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GRAPH EQUATIONS FOR LINE GRAPHS, JUMP GRAPHS, MIDDLE GRAPHS, SPLITTING GRAPHS AND LINE SPLITTING GRAPHS

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ABSTRACT

For a graph G, let $\overline{G}$, L(G), J(G) S(G), $L_s(G)$ and M(G) denote Complement, Line graph, Jump graph, Splitting graph, Line splitting graph and Middle graph respectively.

In this paper, we solve the graph equations L(G) = S(H), M(G) = S(H), L(G) = $L_s(H)$, M(G) =$L_s(H)$, J(G) = S(H), $\overline{M(G)}$ = S(H), J(G) = $L_s(H)$ and M(G) = $L_s(G)$. The equality symbol '=' stands for an isomorphism between two graphs.

Keywords: Line graph, Jump graph, Middle graph, Splitting graph, Line splitting graph.

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1. Introduction

By a graph, we mean a finite, undirected graph without loops or multiple edges. Definitions not given here may be found in [5]. For a graph G, $V(G)$ and $E(G)$ denote its vertex set and edge set respectively.

The *open-neighborhood* $N(u)$ of a vertex u in $V(G)$ is the set of vertices adjacent to u viz. $N(u) = \{v / uv \in E(G)\}$.

For each vertex u_i of G, a new vertex u_i' is taken and the resulting set of vertices is denoted by $V_1(G)$.

The *splitting graph* $S(G)$ of a graph G is defined as the graph having vertex set $V(G) \cup V_1(G)$ and two vertices are adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex u_i' of $V_1(G)$ and the other to a vertex w_i of G and w_i is in $N(u_i)$. This concept was introduced by Sampathkumar and Walikar in [7].

The *open-neighborhood* $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i viz. $N(e_i) = \{e_j / e_i$ *and* e_j *are adjacent in* $G\}$.

For each edge e_i of G, a new vertex e_i¢ is taken and the resulting set of vertices is denoted by $E_1(G)$.

The *line splitting graph* $L_s(G)$ of a graph G is defined as the graph having vertex set $E(G) \cup E_1(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of $E_1(G)$ and the other to an element e_j of $E(G)$ where e_j is in $N(e_i)$. This concept was introduced by Kulli and Biradar in [6].

The *jump graph* $J(G)$ of G is the graph whose vertices are edges of G and two vertices of $J(G)$ are adjacent if and only if they are not adjacent in G. Equivalently, the jump graph $J(G)$ of G is the complement of the line graph $L(G)$ of G. This concept was introduced by Chartrand in [3].

The *middle graph* $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge incident with it. This concept was introduced by Akiyama, Hamada and Yoshimura in [1].

In this paper, we solve the following graph equations :

I. $L(G) = S(H)$	V. $J(G) = S(H)$
II. $M(G) = S(H)$	VI. $\overline{M(G)} = S(H)$
III. $L(G) = L_s(H)$	VII. $J(G) = L_S(H)$
IV. $M(G) = L_s(H)$	VIII. $\overline{M(G)} = L_S(H)$

Beineke has shown in [2] that a graph G is a line graph if and only if G has none of the nine specified graphs F_i, $i = 1,2,\ldots,9$ as an induced subgraph. We depict here three of the nine graphs which we use. They are $F_1 = K_{1,3}$, F_3(see Figure 1(a)], and F_5 [see Figure 1(b)]

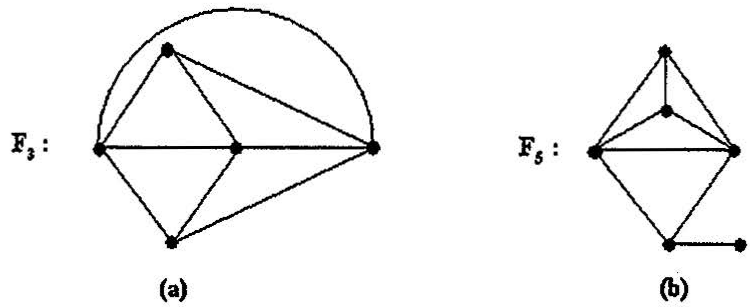

Figure 1

A graph G^+ is the *endedge graph* of a graph G if G^+ is obtained from G by adjoining an endedge u_iu_i¢ at each vertex u_i of G. Hamada and Yoshimura have proved in [4] that $M(G) = L(G^+)$.

2. The Solution of L(G) = S(H)

Any graph H which is a solution of the above equation, satisfies the following properties:

i) H must be a line graph, since H is an induced subgraph of $S(H)$.

ii) H does not contain a cut-vertex, since otherwise, F_1 is an induced subgraph of $S(H)$.

iii) H does not contain a vertex which is adjacent to two nonadjacent vertices, since otherwise, F_1 is an induced subgraph of $S(H)$.

iv) H does not contain C_n, $n \geq 4$ as a subgraph since otherwise, F_1 is an induced subgraph of $S(H)$.

It follows from above observations that H has no cut-vertices. We consider the following cases:

Case 1. Suppose H is disconnected. Then components of H are K_1 or K_2 or K_3. Therefore, $(2nK_2, nK_1)$, $n \geq 2$; (nP_5, nK_2), $n \geq 2$; (nK_3^+, nK_3), $n \geq 2$; $(2mK_2 \cup nP_5, mK_1 \cup nK_2)$, $m, n \geq 1$; $(2mK_2 \cup nK_3^+, mK_1 \cup nK_3)$, $m,n \geq 1$; $(mP_5 \cup nK_3^+, mK_2 \cup nK_3)$, $m, n \geq 1$; and $(2mK_2 \cup nP_5 \cup lK_3^+, mK_1 \cup nK_2 \cup lK_3)$, $m, n, l \geq 1$ are the solutions.

Case 2. Suppose H is connected. Then H is K_1 or K_2 or K_3. The corresponding G is $2K_2$ or P_5 or K_3^+ respectively.

From the above discussion, we conclude the following :

Theorem 1. The following pairs (G,H) are all pairs of graphs satisfying the graph equation $L(G) = S(H)$:

(nP_5, nK_2), $n \geq 1$; $(2nK_2, nK_1)$, $n \geq 1$; (nK_3^+, nK_3), $n \geq 1$; $(2mK_2 \cup nP_5, mK_1 \cup nK_2)$, $m, n \geq 1$; $(2mK_2 \cup nK_3^+, mK_1 \cup nK_3)$, $m, n \geq 1$; $(mP_5 \cup, mK_2 \cup nK_3)$, $m, n \geq 1$; and $(2mK_2 \cup nP_5 \cup, lK_3^+, mK_1\ nK_2 \cup lK_3)$, m, n, $l \geq 1$.

3. The solution of M(G) = S(H)

We have investigated the solutions (G,H) of equation 2 in Theorem 1. Among these solutions $(2nK_2, nK_1)$, $n \geq 1$; (nK_3^+, nK_3), $n \geq 1$; and $(2mK_2 \cup nK_3^+, mK_1 \cup nK_3)$, $m, n \geq 1$ are as (G^+,H). Therefore, the solutions of the equation 3 are $(2nK_1, nK_1)$ $n \geq 1$, (nK_3, nK_3), $n \geq 1$ and $(2mK_1 \cup nK_3, mK_1 \cup nK_3)$, $m, n \geq 1$. Thus we have the following.

Theorem 2. The solutions (G,H) of the graph equation $M(G) = S(H)$ are $(2nK_1, nK_1)$ $n \geq 1$; (nK_3, nK_3), $n \geq 1$; and $(2mK_1 \cup nK_3, mK_1 \cup nK_3)$, $m,n \geq 1$.

4. The Solution of $L(G) = L_S(H)$

We observe that in this case H satisfies the following properties:

i) H does not contain a component having more than one cut-vertex since otherwise, F_1 is an induced subgraph of $L_s(H)$.

ii) H is not a complete graph K_n, $n \geq 4$, since otherwise, F_1 is an induced subgraph of $L_s(H)$.

iii) H does not contain P_4 as a subgraph since otherwise, F_1 is an induced subgraph of $L_s(H)$.

iv) H does not contain $K_{1,4}$ as an induced subgraph since otherwise, F_1 is an induced subgraph of $L_s(H)$.

v) H is not a cycle C_n, $n \geq 4$ since otherwise, F_1 is an induced subgraph of $L_s(H)$.

vi) H does not contain a cut-vertex which lies on blocks other than K_2.

From observation (i) it follows that every component of H has at most one cut-vertex. We consider the following cases.

Case 1. Suppose H has no cut-vertices. Then H is nK_2, $n \geq 1$ or nK_3, $n \geq 1$, or $mK_2 \cup nK_3$, $m, n \geq 1$.

For $H = nK_2$, $n \geq 1$, $\quad G = 2nK_2$

For $H = nK_3$, $n \geq 1$, $\quad G = nK_3^+$

For $H = mK_2 \cup nK_3$, $m, n \geq 1$, $G = 2mK_2 \cup nK_3^+$

Case 2. Suppose H has cut-vertices. We consider the following subcases :

Subcase 2.1. Assume H is connected. Then H is $K_{1,2}$ or $K_{1,3}$. The corresponding G is P_5 or nK_3^+ respectively.

Subcase 2.2. Assume H is disconnected. Then each component of H has atmost one cut-vertex. Then H is $mK_{1,2} \cup nK_{1,3} \cup lK_2 \cup rK_3$, $m \geq 1$ and $n, l, r \geq 0$ or $mK_{1,2} \cup nK_{1,3} \cup lK_2 \cup rK_3$, $n \geq 1$ and $m, l, r \geq 0$. In this case $(mP_5 \cup (n+r)\ K_3^+ \cup 2mK_2,$ $mK_{1,2} \cup nK_{1,3} \cup lK_2 \cup rK_3)$ is the solution.

From above discussions, we conclude the following:

Theorem 3. The following pairs (G,H) are all pairs of graphs satisfying the graph equation $L(G) = L_s(H)$:

$(2nK_2, nK_2), n \geq 1$; $(nK_3^+, nK_3), n \geq 1$; $(2mK_2 \cup nK_3^+, mK_2 \cup nK_3), m, n \geq 1$; $(mP_5 \cup (n+r)\ K_3^+ \cup 2lK_2, mK_{1,2} \cup nK_{1,3} \cup lK_2 \cup rK_3), m \geq 1, n, l, r \geq 0$; and $(mP_5 \cup (n+r)\ K_3^+ \cup 2lK_2, mK_{1,2} \cup nK_{1,3} \cup lK_2 \cup rK_3), n \geq 1, m, l, r \geq 0$.

5. The Solution of $M(G)=L_s(H)$

Theorem 3 provides solutions of the equation $L(G) = L_s(H)$. Among these solutions $(2nK_2, nK_2), n \geq 1$; $(nK_3^+, nK_3), n \geq 1$; $(2mK_2 \cup nK_3^+, mK_2 \cup nK_3), m, n \geq 1$; and $(mP_5 \cup (n+r)\ nK_3^+ \cup 2lK_2, mK_{1,2} \cup nK_{1,3} \cup lK_2 \cup rK_3), r, l \geq 0, n \geq 1$ and $m = 0$ are as (G^+,H). Therefore, solutions of equation 5 are $(2nK_1, nK_2), n \geq 1$; $(nK_3, nK_3), n \geq 1$; $(2mK_1 \cup nK_3, mK_2 \cup nK_3), m, n \geq 1$; and $((n+r)\ K_3 \cup 2lK_1, nK_{1,3} \cup lK_2 \cup rK_3)$, $n \geq 1$ and $r, l \geq 0$. Now we state the following result.

Theorem 4. The solutions (G,H) of the graph equation $M(G) = L_s(H)$ are $(2nK_1, nK_2), n \geq 1$; $(nK_3, nK_3), n \geq 1$; $(2mK_1 \cup nK_3, mK_2 \cup nK_3), m, n \geq 1$; and $((n+r)K_3 \cup 2lK_1, nK_{1,3} \cup lK_2 \cup rK_3), n \geq 1$ and $r, l \geq 0$.

6. The Solution of J(G) = S(H)

First, we observe that in this case H satisfies the following properties:

i) If H has atleast one edge, then it is connected since otherwise, $\overline{F_3}$ is an induced subgraph of $S(H)$.

ii) H does not contain a cut-vertex, since otherwise, $\overline{F_3}$ is an induced subgraph of $S(H)$.

iii) If H is a block, then it is a complete graph since otherwise, $\overline{F_5}$ is an induced subgraph of $S(H)$.

It follows from observations (i), (ii) and (iii), that H is $nK_1, n \geq 1$ or $K_n, n \geq 2$. The corresponding G is $K_{1,2n}$ or , $K_{1,n}^+ - v$ where v is a pendant vertex of $K_{1,n}^+$ which is adjacent to the vertex of maximum degree respectively.

Hence equation 6 is solved and solutions are given in the following theorem.

Theorem 5. The following pairs (G,H) are all pairs of graphs satisfying the graph equation $J(G) = S(H)$:

$(K_{1,2n}, nK_1)$ $n \geq 1$; and $(K_{1,n}^{+}, K_n)$ $n \geq 2$ where v is a pendant vertex of which is adjacent to the vertex of maximum degree.

7. The Solution of $\overline{M(G)} = S(H)$

Theorem 5 gives solutions for the equation $J(G) = S(H)$. But none of these is as (G^{+},H). Hence, there is no solution of the equation $\overline{M(G)} = S(H)$.

Thus we have the following result.

Theorem 6. There is no solution of the graph equation $\overline{M(G)} = S(H)$.

8. The Solution of $J(G) = L_s(H)$

In this case H satisfies the following properties:

i) H does not contain more than one cut-vertex, since otherwise, $\overline{F_5}$ is an induced subgraph of $L_s(H)$.

ii) H does not contain a cut-vertex, which lies on blocks otherthan K_2, since otherwise, $\overline{F_5}$ is an induced subgraph of $L_s(H)$.

iii) H does not contain a cycle C_n, $n \geq 4$, since otherwise, $\overline{F_5}$ is an induced subgraph of $L_s(H)$.

iv) If H is disconnected graph then every component of H is K_2, since otherwise, $\overline{F_3}$ is an induced subgraph of $L_s(H)$.

We consider the following cases.

Case 1. Suppose H is disconnected. Then from observation (iv), H is nK_2, $n \geq 2$. The corresponding G is $2nK_2$.

Case 2. Suppose H is connected. Then from observation (i), H has atmost one cut-vertex. We consider the following subcases.

Subcase 2.1. Assume H has a cut-vertex. Then from observation (ii), H is $K_{1,n}$, $n \geq 2$. The corresponding G is $K_{1,n}^{+} - v$, where v is a pendant vertex adjacent to the vertex of maximum degree.

Subcase 2.2. Assume H is a block. Then H is K_2 or K_3. The corresponding G is $K_{1,2}$ or $K_{1,3}^{+}$, where v is a pendant vertex of $K_{1,3}^{+}$ adjacent to a vertex of maximum degree.

Thus equation 8 is solved and we have the following.

Theorem 7. The following pairs (G,H) are all pairs of graphs satisfying the graph equation $J(G) = L_s(H)$:

$(2nK_2, nK_2)$, $n \geq 2$; $(K_{1,2}, K_2)$; $(K_{1,3}^{+} - v, K_3)$, where v is a pendant vertex adjacent to the vertex of maximum degree; and $(K_{1,n}^{+} - v, K_{1,n})$, $n \geq 2$, where v is a pendant vertex adjacent to the vertex of maximum degree.

9. The Solution of $\overline{M(G)} = L_s(H)$

Theorem 7 gives solutions for the equation $J(G) = L_s(H)$. Among these $(2nK_2, nK_2)$, $n \geq 2$ is of the form (G^{+},H). Therefore, the solution of the equation is $(2nK_1, nK_2)$, $\overline{M(G)} = L_s(H)$ is $(2nK_1, nK_2)$, $n \geq 2$.

Now, we state the following result.

Theorem 8. The solutions of the graph equation are $\overline{M(G)} = L_s(H)$ **are** $(2nK_1, nK_2)$, $n \geq 2$.

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