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Study of Thermal Convection in Micropolar Fluids Occupying a Rectangular Box

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Abstract

This paper is a Fourier–series assisted numerical study of two–dimensional steady thermal convection in micropolar fluid occupying a rectangular box. The horizontal walls of the cavity are uniformly heated to establish a linear temperature in the vertical direction. The vertical walls are insulated. The critical Rayleigh number is obtained numerically as a function of coupling parameter, couple stress parameter and aspect ratio, and the same is plotted graphically. The results of slender vertical, rectangular and square box of finite aspect ratio are obtained as limiting cases of the study.

Keywords: Benard convection, critical Rayleigh number, Fourier series, micropolar fluid, aspect ratio.

1. Introduction

Convective flow in a thin layer of fluid, free at the upper surface and heated from below, is of fundamental importance and a prototype to a more complex configuration in experiments and industrial processes. The convective flows in a liquid layer can be

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driven by buoyancy forces due to temperature gradients and / or thermo capillary forces caused by surface tension gradients. Thermal convective problems have long been studied extensively since the pioneering experimental and theoretical works of Benard [1], Rayleigh [2] and Pearson [3]. Most of the previous studies were concerned with convection in Newtonian fluids. However, much less work has been done on convection in non-Newtonian fluids such as the micropolar fluids. The theory of micropolar fluids, as developed by Eringen [4], has been a field of sprightly research for the last few decades especially in many industrially important fluids like paints, polymeric suspensions, colloidal fluids, and also in physiological fluids such as normal human blood and snivel fluids. Rama Rao [5] studied the effect of a magnetic field on convection in a micropolar fluid. Sharma and Gupta [6] studied convection in micropolar fluids in a porous medium. Siddheshwar and Sri Krishna [7] presented both linear and nonlinear analyses of convection in a micropolar fluid occupying porous medium. Rudraiah and Siddheshwar [8] analysed the effects of non-uniform temperature gradients of parabolic and stepwise types on the onset of Marangoni convection in a micropolar fluid. This study was later extended by Siddheshwar and Pranesh [9] to include the effect of a magnetic field and buoyancy forces. The effect of non uniform temperature gradient on Benard convection in micropolar fluids, using single tem Galerkin expansion technique has been investigated by Siddheshwar and Pranesh [10]. In this paper, we have investigated the effect of couple stress parameter, coupling parameter and aspect ratio on critical Rayleigh number in a rectangular, square and slender vertical box numerically.

Nomenclature

A Aspect ratio

b Breadth of rectangular cavity

$\vec{g}$ Acceleration due to gravity

h Height of rectangular cavity

I Momentum of Inertia

$\vec{\omega}$	Spin
N_1	Coupling parameter
N_2	Inertia parameter
N_3	Couple stress parameter
p	pressure
Pr	Prandtl number
$\vec{q}$	Velocity
Ra	Rayleigh number
Ra_c	Critical Rayleigh number
u	Horizontal velocity component
w	Vertical velocity component
x	Horizontal Cartesian coordinate
z	Vertical Cartesian coordinate

Greek symbols

α	Co-efficient of thermal expansion
ΔT	Temperature difference between the two horizontal plates
χ	Thermal diffusivity
η	Co-efficient of shear kinematic viscosity
η'	Co-efficient of spin viscosity
λ'	Co-efficient of bulk viscosity
ψ	Stream function
ρ	Actual density
ρ_0	Reference density
θ	Deviation from static temperature
ζ	Co-efficient of coupling viscosity or vortex viscosity

∇ Differential operator

∇_A^2 Modified Laplacian operator

Subscripts

c Critical quantity

Superscripts

* Dimensionless quantity

2. Mathematical Formulation

We consider two–dimensional thermal convection in micropolar fluids occupying a rectangular box with height h and width b. We choose a cartesian coordinate system with the x–axis in the horizontal direction and z–axis in the vertical direction. The horizontal walls are at $z = 0$ *and* $z = h$ and the vertical walls are at $x = -\frac{b}{2}$ and $x = \frac{b}{2}$ as shown in Fig. 1. The fluid is heated from below and cooled from above as in a typical Rayleigh-Benard problem, the temperature difference between the bounding walls being ΔT.

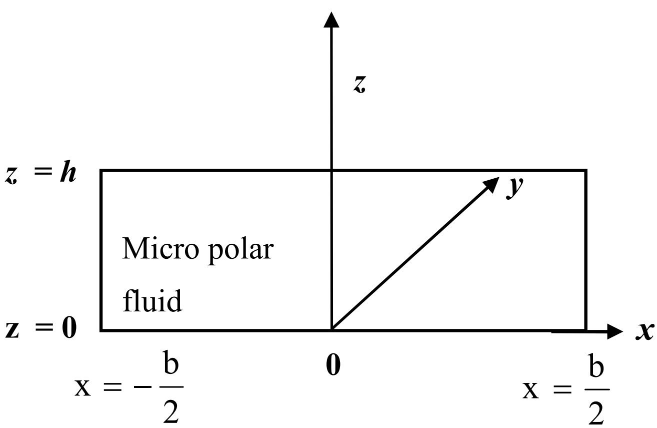

Figure 1: Schematic diagram of the flow configuration.

The governing equations are:

Continuity equation:

$$\nabla \cdot \vec{q} = 0 \tag{1}$$

Conservation of linear momentum:

$$\rho_0 \left[\left(\vec{q} \cdot \nabla \right) \vec{q} \right] = -\nabla p + \rho \vec{g} + (2\zeta + \eta) \nabla^2 \vec{q} + \zeta \left(\nabla \times \vec{\omega} \right) \tag{2}$$

Conservation of angular momentum:

$$\rho_0 I \left[\left(\vec{q} \cdot \nabla \right) \vec{\omega} \right] = (\lambda' + \eta') \nabla \left(\nabla \cdot \vec{\omega} \right) + \eta' \nabla^2 \vec{\omega} + \zeta \left(\nabla \times \vec{q} \right) - 2\zeta \vec{\omega} \tag{3}$$

Conservation of energy:

$$\left(\vec{q} \cdot \nabla \right) T = \chi \nabla^2 T \tag{4}$$

Equation of State:

$$\rho = \rho_0 \left(1 - \alpha \left(T - T_0 \right) \right). \tag{5}$$

In the Micropolar fluid, the particle spin matches with the vorticity of the carrier fluid and this result in an additional biharmonic term in equation (2). This is a drag term contributed to by the suspended particles. The effect of suspended particles in equation (4) on temperature T comes through the velocity $\vec{q}$. The upper and lower boundaries are at isothermal temperatures T_0 and $T_0 + \Delta T$ respectively, where ΔT is positive temperature difference. All the boundaries are assumed to be impermeable and perfectly heat conducting. From the governing equations (1) – (5) it follows that a motionless conduction state exists only if the static temperature distribution is independent of *x* and depends linearly on *z*. The present study is restricted to this case.

It is convenient to express the temperature in the form

$$T = T_0 + \Delta T\left(1 - \frac{z}{h}\right) + \theta \tag{6}$$

where θ is the deviation from the static temperature.

The component forms of equations (1) - (5) for steady flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{7}$$

$$\frac{1}{\rho_o}\frac{\partial p}{\partial x} - \frac{(2\zeta+\eta)}{\rho_0}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\zeta}{\rho_0}\frac{\partial \omega_y}{\partial z} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = 0, \tag{8}$$

$$\frac{1}{\rho_o}\frac{\partial p}{\partial z} - \frac{(2\zeta+\eta)}{\rho_0}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\zeta}{\rho_0}\frac{\partial \omega_y}{\partial x} + \frac{\rho}{\rho_0}g + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = 0 \tag{9}$$

$$\eta'\left(\frac{\partial^2 \omega_y}{\partial x^2} + \frac{\partial^2 \omega_y}{\partial z^2}\right) + \zeta\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - 2\zeta\omega_y - \rho I\left(u\frac{\partial \omega_y}{\partial x} + w\frac{\partial \omega_y}{\partial z}\right) = 0 \tag{10}$$

$$\chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) - u\frac{\partial T}{\partial x} - w\frac{\partial T}{\partial z} = 0 \tag{11}$$

$$\rho = \rho_o\left[1 - \alpha\,\Delta T\left(1 - \frac{z}{h}\right) - \alpha\theta\right]. \tag{12}$$

Since the flow is two-dimensional, we introduce the stream function ψ by

$$u = \frac{\partial\psi}{\partial z}, \quad w = -\frac{\partial\psi}{\partial x}. \tag{13}$$

We also define non-dimensional variables denoted by asterisks

$$(x^*, z^*) = \left(\frac{x}{b}, \frac{z}{h}\right), \; T^* = \frac{T}{\Delta T}, \; \omega_y^* = \frac{h^2}{\chi}\omega_y, \; \theta^* = \frac{\theta}{\Delta T}, \; \psi^* = \frac{\psi}{\chi}. \tag{14}$$

By eliminating the pressure from (8) and (9), introducing the expressions (13) and (14) into the resulting equation and equation (10) and (11) and dropping the asterisk, we get the governing equations in the following form:

$$(1+N_1)\nabla_A^2\psi - A^4 Ra\frac{\partial\theta}{\partial x} - N_1\nabla_A^2\omega_y + \frac{A}{\Pr}\frac{\partial(\psi,\nabla_A^2\psi)}{\partial(x,z)} = 0\,, \tag{15}$$

$$N_3\nabla_A^2\omega_y + A^2 N_1\nabla_A^2\psi - 2A^2 N_1\omega_y + \frac{AN_2}{\Pr}\frac{\partial(\psi,\omega_y)}{\partial(x,z)} = 0\,, \tag{16}$$

$$A^2\frac{\partial\psi}{\partial x} - A\nabla_A^2\theta - A^2\frac{\partial(\psi,\theta)}{\partial(x,z)} = 0\,, \tag{17}$$

where

$$A = \frac{h}{b},\ \nabla_A^2 = A^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2},\ N_1 = \frac{\zeta}{\zeta+\eta},\ N_2 = \frac{I}{b^2},$$

$$N_3 = \frac{\eta'}{(\zeta+\eta)b^2},\ \Pr = \frac{\rho_0\psi}{\zeta+\eta},\ Ra = \frac{\rho_0\alpha g\,\Delta T b^3}{\chi(\zeta+\eta)}$$

and boundary conditions are

$$\psi = \nabla_A^2\psi = \theta = \omega_y = 0 \ \text{at} \begin{cases} x = -\dfrac{1}{2}, \quad x = \dfrac{1}{2}, \quad 0 < z < 1 \\ z = 0, \quad z = 1, \ -\dfrac{1}{2} < x < \dfrac{1}{2} \end{cases}. \tag{18}$$

3. Linear Stability

The onset of thermal convection is described by the linear versions of equations (15), (16) and (17). To make this study we neglect the Jacobians in equations (15), (16) and (17) leads to

$$(1+N_1)\nabla_A^2\psi - A^4 Ra\frac{\partial\theta}{\partial x} - N_1\nabla_A^2\omega_y = 0\,, \tag{19}$$

$$A^2 N_1\nabla_A^2\psi + \left(N_3\nabla_A^2 - 2A^2 N_1\right)\omega_y = 0\,, \tag{20}$$

$$\nabla_A^2\theta - A\frac{\partial\psi}{\partial x} = 0\,. \tag{21}$$

By eliminating ω_y between (19) and (20) leads us to differential equation of sixth order as

$$(1+N_1)N_3\nabla_A^6\psi - A^2(2+N_1)N_1\nabla_A^4\psi + 2N_1A^6Ra\frac{\partial\theta}{\partial x}$$
$$-A^4N_3Ra\left(A^2\frac{\partial^3\theta}{\partial x^3}+\frac{\partial^3\theta}{\partial x\partial z^2}\right)=0. \tag{22}$$

The solutions of (21) and (22) can be expanded in the half range Fourier sine series:

$$\psi = \sum_{n=1}^{\infty} C_n(x)\sin n\pi z \tag{23}$$

$$\theta = \sum_{n=1}^{\infty} F_n(x)\sin n\pi z \tag{24}$$

Equations (21) and (22) are solved for stress free, isothermal boundaries, hence we have the boundary conditions in terms of $C_n(x)$ and $F_n(x)$ are:

$$C_n(x) = \frac{d^2}{dx^2}[C_n(x)] = \frac{d^4}{dx^4}[C_n(x)] = F_n(x) = 0 \text{ at } x = \pm\frac{1}{2} \tag{25}$$

Using (23) and (24) with the boundary condition (25) into the linearized versions of the governing equations (21) and (22), and equating coefficients of $\sin n\pi z$, leads to set of ordinary differential equations

$$\left(A^2\frac{d^2}{dx^2}-n^2\pi^2\right)F_n - A\frac{dC_n}{dx} = 0 \tag{26}$$

$$A^6\left[(1+N_1)N_3\right]\frac{d^6C_n}{dx^6}-A^4\left[3n^2\pi^2(1+N_1)N_3+A^2(2+N_1)N_1\right]\frac{d^4C_n}{dx^4}+$$
$$A^2\left[3n^4\pi^4(1+N_1)N_3+2n^2\pi^2A^2(2+N_1)N_1\right]\frac{d^2C_n}{dx^2}$$
$$-\left[n^6\pi^6(1+N_1)N_3+n^4\pi^4A^2(2+N_1)N_1\right]C_n \tag{27}$$
$$+A^4\left[2N_1A^2Ra\frac{d}{dx}+n^2\pi^2N_3Ra\frac{d}{dx}-N_3A^2Ra\frac{d^3}{dx^3}\right]F_n=0.$$

By eliminating F_n between (26) and (27) leads us to an ordinary differential equation of eighth order in C_n as

$$A^8\left[(1+N_1)N_3\right]\frac{d^8C_n}{dx^8}-A^6\left[4n^2\pi^2(1+N_1)N_3+A^2(2+N_1)N_1\right]\frac{d^6C_n}{dx^6}+$$
$$A^4\left[3n^2\pi^2A^2(2+N_1)N_1+6n^4\pi^4(1+N_1)N_3-N_3A^3Ra\right]\frac{d^4C_n}{dx^4}-$$
$$A^2\left[\begin{matrix}4n^6\pi^6(1+N_1)N_3+3n^4\pi^4A^2(2+N_1)N_1\\-2N_1A^5Ra-n^2\pi^2N_3A^3Ra\end{matrix}\right]\frac{d^2C_n}{dx^2}+ \tag{28}$$
$$\left[n^8\pi^8(1+N_1)N_3+n^6\pi^6A^2(2+N_1)N_1\right]C_n=0,$$

with boundary conditions

$$C_n\left(\pm\frac{1}{2}\right)=\frac{dC_n}{dx}\left(\pm\frac{1}{2}\right)=\frac{d^2C_n}{dx^2}\left(\pm\frac{1}{2}\right)=\frac{d^4C_n}{dx^4}\left(\pm\frac{1}{2}\right)=0. \tag{29}$$

The general solution of differential equation (28) for $n = 1$ is

$$C_1(x)=a_1e^{m_1x}+a_2e^{m_2x}+a_3e^{m_3x}+a_4e^{m_4x}+a_5e^{m_5x}$$
$$+a_6e^{m_6x}+a_7e^{m_7x}+a_8e^{m_8x} \tag{30}$$

where a_i's are arbitrary constants and m_i's are roots of the auxiliary equation of (28) with $m_2 = -m_1$, $m_4 = -m_3$, $m_6 = -m_5$, $m_8 = -m_7$.

Equations (29) and (30) give us eight homogenous equations in the eight unknowns $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$. For a non–trivial solution for the system we require

$$\begin{vmatrix} e^{0.5m_1} & e^{-0.5m_1} & e^{0.5m_3} & e^{-0.5m_3} & e^{0.5m_5} & e^{-0.5m_5} & e^{0.5m_7} & e^{-0.5m_7} \\ e^{-0.5m_1} & e^{0.5m_1} & e^{-0.5m_3} & e^{0.5m_3} & e^{-0.5m_5} & e^{0.5m_5} & e^{-0.5m_7} & e^{0.5m_7} \\ m_1e^{0.5m_1} & -m_1e^{-0.5m_1} & m_3e^{0.5m_3} & -m_3e^{-0.5m_3} & m_5e^{0.5m_5} & -m_5e^{-0.5m_5} & m_7e^{0.5m_7} & -m_7e^{-0.5m_7} \\ m_1e^{-0.5m_1} & -m_1e^{0.5m_1} & m_3e^{-0.5m_3} & -m_3e^{0.5m_3} & m_5e^{-0.5m_5} & -m_5e^{0.5m_5} & m_7e^{-0.5m_7} & -m_7e^{0.5m_7} \\ m_1^2e^{0.5m_1} & m_1^2e^{-0.5m_1} & m_3^2e^{0.5m_3} & m_3^2e^{-0.5m_3} & m_5^2e^{0.5m_5} & m_5^2e^{-0.5m_5} & m_7^2e^{0.5m_7} & m_7^2e^{-0.5m_7} \\ m_1^2e^{-0.5m_1} & m_1^2e^{0.5m_1} & m_3^2e^{-0.5m_3} & m_3^2e^{0.5m_3} & m_5^2e^{-0.5m_5} & m_5^2e^{0.5m_5} & m_7^2e^{-0.5m_7} & m_7^2e^{0.5m_7} \\ m_1^4e^{0.5m_1} & m_1^4e^{-0.5m_1} & m_3^4e^{0.5m_3} & m_3^4e^{-0.5m_3} & m_5^4e^{0.5m_5} & m_5^4e^{-0.5m_5} & m_7^4e^{0.5m_7} & m_7^4e^{-0.5m_7} \\ m_1^4e^{-0.5m_1} & m_1^4e^{0.5m_1} & m_3^4e^{-0.5m_3} & m_3^4e^{0.5m_3} & m_5^4e^{-0.5m_5} & m_5^4e^{0.5m_5} & m_7^4e^{-0.5m_7} & m_7^4e^{0.5m_7} \end{vmatrix} = 0 \qquad (31)$$

The left hand side of (31) may be viewed as a function Ra_c, say f(Ra_c), with Ra_c depending on A, N_1 and N_3, hence equation (31) can be written as f(Ra_c) = 0. Using Newton–Raphson method for various values of A, N_1 and N_3, Ra_c can be calculated numerically, using the iterative formula

$$(Ra_c)_{k+1} = (Ra_c)_k - \frac{f((Ra_c)_k)}{f'((Ra_c)_k)} \qquad (32)$$

where prime denotes differential of f(Ra_c) with respect to Ra_c and the paper calculation based on Newton–Raphson method were done using Matlab.

4. Results and Discussion

The thermal convection in micropolar fluids occupying a rectangular box is investigated numerically using linear stability theory assisted by Fourier series. Different aspect ratios are considered to cover the results of slender, square and rectangular cavity. In the present paper, the behavior of the system as a function of the critical Rayleigh number, $\mathrm{Ra_c}$, depends upon the aspect ratio A, coupling parameter N_1 and couple stress parameter N_3. For fixed N_3, the variation of Ra_c for steady thermal convection with N_1 *for* different values of A is shown in figure 2. From this we observe that the Ra_c increases with increases with N_1 and decreases with increase in the value of A. From the plot of Ra_c^* [$Ra_c^* = Ra_c - Ra_c$ (at $N_3 = 1$)] versus N_3 for different N_1 in a rectangular, square and vertical slender box, we observe that Ra_c^* increases with increase in N_1 and N_3 and decreases with increase in A as shown in figure 3. In the absence of coupling parameter, i.e., for $N_1 = 0$, the value of $Ra_c = 657.51$ (classical Rayleigh-Benard result) is obtained for a square box ($A = 1$) and for $N_3 \neq 0$.

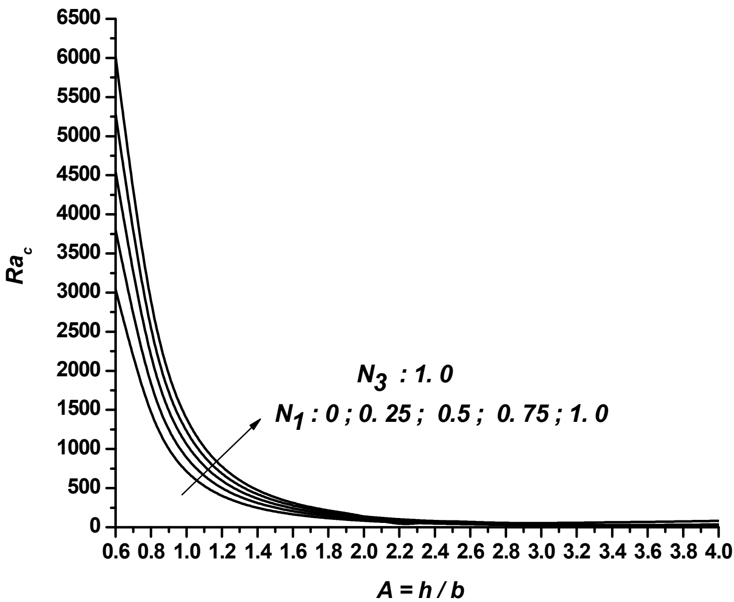

Figure 2: Plot of critical Ra_c versus A, for different values of N1 and for N3 = 1.0, for A < 1(shallow box), A = 1 (square box) and A > 1 (slender vertical box).

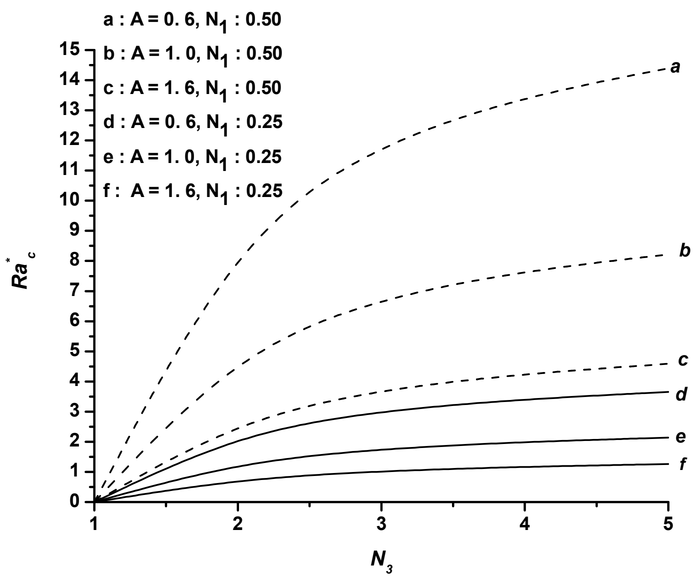

Figure 3: Plot of Ra_c^* versus N_3, for a various values of N_1 and A.

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