ISSN 0975-3303
Mapana J Sci, **11**, 4(2012), 1-14
https://doi.org/10.12725/mjs.23.1

Sign-Compatibility of Some Derived Signed Graphs

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Abstract

A *signed graph* (or *sigraph* in short) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \to \{+1, -1\}$ is a function from the edge set E of S^u into the set $\{+1, -1\}$, called the *signature* of S. A sigraph S is *sign-compatible* if there exists a *marking* μ of its vertices such that the end vertices of every negative edge receive '-1' marks in μ and no positive edge does so. In this paper, we characterize S such that its ×-line sigraphs, semi-total line sigraphs, semi-total point sigraphs and total sigraphs are sign-compatible.

Keywords: Sign-compatible, ×-line sigraph, semi-total line sigraph, semi- total point sigraph, total sigraph.

1. Introduction

For standard terminology and notation in graph theory we refer to Harary [6] and West [13] and Zaslavsky [14, 15] for sigraphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

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Received: July 2012, Reviewed: Aug. 2012

A *signed graph* (or *sigraph* in short; see [5]) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \rightarrow \{+1, -1\}$ is a function from the edge set E of S^u into the set $\{+1, -1\}$, called the *signature* of S. Let $E^+(S) = \{e \in E(G) : \sigma(e) = +1\}$ and $E^-(S) = \{e \in E(G): \sigma(e) = -1\}$. The elements of $E^+(S)$ and $E^-(S)$ are called *positive* and *negative* edges of S, respectively. A sigraph is *all-positive* (*all-negative*) if all its edges are positive (negative); further, it is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise. The *positive* (*negative*) degree of a vertex $v \in V(S)$ denoted by $d^+(v)(d^-(v))$ is the number of positive (negative) edges incident on the vertex v and $d(v) = d^+(v) + d^-(v)$.

A *marked sigraph* is an ordered pair $S_\mu = (S, \mu)$ where $S = (S^u, \sigma)$ is a sigraph and $\mu : V(S) \rightarrow \{+1, -1\}$is a function from the vertex set $V(S)$ of S into the set $\{+1, -1\}$, called a *marking* of S.

For a sigraph S, Behzad and Chartrand [1] defined its *line sigraph*, $L(S)$ as the sigraph in which the edges of S are represented as vertices, two of these vertices are defined adjacent whenever the corresponding edges in S have a vertex in common, any such edge *ef* is defined to be negative whenever both e and f are negative edges in S.

For a sigraph S, Gill [4] defined its ×*-line sigraph* $L_\times(S)$ as follows: the $L_\times(S)$ is a sigraph defined on the line graph $L(S^u)$ of the graph S^u by assigning to each edge *ef* of $L(S^u)$, the product of signs of the adjacent edges e and f of S.

The *semi-total line graph* $T_1(G)$ [7] of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ where $V(G)$ and $E(G)$ are vertex set and edge set of G, respectively and in $T_1(G)$ two vertices are adjacent if and only if (i) they are adjacent edges in G, or (ii) one is a vertex and the other is an edge in G incident to it.

The semi-total point graph $T_2(G)$ [7] of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ where $V(G)$ and $E(G)$ are vertex set and edge set of G respectively and in $T_2(G)$ two vertices are adjacent if and only if (i) they are adjacent vertices in G, or (ii) one is a vertex and the other is an edge in G incident to it.

Let $S = (V, E, \sigma)$ be any sigraph. Its *semi-total line sigraph* $T_1(S)$ [as shown in Figure 1] has $T_1(S^u)$ as its underlying graph and for any edge uv of $T_1(S^u)$

$$\sigma_{T_1}(uv) = \begin{cases} \sigma(u)\sigma(v) \text{ if } u, v \in E \\ \sigma(u) \text{ if } u \in E \text{ and } v \in E. \end{cases}$$

Let $S = (V, E, \sigma)$ be any sigraph. Its *semi-total point sigraph* $T_2(S)$ [as shown in Figure 1] has $T_2(S^u)$ as its underlying graph and for any edge uv of $T_2(S^u)$

$$\sigma_{T_2}(uv) = \begin{cases} \sigma(uv) \text{ if } u, v \in E \\ \sigma(u) \prod\limits_{e_j \in E_v} \sigma(e_j) \text{ if } u \in E \text{ and } v \in E. \end{cases}$$

We observe that $L_\times(S)$ is an induced subsigraph of $T_1(S)$ and S is an induced subsigraph of $T_2(S)$.

The *total graph* $T(G)$ [2, 3] of a graph G is that graph whose vertex set is $V(G) \cup E(G)$ where $V(G)$ and $E(G)$ are vertex set and edge set of G respectively and in $T(G)$ two vertices are adjacent if and only if they are adjacent or incident in G.

Let $S = (V, E, \sigma)$ be any sigraph. Its *total sigraph* $T(S)$ [as shown in Figure 1] has $T(S^u)$ as its underlying graph and for any edge uv of $T(S^u)$ the sign of the edge is defined as (*i*) if $u, v \in V$, then $\sigma_T(uv) = \sigma(uv)$ (*ii*) if $u, v \in E$, then $\sigma_T(uv) = \sigma(u)\sigma(v)$ (*iii*) if $u \in E$ and $v \in V$, then $\sigma_T(uv) = \sigma(u) \prod_{e_j \in E_v} \sigma(e_j)$.

We observe that S and $L_\times(S)$ are the induced subsigraphs of $T(S)$. Also, $T_2(S)$ is a subsigraph of $T(S)$. Much of the work has already been done on the structures $T_1(S)$, $T_2(S)$ and $T(S)$. (see [9, 10, 11])

Looking into the structures of the line sigraphs it was observed [8] that the vertices of a line sigraph can be marked so that both the ends of every negative edge receive negative marks and no positive edge receives negative marks at both of its ends. Since the converse of the above statement does not hold, as can be seen in Figure 2 and Figure 3, therefore it becomes important to investigate the notion of '*sign-compatibility*' in sigraphs.

A sigraph S is *sign-compatible* [8] if there exists a marking μ of its vertices such that the end vertices of every negative edge receive '−1' marks in μ and no positive edge in S has both of its ends assigned '−1' marks in μ, *sign-incompatible* otherwise.

Theorem 1. [8] A sigraph S is sign-compatible if and only if its vertices can be partitioned into two subsets V_1 and V_2 such that the all-negative subsigraph of S is precisely the subsigraph induced by exactly one of the subset V_1 or V_2.

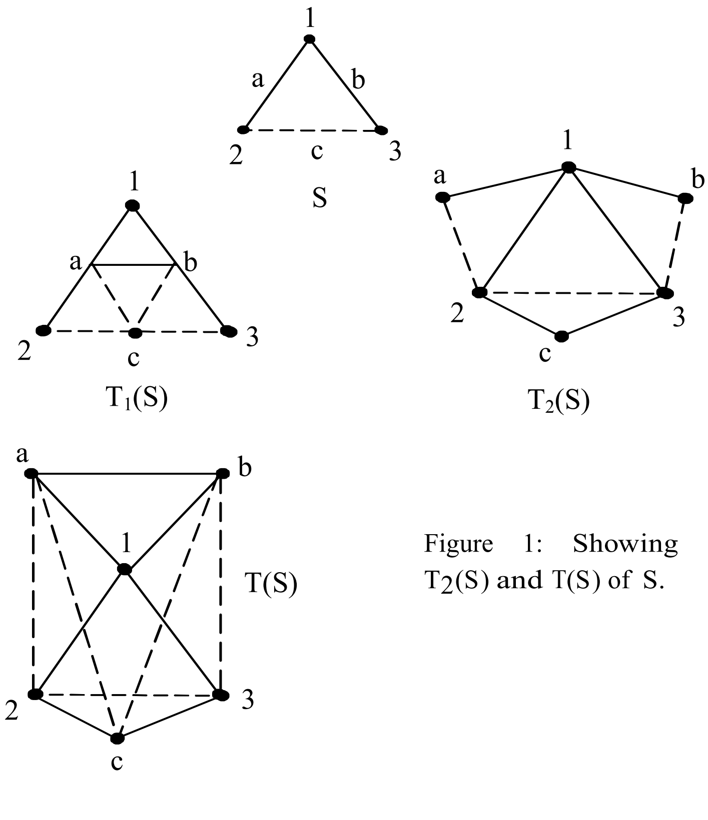

Figure 1: Showing $T_1(S)$, $T_2(S)$ and T(S) of S.

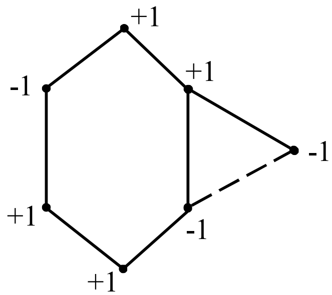

Figure 2: A line sigraph which is sign-compatible.

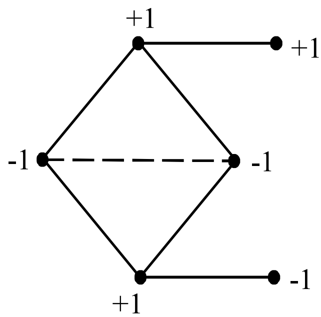

Figure 3: A sigraph which is sign-compatible but not a line sigraph.

Proposition 2. Every subsigraph of a sign-compatible sigraph is sign-incompatible.

Theorem 3. [12] A sigraph S is sign-compatible if and only if S does not contain a subsigraph isomorphic to either of the two sigraphs, S_1 formed by taking the path $P_4=(x, u, v, y)$ with both the edges xu and vy negative and the edge uv positive and S_2 formed by taking S_1 and identifying the vertices x and y (Figure 4).

2. Sign-Compatibility of ×-line Sigraphs

In this section we establish a characterization of sign-compatible ×-line sigraphs. The observations, which becomes ready reference for the proof of the theorems, are stated as under.

Observation 4. For a given sigraph S = (S^u,σ), if $L_{\times}(S)$ is heterogeneous, then S is heterogeneous but converse is not true, as can be seen in Figure 5.

Observation 5. L×(S) is all-negative if and only if S is either an even cycle or a path with alternative '+1' and '−1' signs.

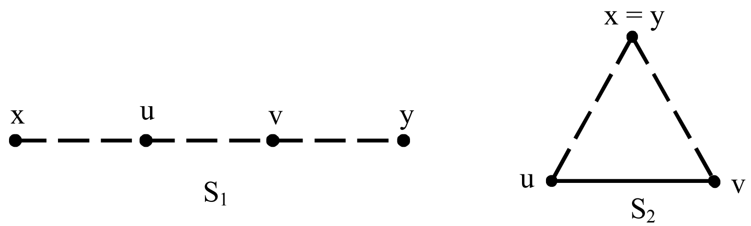

Figure 4: Acharya and Sinha forbidden subsigraphs for a sign-compatible sigraph

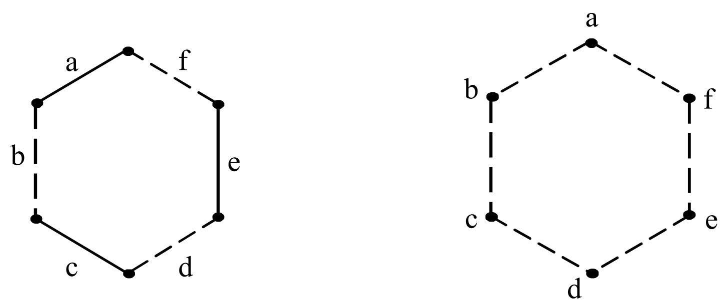

Figure 5: A sigraph and its x-line sigraph

Observation 6. L×(S) is all-positive if and only if S is either all-positive or all-negative.

The following theorem determines the solution when a ×-line sigraph is sign- compatible.

Theorem 7. For a given sigraph S = (S^u, σ), $L_{\times}(S)$ is sign-compatible if and only if for any two adjacent positive (negative) edges, say e_i and e_j, either there is no negative (positive) edge adjacent with e_i or there is no negative (positive) edge adjacent with e_j in S.

Proof. **Necessity:** Suppose $L_\times(S)$ is sign-compatible. Then, by Theorem 3, $L_\times(S)$ does not contain a subsigraph isomorphic to either S_1 or S_2 in Figure 4.

Let there be two adjacent positive (negative) edges, say e_i and e_j, in S and there are negative (positive) edges adjacent with e_i and e_j. Either there is one negative (positive) edge, which is adjacent with both e_i and e_j or there are distinct negative (positive) edges adjacent with e_i and e_j. If there is one negative (positive) edge adjacent with both e_i and e_j, then by the definition of $L_\times(S)$, $L_\times(S)$ contains a subsigraph isomorphic to S_2, a contradiction to our assumption. Now, let e_k be a negative (positive) edge adjacent with e_i and e_l is a negative (positive) edge adjacent with e_j in S, then by the definition of $L_\times(S)$, we have a path $P_4 = (e_k, e_i, e_j, e_l)$ in $L_\times(S)$ such that $e_k e_i$ and $e_j e_l$ are negative edges and $e_i e_j$ is a positive edge. Thus, $L_\times(S)$ contains a subsigraph isomorphic to S_1, a contradiction to our assumption. Thus, the condition follows.

Sufficiency: Suppose condition in the statement of the theorem holds for a sigraph S. We want to show that $L_\times(S)$ is sign-compatible. Let on contrary, $L_\times(S)$ is not sign-compatible. Then, by Theorem 3, $L_\times(S)$ contains a subsigraph isomorphic to either S_1 or S_2.

Case I: Let $L_\times(S)$ contains a subsigraph, say P_4', isomorphic to S_1. Let $P_4' = (e_i, e_j, e_k, e_l)$ such that $e_i e_j$ and $e_k e_l$ are negative edges and $e_j e_k$ is a positive edge in $L_\times(S)$. Then, by the definition of $L_\times(S)$, we have a path $P_5 = (u, v, w, x, y)$ in S such that $e_i = uv$, $e_j = vw$, $e_k = wx$ and $e_l = xy$. Also, e_j, e_k are positive (negative) edges and e_i, e_l are negative (positive) edges in S, a contradiction to the hypothesis. Thus, $L_\times(S)$ does not contain a subsigraph isomorphic to S_1.

Case II: Now, let $L\times(S)$ contains a subsigraph isomorphic to S_2, then this triangle is either due to the edges of a triangle or due to a vertex $v \in V(S)$ in S with $d(v) \geq 3$.

II (a): Let Z' be a triangle in $L\times(S)$ which is isomorphic to S_2. If all the vertices of Z' are due to the adjacent edges of a single triangle Z in S, then by the definition of $L\times(S)$, we have a triangle Z in S with two positive (negative) and one negative (positive) edges. Thus, for two adjacent positive (negative) edges in S, we have a negative (positive) edge adjacent with these edges, a contradiction to hypothesis.

II (b): Now, since this triangle is not due to any triangle of S, therefore Z' must contain a vertex, say e_p, which corresponds to an edge e_p not lying on any cycle but incident to a vertex v with $d(v) \geq 3$ in S. Then, by the definition of $L\times(S)$, there are two positive (negative) and one negative (positive) edges incident on v in S. Thus, we have a contradiction to the hypothesis. Hence, $L\times(S)$ does not contain a subsigraph isomorphic to S_2. Hence, by Theorem 3, $L\times(S)$ is sign- compatible.

Note: In the all upcoming theorems, e_i, e_j, e_k, e_l and e_m denote the edges of S and v_i, v_j, v_k, v_l, v_m and v_n denote the vertices of S.

3. Sign-compatibility of Semi-total Line Sigraphs

Next, for studying the characterization of yet another structure, semi-toatl line sigraphs, we have the following observation.

Observation 8. For a given sigraph $S = (S^u, \sigma)$, $T_1(S)$ is never all-negative.

The following theorem determines the solution when a semi-total line sigraph is sign-compatible.

Theorem 9. For a given sigraph $S = (S^u, \sigma)$, $T_1(S)$ is sign-compatible if and only if S is all-positive.

Proof. **Necessity:** Let $T_1(S)$ be sign-compatible. Then, by Theorem 3, $T_1(S)$ does not contain a subsigraph isomorphic to either S_1 or S_2.

Let $e_l = v_i v_j$ and $e_m = v_j v_k$ be two adjacent edges in S. If one is positive and other is negative or both are negative, then by the definition of $T_1(S)$, we have a triangle (e_l, e_m, v_j, e_l) in $T_1(S)$ with one positive and two negative edges. Thus, $T_1(S)$ contains a subsigraph isomorphic to S_2, a contradiction to the hypothesis. Hence, no two adjacent edges can be heterogeneous or both negative in S. Hence, S is all-positive.

Sufficiency: Let S be all-positive, then by the definition of $T_1(S)$, $T_1(S)$ is also all-positive. Hence, $T_1(S)$ is sign-compatible.

4. Sign-compatibility ofSsemi-total Point Sigraphs

With the Observation 10 and Observation 11, we now venture for the characterization of sign-compatible semi-total point-sigraphs.

Observation 10. For a given sigraph $S = (S^u, \sigma)$, if S is heterogeneous then $T_2(S)$ is also heterogeneous but converse is not true, as can be seen in **Figure 6**.

Observation 11. $T_2(S)$ is all-negative if and only if S is all-negative and $d(v)$ is even $\forall\, v \in V(S)$.

The following theorem determines the solution when a semi-total point sigraph is sign-compatible.

Theorem 12. For a given sigraph $S = (S^u, \sigma)$, $T_2(S)$ is sign-compatible if and only if for any edge $e_k = v_i v_j$ in S,

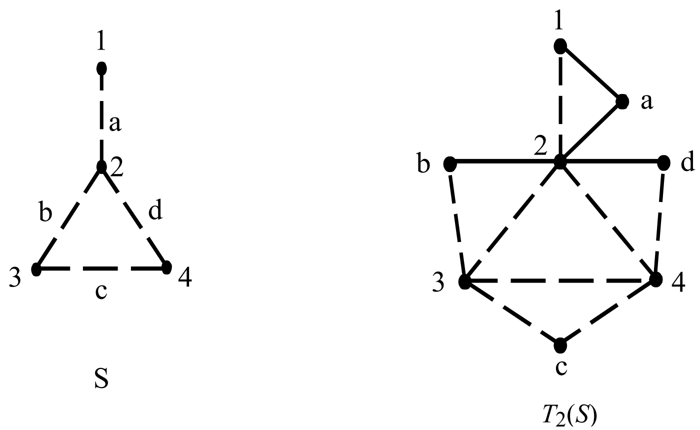

Figure 6: A sigraph and its semi-total point sigraph.

i. if e_k is a positive edge, then either there is no negative edge on v_i or there is no negative edge on v_j and

ii. if e_k is a negative edge, then $d^-(v_i)$ and $d^-(v_j)$ are of same parity.

Proof. **Necessity:** Let $T_2(S)$ be sign-compatible. Then, by Theorem 3, $T_2(S)$ does not contain a subsigraph isomorphic to either S_1 or S_2.

Let $e_k = v_i v_j$ be any edge in S. First we assume that e_k is a positive edge. Let for a positive edge $e_k = v_i v_j$, there be negative edges on v_i and v_j . This implies that S contains a subsigraph isomorphic to either S_1 or S_2. Thus, $T_2(S)$ contains a subsigraph isomorphic to either S_1 or S_2 , a contradiction to our assumption. Hence, (i) follows.

Next, let $e_k = v_i v_j$ be negative edge in S and $d^-(v_i)$ and $d^-(v_j)$ are of opposite parity i.e., $d^-(v_i)$ is odd and $d^-(v_j)$ is even or vice-versa. Then, by the definition of $T_2(S)$, we have a triangle (v_i , v_j , e_k , v_i), with one positive and two negative edges in $T_2(S)$. Thus, $T_2(S)$ contains a subsigraph isomorphic to S_2, a contradiction to our assumption. Hence, (ii) follows.

Sufficiency: Suppose conditions (i) and (ii) hold for a given sigraph S. We shall show that $T_2(S)$ is sign-compatible. Suppose on contrary, $T_2(S)$ is not sign-compatible. Then, by Theorem 3, $T_2(S)$ contains a subsigraph isomorphic to either S_1 or S_2.

Case I: Let $T_2(S)$ contains a subsigraph, say P_4, isomorphic to S_1. Now, P_4 will be one out of (v_i, v_j, v_k, v_l), (v_i, v_j, v_k, e_l), (e_i, v_j, v_k, e_l), (v_i, v_j, e_k, v_l) and (e_i, v_j, e_k, v_l). Here $P_4 = (v_i, v_j, v_k, v_l)$, otherwise we have a contradiction to (i).

I(a): Let $P_4 = (v_i, v_j, e_k, v_l)$ or (e_i, v_j, e_k, v_l). In both the cases e_k is not a negative edge in S otherwise we have a subsigraph, (v_j, e_k, v_l, v_j), isomorphic to S_2 in $T_1(S)$. Thus, we have one negative edge, e_k, in S such that the negative degree of the end vertices of e_k have opposite parity, a contradiction to (ii). Hence, e_k is a positive edge in S. When $P_4 = (v_i, v_j, e_k, v_l)$, then $e_k v_l$ is a negative edge in $T_2(S)$, then by the definition of $T_2(S)$, there are odd no of negative edges incident on v_l in S. Thus, we have a positive edge $e_k = v_j v_l$ in S such that there are negative edges incident on v_j and v_l, a contradiction to (i).

Now, when $P_4 = (e_i, v_j, e_k, v_l)$ either e_i is a negative or a positive edge in S, but in both the cases, by the definition of $T_2(S)$, we have a positive edge $e_k = v_j v_l$ in S such that there are negative edges incident on v_j and v_l, a contradiction to (i).

I(b): Let $P_4 = (v_i, v_j, v_k, e_l)$ or $P_4 = (e_i, v_j, v_k, e_l)$, in both the cases whatever be the sign of e_i and e_l in S, by the definition of $T_2(S)$, we have a positive edge $v_j v_k$ in S such that there are negative edges incident on v_j and v_k, a contradiction to (i). Thus, $T_2(S)$ does not contain a subsigraph isomorphic to S_1.

Case II: Now, let $T_2(S)$ contains a subsigraph isomorphic to S_2. Then, either this triangle is due to the vertices of a triangle of S or due to two adjacent vertices of S.

II (a): Let Z' be a triangle isomorphic to S_2 in $T_2(S)$ and this is due to the vertices of a triangle of S. Then, we have a triangle with one positive and two negative edges in S, a contradiction to (i).

II (b): Let Z' be due to two adjacent vertices, say v_i and v_j, in S and let $e_k = v_i v_j$. Now, either e_k is a positive or negative edge in S. If e_k is a positive edge, then by the definition of $T_2(S)$, we have a positive edge $e_k = v_i v_j$ in S such that there are odd number of negative edges on both the vertices, v_i and v_j . This contradicts (i). If e_k is a negative edge in S, then by the definition of $T_2(S)$, we have a negative edge e_k in S such that $d^-(v_i)$ and $d^-(v_j)$ are of opposite parity. This contradicts (ii). Thus, $T_2(S)$ does not contain a sub-sigraph isomorphic to S_2. Hence, $T_2(S)$ is sign-compatible.

5. Sign-compatibility of Total Sigraphs

Having characterized about semi-total line sigraphs and semi-total point sigraphs, the natural thing that arises in ones mind is that what if both S and ×-line sigraph lie as an induced subsigraph in some sigraph. Yes, when the sigraph is a total sigraph. Now, we present a characterization of sign-compatible total sigraphs with the help of Observation 13.

Observation 13. $T(S)$ is never all negative.

The above observation easily follows from the fact that S and $L_\times(S)$ are the induced subsigraphs of $T(S)$.

The following theorem determines the solution when a total sigraph is sign-compatible.

Theorem 14. For a given sigraph $S = (S^u, \sigma)$, $T(S)$ is sign-compatible if and only if S is either all-positive or all-negative such that $d(v)$ is odd $\forall\, v \in V(S)$.

Proof. **Necessity:** Let $T(S)$ be sign-compatible. Then, by Theorem 3, $T(S)$ does not contain a subsigraph isomorphic to either S_1 or S_2.

Now, either $T(S)$ is homogeneous or heterogeneous. By Observation 13, $T(S)$ will be either all-positive or heterogeneous. Since S is an induced subsigraph of $T(S)$, therefore if $T(S)$ is all-positive, then S is also all-positive. If $T(S)$ is heterogeneous, then either S is heterogeneous or S is all-negative. Let e_i and e_j be two edges of opposite signs in S and incident on a vertex, say v_k. Then, by the definition of $T(S)$, we have a triangle (e_i, e_j, v_k, e_i) in $T(S)$ which is isomorphic to S_2. Thus, we get a contradiction to our assumption. Hence, $T(S)$ is not heterogeneous.

Next, let S be all-negative. Since $T_2(S)$ is a subsigraph of $T(S)$, therefore $T_2(S)$ is also all-negative. Now, by Proposition 2 and Theorem 12, degree of every vertex of $T(S)$ is either odd or even. Suppose degree of any vertex, say v_k, is even in S. Let e_i and e_j be two adjacent edges on the vertex v_k. Then, by the definition of $T(S)$, we have a triangle (e_i, e_j, v_k, e_i) with one positive and two negative edges. Thus, $T(S)$ contains a subsigraph isomorphic to S_2, a contradiction to our assumption. Hence, S is all-negative such that $d(v)$ is odd $\forall\, v \in V(S)$.

Sufficiency: Suppose the condition in the theorem holds for a given sigraph S. If S is all-positive, then by the definition of $T(S)$, $T(S)$ is also all-positive. Hence $T(S)$ is sign-compatible.

Now, let S be all-negative such that degree of every vertex is odd. By the definition of $T(S)$, any positive edge in $T(S)$ will be either due to two adjacent edges in S or due to a vertex and an edge incident on it and a negative edge in $T(S)$ is only due to two adjacent vertices i.e., any positive edge is of the form $e_i e_j$ or $v_l e_k$ and negative edge is of the form $v_m v_n$. Thus, for a positive edge in $T_2(S)$, there are not negative edges incident on both its end vertices. Thus, $T(S)$ does not contain a subsigraph isomorphic to either S_1 or S_2. Thus, by Theorem 3, $T(S)$ is sign-compatible. Hence the theorem.

Acknowledgement

The authors express gratitude to Mr. Pravin Garg who was always there in the prior discussion of the paper to give his comments. The research is supported by Department of Science and Technology (Govt. of India), New Delhi, India under the Project SR/S4/MS: 409/06.

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