a uniform magnetic field. Mixed convection of non-Newtonian fluids from a vertical plate embedded in a porous medium is studied by Wang *et al.*, [3]. Orhan and Ahmet [4] studied the steady, laminar, mixed convection heat transfer from an isoflux vertical impermeable plate embedded in a fluid-saturated porous medium. Unsteady two-dimensional laminar free convection flow of an incompressible, viscous fluid through a porous medium bounded by an infinite vertical plane surface of constant temperature has been studied by Kamal [5].

In most of the studies mentioned above the permeability of the porous medium has been assumed as constant. In fact, a porous material containing the fluid is a non-homogeneous medium and there can be numerous inhomogeneities present in a porous medium. Therefore, the permeability of the porous medium may not necessarily be constant. Also, free convection along inclined sur-faces has received less attention than the cases of vertical and horizontal plates. Rees and Riley [6], Ingham *et al.*, [7] and Kumari *et al.*, [8] have presented detailed analytical and numerical solutions to the problem of free convection along a flat plate in a porous medium which are valid only for inclined plates at small angles to the horizontal. These sol-utions are, however, not valid uniformly from the horizontal limit to the vertical limit, respectively. The problem of thermal diffusion and magnetic field effects on combined free-forced convection and mass transfer flow past a vertical porous flat plate, in the presence of heat generation is studied by Abdel-Rahman [9]. The study of unsteady hydro magnetic free convective flow of viscous incompressible and electrically conducting fluids past an infinite vertical porous plate in the presence of constant suction and heat absorbing sinks has been made by Sahoo *et al.*, [10]. Through the present paper an attempt has been made to study the effects of unsteady free convective MHD non-Newtonian flow through a porous medium bounded by an infinite inclined porous plate.

2. Formulation of the Problem:

Let us consider x- axis in the direction along the infinite inclined plate and y- axis in the direction perpendicular to the fluid flow. The inclined plate makes an angle Φ with the horizontal. In the investigation the following assumptions are made:

1. All the fluid properties are constant except the density in the buoyancy force term.

2. The influence of the density variation in terms of momentum and energy equations, and the variation of the expansion coefficient with temperature, is negligible.

3. The Eckert number Ec and the magnetic Reynolds number R_m are small, so that the induced magnetic field can be neglected.

Using Boussineqs approximation with the above assumptions and following Sahoo *et.al* [7], the basic flow equations through porous medium are:

Equation of Continuity

$$\frac{\partial v}{\partial y} = 0 \Rightarrow v = -v_0 (v_0 > 0) \qquad ....(1)$$

Equation of Motion

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = g\sin\Phi\beta(T - T_\infty) + v\frac{\partial^2 u}{\partial y^2} + B_1\left(\frac{\partial^3 u}{\partial t \partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right) - \frac{\sigma B_0^2}{\rho}u - \frac{v}{K}u \qquad ....(2)$$

Equation of Energy

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y} = \frac{k'}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + A(T - T_\infty) + \frac{v}{C_p}\left(\frac{\partial u}{\partial y}\right)^2 \qquad ....(3)$$

By disregarding Joulean heat dissipation, the boundary conditions of the problem are:

$$\left.\begin{array}{l} y=0;\ u=0,\ v=-v_0,\ T = T_\omega + \varepsilon (T_\omega - T_\infty) e^{i\omega t} \\ y \to \infty;\ u \to 0,\ T \to T_\infty \end{array}\right\} \qquad ....(4)$$

3. Method of Solution:

Introduce the following non-dimensional quantities into the equations (2) and (3),

$$y^* = y v_0 / \nu,\ t^* = t v_0^2 / 4\nu,\ \omega^* = 4\nu\omega / v_0^2,\ u^* = u / v_0,\ \nu = \mu / \rho,$$
$$\Pr = \nu / k^*,\ A^* = 4A\nu / v_0^2,\ k^* = k' / \rho C_p,\ T^* = (T - T_\infty)/(T_w - T_\infty),$$
$$Gr = \nu g \beta (T_w - T_\infty)/ v_0^3,\ k = K v_0^2 / \nu^2,\ Ec = v_0^2 / C_p (T_w - T_\infty), \qquad \ldots (5)$$
$$M = (\sigma B_0^2 / \rho)\nu / v_0^2,\ R_m = B_1 v_0^2 / \nu^2$$

Then we get,

$$\frac{1}{4}(\partial u / \partial t) - \partial u / \partial y = Gr \sin \Phi T + \partial^2 u / \partial y^2 + R_m (1/4).(\partial^3 u / \partial t \partial y^2)$$
$$-\partial^3 u / \partial y^3 - k^{-1} u \qquad \ldots.(6)$$

$$(\Pr/4).(\partial T / \partial t) - \Pr(\partial T / \partial y) = \partial^2 T / \partial y^2 + (\Pr/4) A T + \Pr Ec (\partial T / \partial y)^2 \qquad \ldots.(7)$$

(after dropping the asterisks)

The corresponding boundary conditions in non-dimensional form are:

$$y = 0: \quad u = 0, \quad T = 1 + \varepsilon e^{i\omega t}$$
$$y \to \infty: \quad u \to 0, \quad T \to 0 \qquad \ldots.(8)$$

To solve the equations (6) and (7), subject to the boundary conditions (8), the velocity u and temperature T in the neighborhood of the plate are assumed to be of the form,

$$u(y,t) = u_0(y) + \varepsilon e^{i\omega t} u_1(y)$$
$$T(y,t) = T_0(y) + \varepsilon e^{i\omega t} T_1(y) \qquad \ldots.(9)$$

Substituting equation (9) in the equations (6) and (7), and equating harmonic and non-harmonic terms for velocity and temperature, after neglecting coefficients of δ^2, the following set of equations are obtained:

$$R_m u_0''' - u_0'' - u_0' + (1/k)u_0 = Gr \sin\Phi T_0 \quad ....(10)$$

$$R_m u_1''' - (1 + iR_m\omega/4)u_1'' - u_1' + (k^{-1} + i\omega/4)u_1 = Gr \sin\Phi T_1 \quad ....(11)$$

$$T_0'' + \Pr T_0' + (\Pr A/4)T_0 = -\Pr Ec (u_0')^2 \quad ....(12)$$

$$T_1'' + \Pr T_1' + (\Pr/4)(A - i\omega)T_1 = -2\Pr Ec u_0' u_1' \quad ....(13)$$

The corresponding boundary conditions are:

$$\begin{aligned} &y = 0;\ u_0 = u_1 = 0,\ T_0 = T_1 = 1 \\ &y \to \infty;\ u_0 \to 0,\ u_1 \to 0,\ T_0 \to 0,\ T_1 \to 0 \end{aligned} \quad .... (14)$$

In equations (10) and (11), due to presence of elasticity, we get third order differential equations. To solve these equations, we need three boundary conditions but we have two. So, following Beard and Walters [11], we assume the solutions as

$$u_0 = u_{00} + R_m u_{01} + O(R_m^2) \quad ....(15)$$

$$u_1 = u_{10} + R_m u_{11} + O(R_m^2) \quad ....(16)$$

$$T_0 = T_{00} + R_m T_{01} + O(R_m^2) \quad ....(17)$$

$$T_1 = T_{10} + R_m T_{11} + O(R_m^2) \quad ....(18)$$

Zero – order of R_m:

$$u_{00}'' + u_{00}' - (k^{-1} + M)u_{00} = -Gr \sin\Phi T_{00} \quad ....(19)$$

$$u_{10}'' + u_{10}' - (k^{-1} + M + i\omega/4)u_{10} = -Gr \sin\Phi T_{10} \quad ....(20)$$

$$T_{00}'' + \Pr T_{00}' + (\Pr/4) A T_{00} = -\Pr Ec (u_{00}')^2 \quad ....(21)$$

$$T_{10}' + \Pr T_{10}' + (\Pr/4)(A - i\omega) T_{10} = -2 \Pr Ec u_{00}' u_{10}' \quad ....(22)$$

First – order of R_{m}:

$$u_{01}'' + u_{01}' - (k^{-1} + M) u_{01} = u_{00}''' - Gr \sin\Phi \, T_{01} \quad ....(23)$$

$$u_{11}'' + u_{11}' - (k^{-1} + M + i\omega/4) u_{11} = u_{10}''' - Gr \sin\Phi \, T_{11} - i(\omega/4) u_{10}'' \quad ....(24)$$

$$T_{01}'' + \Pr T_{01}' + (\Pr/4) A T_{01} = -2 \Pr Ec u_{00}' u_{01}' \quad ....(25)$$

$$T_{11}'' + \Pr T_{11}' + (\Pr/4)(A - i\omega) T_{11} = -2 \Pr Ec(u_{10}' u_{01}' + u_{00}' u_{11}') \quad ....(26)$$

In order to obtain a solution of above coupled nonlinear system of equations (19) to (26), we expand $u_{00}, u_{01}, u_{10}, u_{11}, T_{00}, T_{01}, T_{10}, \, and \; T_{11}$ in powers of Eckert number Ec. This is valid as Ec is very small (Ec<<]) for all incompressible fluids. So, we assumed that

$$u_{00} = u_{000} + Ec u_{001} + O(Ec^2) \quad ....(27)$$

$$u_{01} = u_{010} + Ec u_{011} + O(Ec^2) \quad ....(28)$$

$$u_{10} = u_{100} + Ec u_{101} + O(Ec^2) \quad ....(29)$$

$$u_{11} = u_{110} + Ec u_{111} + O(Ec^2) e \quad ....(30)$$

$$T_{00} = T_{000} + Ec T_{001} + O(Ec^2) \quad ....(31)$$

$$T_{01} = T_{010} + Ec T_{011} + O(Ec^2) \quad ....(32)$$

$$T_{10} = T_{100} + Ec T_{101} + O(Ec^2) \quad ....(33)$$

$$T_{11} = T_{110} + Ec T_{111} + O(Ec^2) \quad ....(34)$$

Using equations (27) to (34) in the equations (19) to (26) and equating the coefficient of Ec^0 and Ec^1, we get the following sets of differential equations.

Zero – order of Ec :

$$u_{000}'' + u_{000}' - (k^{-1} + M) u_{000} = -Gr \sin\Phi\, T_{000} \quad ....(35)$$

$$u_{010}'' + u_{010}' - (k^{-1} + M) u_{010} = -Gr \sin\Phi\, T_{010} + u_{000}''' \quad ....(36)$$

$$u_{100}'' + u_{100}' - (k^{-1} + M + i\omega/4) u_{100} = -Gr \sin\Phi\, T_{100} \quad ....(37)$$

$$u_{110}'' + u_{110}' - (k^{-1} + M + i\omega/4) u_{110} = -Gr \sin\Phi\, T_{110} + u_{100}''' - i(\omega/4) u_{100}'' \quad ....(38)$$

$$T_{000}'' + \Pr T_{000}' + (\Pr/4) A T_{000} = 0 \quad ....(39)$$

$$T_{010}'' + \Pr T_{010}' + (\Pr/4) A T_{010} = 0 \quad ....(40)$$

$$T_{100}'' + \Pr T_{100}' + (\Pr/4)(A - i\omega) T_{100} = 0 \quad ....(41)$$

$$T_{110}'' + \Pr T_{110}' + (\Pr/4)(A - i\omega) T_{110} = 0 \quad ....(42)$$

First – order of Ec :

$$u_{001}'' + u_{001}' - (k^{-1} + M) u_{001} = -Gr \sin\Phi\, T_{001} \quad ....(43)$$

$$u_{011}'' + u_{011}' - (k^{-1} + M) u_{011} = -Gr \sin\Phi\, T_{011} + u_{001}''' \quad ....(44)$$

$$u_{101}'' + u_{101}' - (k^{-1} + M + i\omega/4)u_{101} = -Gr\sin\Phi T_{101} \qquad ....(45)$$

$$u_{111}'' + u_{111}' - (k^{-1} + M + i\omega/4)u_{111} = -Gr\sin\Phi T_{111} + u_{101}''' - i(\omega/4)u_{101}'' \qquad ....(46)$$

$$T_{001}'' + \Pr T_{001}' + (\Pr/4)AT_{001} = -\Pr(u_{000}')^2 \qquad ....(47)$$

$$T_{011}'' + \Pr T_{011}' + (\Pr/4)A\,T_{011} = -2\Pr u_{000}'u_{010}' \qquad ....(48)$$

$$T_{101}'' + \Pr T_{101}' + (\Pr/4)(A - i\omega)T_{101} = -2\Pr u_{000}'u_{100}' \qquad ....(49)$$

$$T_{111}'' + \Pr T_{111}' + (\Pr/4)(A - i\omega)T_{111} = -2\Pr(u_{100}'u_{010}' + u_{000}'u_{110}') \qquad ....(50)$$

The corresponding boundary conditions are:

$$y = 0:\ u_{000} = u_{010} = u_{001} = u_{011} = 0,\quad T_{000} = 1;\ T_{010} = T_{001} = T_{011} = 0$$
$$u_{100} = u_{110} = u_{101} = u_{111} = 0,\quad T_{100} = 1;\ T_{110} = T_{101} = T_{111} = 0$$

$$y \to \infty:\ u_{000} \to u_{010} \to u_{001} \to u_{011} \to 0,\quad T_{000} \to T_{010} \to T_{001} \to T_{011} \to 0$$
$$u_{100} \to u_{110} \to u_{101} \to u_{111} \to 0,\quad T_{100} \to T_{110} \to T_{101} \to T_{111} \to 0$$

4. Solution of Problem:

Solving these differential equations from (35) to (50) using the above boundary conditions, making use of equations (27) to (34), making the appropriate substitutions in equations (15) to (18) and finally we obtain the expressions for velocity u and temperature T from the equation (9).

$$u(y,t) = A_1e^{-t_1y} + A_2e^{-t_2y} + A_3e^{-2t_1y} + A_4e^{-2t_2y} + A_5e^{-t_3y} + A_6e^{-t_3y} + A_7e^{-t_5y}$$
$$+ A_8e^{-t_6y} + A_9e^{-t_7y} + A_{10}e^{-t_8y} + A_{11}e^{-t_9y}$$

$$T(y,t) = D_1 e^{-l_1 y} + D_2 e^{-2l_1 y} + D_3 e^{-2l_2 y} + D_4 e^{-l_3 y} + D_5 e^{-l_4 y} + D_6 e^{-l_6 y} + D_7 e^{-l_7 y} + D_8 e^{-l_8 y} + D_9 e^{-l_9 y}$$

4.1 Skin Friction: The skin friction at the plate in dimensionless form is given by

$$\tau = \left(\frac{\partial u}{\partial y}\right)_{y=0} = u_0'(0) + \varepsilon e^{i\omega t} u_1'(0)$$

4.2 Rate of Heat Transfer: The rate of heat transfer at the plate in dimensionless form in terms of Nusselt number is given by

$$Nu = \left(\frac{\partial T}{\partial y}\right)_{y=0} = T_0'(0) + \varepsilon e^{i\omega t} T_1'(0)$$

5. Results and Discussions:

In order to get physical insight into the problem, the velocity, temperature fields, skin friction and rate of heat transfer have been discussed by assigning numerical values for M, Gr, Pr, A, k and Φ while keeping $R_m = 0.05$, $\omega = 5.0$, $\varepsilon = 0.2$, $\omega t = \pi/2$ and Ec=0.001 constant. The results obtained are illustrated through the figs 1 - 4 and tables 1 and 2.

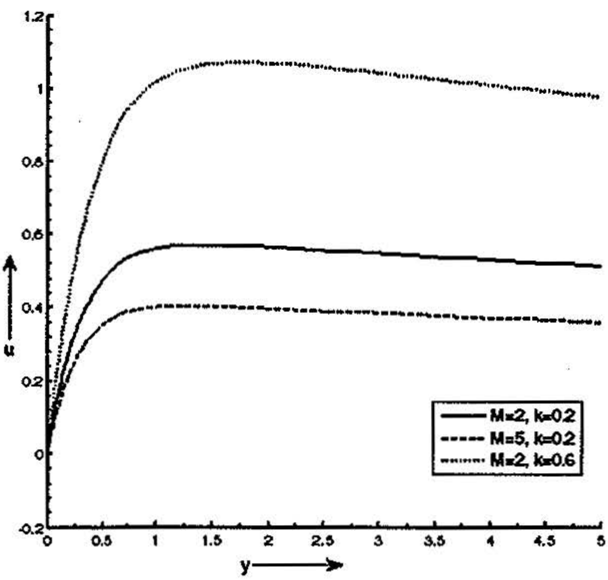

Fig.1: Velocity profiles for variations in M and k

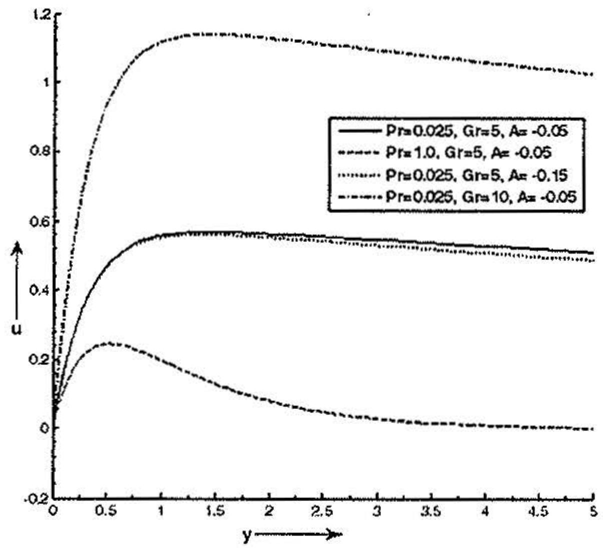

Fig.2: Velocity profiles for variations in Pr, Gr and A

All the results are in excellent agreement with those of Sahoo[10] in absence of magnetic Reynolds number (R_m), permeability parameter(k) and the plate being vertical ($\Phi = \pi/2$). From Fig.1, it is observed that as the permeability parameter (k) increases, velocity (u) also increases. But as the external magnetic field strength (M) increases, the velocity (u) decreases. It indicates that magnetic field suppresses the free convection.

It is also clear from Fig.2 that velocity (u) is greater for mercury (Pr=0.025) than that of electrolytic solution (Pr=1.0) i.e., velocity (u) for viscous fluid is more than the visco-elastic one. As sink strength (A) increases, the velocity (u) decreases. The effect of Gr on velocity (u) is directly proportional. It is noted that as the angle of inclination (Φ) with the horizontal is increased, the velocity (u) also increased as shown in Fig.3.

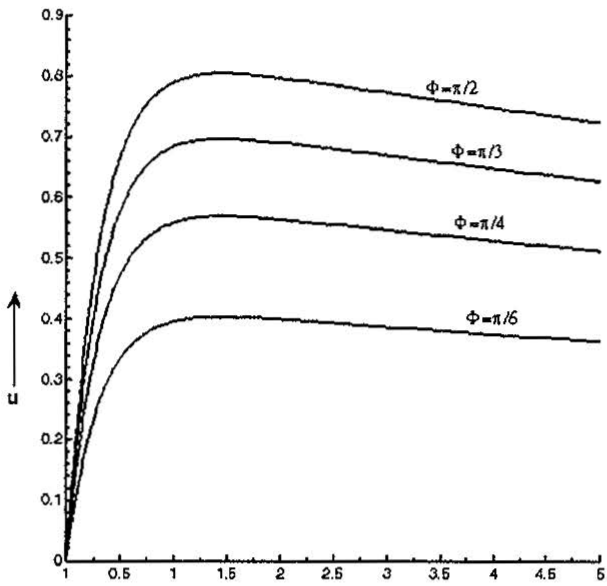

Fig.3: Velocity profiles for variations in Φ

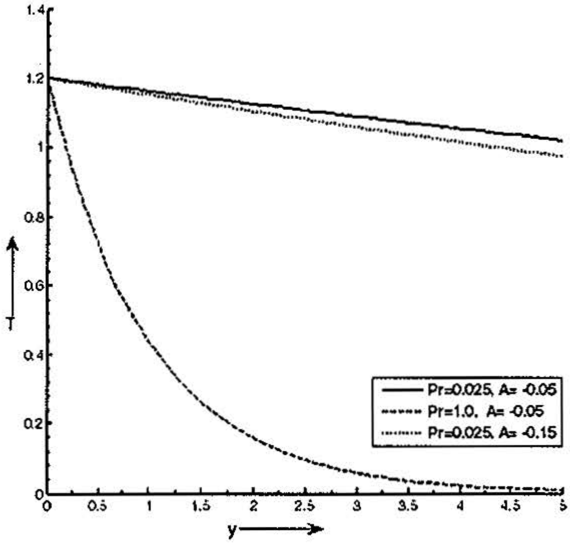

Fig.4: Temperature profiles for variations in Pr and A

Fig.4 depicts the effect of Prandtl number (Pr) and sink parameter (A) on fluid temperature. It is worth mentioning to note that temperature (T) is more for mercury (Pr=0.025) than for electrolytic solution (Pr=1.0). It is noted that temperature (T) decreases as sink strength (A) increases which is in accordance with the existing results in literature.

From Table.1, it is evident that an increase in M and A decreases the skin-friction for both mercury and electrolytic solution. But an increase in k and Φ increases the skin-friction for both mercury and electrolytic solution. Skin friction is more for viscous flow than a non-Newtonian flow. From Table.2. Rate of heat transfer decreases with increase in the sink strength while the effect of permeability parameter is negligible.

Table.2: Variations in Rate of heat transfer

M	A	k	Φ	Skin	Friction
				Pr=0.025	Pr=1.0
2	-0.05	0.2	π/4	1.9066	1.3360
5	-0.05	0.2	π/4	1.5517	1.1499
2	-0.10	0.2	π/4	1.9022	1.3323
2	-0.05	0.6	π/4	2.8066	1.7323
2	-0.05	0.2	π/2	2.6964	1.8894

Table.2: Variations in Rate of heat transfer

A	k	Nusselt Number Nu	
		Pr=0.025 Mercury	Pr=1.0 Electrolytic solution
-0.05	0.2	-0.0391	-1.2122
-0.10	0.2	-0.0454	-1.2233
-0.05	0.6	-0.0391	-1.2122

6. Conclusions:

In this paper, the effect of unsteady free convective MHD non-Newtonian flow through a porous medium bounded by an infinite inclined porous plate has been studied numerically. Neglecting the induced magnetic field, the equations governing the velocity and temperature of the fluid are solved by multi-parameter perturbation technique in terms of dimensionless parameters. The following conclusions are summarized:

- external magnetic field retards the free convection flow
- velocity of viscous fluid is more than that of visco-elastic fluid
- angle of inclination of the plate with the horizontal is directly proportional to the velocity of the fluid
- increase in permeability parameter strengths the fluid flow and skin friction
- temperature of the fluid is more for mercury than for an electrolytic solution

These results are most applicable in the studies of geothermal activities, underground transport of pollutants, paper processing, building insulation, drying of grains, and solar pond designs that of heating from horizontal, vertical and inclined surfaces, if the study is extended to different media. Porous media are widely used in high temperature heat exchangers, turbine blades jet nozzles etc. In practice cooling of porous structure is achieved by forcing the liquid or gas through capillaries of solid.

7. References:

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Appendix

$$t_1 = \frac{\Pr + \sqrt{\Pr^2 - (\Pr.A)}}{2};\qquad t_2 = \frac{1 + \sqrt{1 + 4(1/k + M)}}{2};$$

$$t_3 = t_1 + t_2;\quad B_1 = \Pr(A - i\omega);$$

$$t_4 = \frac{\Pr + \sqrt{\Pr^2 - B}}{2};\qquad t_5 = \frac{1 + \sqrt{1 + 4(1/k + M + i\omega/4)}}{2};$$

$$t_6 = t_2 + t_4;\quad t_7 = t_2 + t_5;\ t_8 = t_1 + t_4;\ t_9 = t_1 + t_5;$$

$$A_1 = -c_1 + c_{10}Ec + c_2 R_m + (c_{13} + c_{18}) R_m Ec;\quad A_2 = c_1 + c_{11}Ec - c_2 R_m + c_{14} R_m Ec;$$

$$A_3 = c_8 Ec + c_{16} R_m Ec;\qquad A_4 = c_7 Ec + c_{17} R_m Ec;$$

$$A_5 = c_9 Ec + c_{15} R_m Ec;\qquad A_6 = k_1 + k_8 Ec + k_{13} R_m + k_{19} R_m Ec;$$

$$A_7 = -k_1 + k_7 Ec - k_{13} R_m + k_{20} R_m Ec;\quad A_8 = k_9 Ec + k_{21} R_m Ec;$$

$$A_9 = k_{10} Ec + k_{22} R_m Ec;\qquad A_{10} = k_{11} Ec + k_{23} R_m Ec;$$

$$A_{11} = k_{12} Ec + k_{24} R_m Ec;$$

$$D_1 = 1 + R_m + c_{12} R_m Ec;\qquad D_2 = c_4 Ec - 2c_4 (c_2 / c_1) R_m Ec;$$

$$D_3 = c_3 Ec - 2c_3 (c_2 / c_1) R_m Ec;\qquad D_4 = c_5 Ec + c_5 (c_2 / c_1) R_m Ec;$$

$$D_5 = \varepsilon e^{iwt}(1 + k_2 Ec + k_{18} R_m Ec);\quad D_6 = \varepsilon e^{iwt}(k_3 Ec + k_{14} R_m Ec);$$

$$D_7 = \varepsilon e^{iwt}(k_4 Ec + k_{15} R_m Ec);\quad D_8 = \varepsilon e^{iwt}(k_5 Ec + k_{16} R_m Ec);$$

$$D_9 = \varepsilon e^{iwt}(k_9 Ec + k_{17} R_m Ec)$$

Where $c_1, c_2, \ldots\ldots c_{17} c_{18}, k_1, k_2, \ldots\ldots k_{24}$ are the constants not mentioned here because of brevity.