ISSN 0975-3303
Mapana J Sci, **11**, 4(2012), 15-42
https://doi.org/10.12725/mjs.23.2

$L(2,1)-$ Labelling of Cactus Graphs

Nasreen Khan,[*] Madhumangal Pal[†] and Anita Pal[‡]

Abstract

The $L(2,1)$-labelling of a graph G is an abstraction of assigning integer frequencies to radio transmitters such that the transmitters that are one unit of distance apart receive frequencies that differ by at least two, and transmitters that are two units of distance apart receive frequencies that differ by at least one. The span of an $L(2,1)$-labelling is the difference between the largest and the smallest assigned frequency. The $L(2,1)$-labelling number of a graph G, denoted by $\lambda(G)$, is the least integer k such that G has an $L(2,1)$-labelling of span k. A cactus graph is a connected graph in which every block is either an edge or a cycle. The goal of the problem is to show that for a cactus graph $\Delta+1 \le \lambda(G) \le \Delta+2$, where Δ is the degree of G. An optimal algorithm is also presented here to label the vertices of cactus graph using $L(2,1)$-labelling technique in $O(n)$ time, where n is the total number of vertices of the cactus graph.

[*] Department of Mathematics, Global Institute of Management and Technology, Krishnagar-741102, West Bengal, India, afsaruddinnkhan@gmail.com

[†] Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore-721102, West Bengal, India, mmpalvu@gmail.com

[‡] Department of Mathematics, National Institute of Technology Durgapur, Durgapur-713209, West Bengal, India, anita.buie@gmail.com

Received: July 2012, Reviewed: Aug. 2012

Keywords: Graph labelling, $L(h,k)$-labelling, $L(2,1)$-labelling, cactus graph, frequency assignment, radiocoloring, design of algorithms, analysis of algorithms

1. Introduction

Cactus graph is a connected graph in which every block is a cycle or an edge, in other words, no edge belongs to more than one cycle. Cactus graph have extensively studied and used as models for many real world problems. This graph is one of the most useful discrete mathematical structure for modelling problem arising in the real world. It has many applications in various fields like computer scheduling, radio communication system etc. Cactus graph have studied from both theoretical and algorithmic points of view. This graph is a subclass of planar graph and superclass of tree.

Recently $L(2,1)$-labelling problem is attracted by many researchers due to its importance in real life applications. An $L(2,1)$-labelling of a graph $G=(V,E)$ is a function of f from its vertex set V to the set of non-negative integers such that $|f(x)-f(y)|\geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)|\geq 1$ if $d(x,y)=2$, where $d(x,y)$ is the distance between the vertices x and y, i.e., the number of edges between x and y. The span of an $L(2,1)$-labelling f of G is $max\{f(v): v\in V\}$. The $L(2,1)$-labelling $\lambda(G)$ of G is the smallest k such that G has a $L(2,1)$-labelling of span k. The number λ is known as minimum span.

This problem was introduced by Griggs and Yeh [12] (proposed by Roberts) as a variation of channel (frequency) assignment problem, which is stated below.

1.1 Application of $L(2,1)$-labelling

The channel assignment problem is to assign a channel (non-negative integer) to each radio transmitter (radio, T.V, cell phone, etc.) so that interfering transmitters are assigned channels whose separations is not in a set of disallowed separations. Hale [13] formulated this problem into the notion of the T-colouring of a graph. A variation of the channel assignment problem in which ``close" transmitters must receive different channels and ``very close" transmitters must receive channels that are at least two channels apart. This problem can be modelled as a graph labelling/ colouring problem where the vertices represent the transmitters two vertices are ``very close" if they are adjacent and ``close" if they are of distance two in the graph. This type of channel assignment is known in the literature as an $L(2,1)$-labelling.

An $L(2,1)$-labelling of a graph G that uses labels in the set $\{0,1,.....,k\}$ is called a k-labelling. In the context of the channel assignment problem the main aim is to minimize k. Griggs and Yeh [12] conjectured that $\lambda(G)\leq\Delta^2$, where Δ represents the maximum degree of a vertex of G. Since $L(2,1)$-labelling problem i.e., to determine $\lambda(G)$ is NP-hard [10], the people focuses on verifying Grrigs and Yeh's conjecture and finding exact values for Δ for particular classes of graphs. A survey of recent literature is given below.

2. Review of Previous Works

Several results are known for $L(2,1)$-labelling of graphs, but, to the best of our knowledge no result is known for cactus graph. In this section, the known result for general graphs and some related graphs of cactus graph are presented.

The lower bound for $\lambda(G)$ is $\Delta+1$, which is achieved for the star $K_{1,\Delta}$. Griggs and Yeh [12] prove that $\lambda(G)\leq\Delta^2+2\Delta$ for general graph and improve this upper bound to $\lambda(G)\leq\Delta^2+2\Delta-3$ when G is 3- connected and $\lambda(G)\leq\Delta^2$ when G is diameter 2 (diameter 2 graph is a graph where all nodes have either distance 1 or 2 each other). Jonas [15] improves the upper bound to $\lambda(G)\leq\Delta^2+2\Delta-4$ if $\Delta\geq 2$, by constructive labelling schemes. Chang and Kuo [2] further decrease the bound to $\Delta^2+\Delta$. Further, Kral and Skrekovski [16] improves this bound $\lambda(G)\leq\Delta^2+\Delta-1$ for any graph G. The best known result till date is $\lambda(G)\leq\Delta^2+\Delta-2$ due to Goncalves [11]

The problem is simple for paths P_n of n vertices. It can easily be verified that $\lambda(P_1)=0$, $\lambda(P_2)=2$, $\lambda(P_3)=\lambda(P_4)=3$, and $\lambda(P_n)=4$ for $n\geq 5$.

When first and last vertices of P_n are merged then P_n becomes C_{n-1}. In [12], Griggs and Yeh shown that $\lambda(C_n)=4$ for any n.

The wheel W_n, is obtained by joining C_n and K_1, i.e., $W_n = C_n + K_1$. In [23], Yeh shown that $\lambda(W_n)=n+1$.

For the complete graph K_n and Kurotoski graph $K_{n,m}$ it can be shown that $\lambda(K_n)=2n-2$, $n\geq 1$ and $\lambda(K_{n,m})=m+n$.

For any tree T, Griggs and Yeh [12] shown that $\lambda(T)$ is either $\Delta+1$ or $\Delta+2$. Heuvel and Mc Guinness prove that $\lambda(G)\leq 2\Delta+35$ [14] for planar graph. Molloy and Salavatipour [18] reduced this upper bound to $5\Delta/3+90$. Wang and Lih [21] proved that if G is a planar graph of girth (girth is defined to be the length of a shortest cycle in G) at least 5, then $\lambda(G)=\Delta+21$.

The $L(2,1)$-labelling for chordal graphs has been first investigate by Sakai [20] and he proved that $\lambda(G)=\dfrac{(\Delta+3)^2}{4}$. For unit interval graph (a subclass of chordal graph) he shown that $\lambda(G)\leq 2(\Delta+1)$. Adams et al. [1], give different bounds for certain generalized petersen graphs. A study on $L(d,1)$-labelling of cartesian product of a cycle and a path is done by Chiang and Yan [4].

For further studies on the $L(2,1)$-labelling, see [3, 10, 7, 9, 10, 12, 17, 20].

3. Preliminaries

In this section, some basic results are presented. Let $G=(V_G,E_G)$ and $H=(V_H,E_H)$ be two graphs with disjoin vertex sets. The **union** of a graph G and H, denoted by $G\bigcup H$, is the graph whose vertex set is $V_G \bigcup V_H$ and edge set is $E_G \bigcup E_H$. The **join** of G and H, denoted by $G+H$, is the graph obtained from $G \bigcup H$ by adding all edges between vertices in V_G and V_H, i.e., $V_{G+H} = V_G \bigcup V_H$ and $E_{G+H} = E_G \bigcup E_H \bigcup \{(u,v) : u \in V_G \,\&\, v \in V_H\}$.

The following important lemmas are useful to establish the lower bounds of $\lambda(G)$.

Lemma 1 [2] If H is a subgraph of G then $\lambda(H) \le \lambda(G)$.

Lemma 2 [2] If $V_G \bigcap V_H = \phi$ then $\lambda(G \bigcup H) = max\{\lambda(G), \lambda(H)\}$ and $\lambda(G+H) = max\{|V_H|-1, \lambda(G)\} + max\{|V_H|-1, \lambda(H)\} + 2$.

A particular type of graph union denoted by $\bigcup_v$ is defined as follows.

Let $G = (V_G, E_G)$ and $H = (H_G, E_G)$ be two graphs having *only one* common vertex v, i.e., $V_G \bigcap V_H = \{v\}$. Let $F = G \bigcup_v H$, where $V_F = V_G \bigcup V_H$ and $E_F = E_G \bigcup E_H$. It is easy to verify that $|V_F| = |V_G| + |V_H| - 1$ and $|E_F| = |E_G| + |E_H|$. Also, the graphs G and H both are subgraphs of $G \bigcup_v H$. We refer this union as *v-union*.

By Lemma 1, $\lambda(G) \le \lambda(G \bigcup_v H)$ and $\lambda(H) \le \lambda(G \bigcup_v H)$. Thus we conclude the following result.

Lemma 3 $\lambda(G \bigcup_v H) \ge max\{\lambda(G), \lambda(H)\}$, where $\{v\} = V_G \bigcap V_H$.

4. The $L(2,1)$-labelling of Induced Subgraphs of Cactus Graphs

Let $G = (V, E)$ be a given graph and subset U of V the **induced subgraph** by U, denoted by $G[U]$, is the given graph $G' = (U, E')$, where $E' = \{(u,v) : u, v \in U$ and $(u,v) \in E\}$.

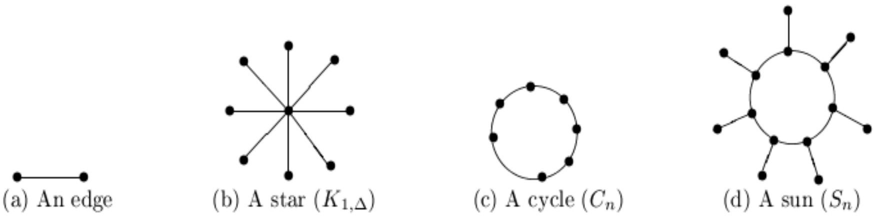

Figure 1: Induced subgraphs of cactus graph.

The cactus graph is a subclass of planar graph and superclass of tree. From the result of [12] and [4] we can conclude the following.

Lemma 4 For any cactus graph G, $\Delta+1 \leq \lambda(G) \leq 2\Delta+35$.

The cactus graph have many interested subgraphs, those are illustrated below. An edge is a nothing but P_2, so $\lambda(an\ edge)=2$. The star graph $K_{1,\Delta}$ is a subgraph of $K_{n,m}$ therefore one can conclude the following result.

Lemma 5 For any star graph $K_{1,\Delta}$, $\lambda(K_{1,\Delta})=\Delta+1$, which is equal to n, where n is the number of vertices.

In [12], Griggs and Yeh have label C_n by $L(2,1)$-labelling and they have obtained the following result. Here we have given a constructive prove of this result.

Lemma 6 [12] For any cycle C_n of length n, $\lambda(C_n)=4=\Delta+2$.

Proof. Let C_n be a cycle of length n. We classify C_n into three groups, viz., C_{3k}, C_{3k+1}, C_{3k+2}. The $L(2,1)$-labelling schemes of C_{3k} are same for any k. Similarly for C_{3k+1}, C_{3k+2}.

Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be the vertices of C_n. Then the labelling process are as follows.

Case 1. Let $n=3k\equiv 0$ (mod 3), i.e., C_{3k}.

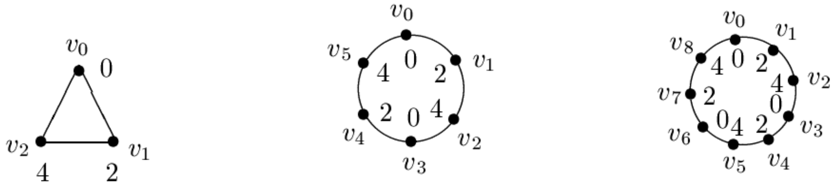

Figure 2: Cases of C_{3k} .

$$f(v_i)=\begin{cases}0, & \text{if } i\equiv 0 \,(\mathrm{mod}\ 3);\\ 2, & \text{if } i\equiv 1 \,(\mathrm{mod}\ 3);\\ 4, & \text{if } i\equiv 2 \,(\mathrm{mod}\ 3).\end{cases}$$

Case 2. Let $n=3k+1\equiv 1$ (mod 3), i.e., C_{3k+1}.

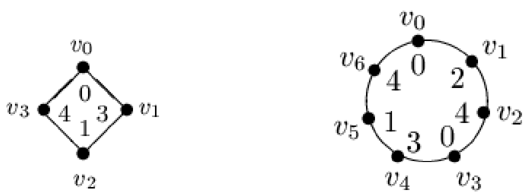

Figure 3: Cases of C_{3k+1}

In this case, the labelling of first $3k-3$ vertices $v_0,v_1,v_2,\ldots,v_{3k-3}=v_{n-5}$ are same as in Case 1. For the last four vertices, viz., v_{n-4},v_{n-3},v_{n-2} and v_{n-1} the f is redefined as $f(v_{n-4})=0$, $f(v_{n-3})=3$, $f(v_{n-2})=1$ and $f(v_{n-1})=4$ respectively.

Case 3. Let $n=3k+2\equiv 2$ (mod 3), i.e., C_{3k+2}.

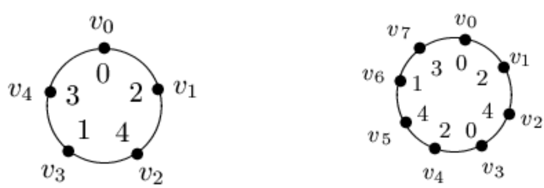

Figure 4: Cases of C_{3k+2}

In this case the labelling procedure of the vertices v_0, v_1, …, v_{n-3} are same as the labelling process of v_0, v_1, …, v_{n-1} of C_{3k} in case 1. Then we label remaining two vertices v_{n-2} and v_{n-1} as $f(v_{n-2})=1$ and $f(v_{n-1})=3$.

Thus, from three cases, it follows that $\lambda(C_n)=4$.

Let us consider the sun S_{2n} of $2n$ vertices. This graph is obtained by adding an edge to each vertex of a cycle C_n. So C_n is a subgraph of S_{2n}. Now the question is what is the value of $\lambda(S_{2n})$? Griggs and Yeh [12] prove the following result for the graph similar to S_{2n}.

Lemma 7 [12] If a graph G contains three vertices with maximum degree $\Delta \geq 2$ and one of them is adjacent to the other two vertices, then $\lambda(G)$ is at least $\Delta+2$.

For S_{2n}, $\Delta=3$ hence by Lemma 7, $\lambda(S_{2n}) \geq 5$. Here we shall show that $\lambda(S_{2n})=5$.

Lemma 8 For any sun S_{2n}, $\lambda(S_{2n})=5=\Delta+2$.

Proof. Let S_{2n} be constructed from C_n by adding an edge to each vertex. To label this graph we consider the three cases.

Let $v_0, v_1, v_2, \ldots, v_{n-1}$ be the vertices of C_n and v_i is adjacent to v_{i+1} and v_{n-1} is adjacent to v_0. To compute S_{2n}, we add an edge (v_i, v_i') to the vertex v_i, i.e., v_i' are the pendent vertices. We first label C_n by using Lemma 6. Then we label the pendent vertices as follows.

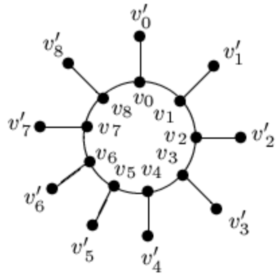

Figure 5: Sun S_{18}

Case 1. Let $n = 3k \equiv 0\ (\mathrm{m}od\ 3)$.

The labelling of v_i' is assign as $f(v_i') = \begin{cases} 3, & if\ i \equiv 0\ (\mathrm{m}od\ 3); \\ 5, & if\ i \equiv 1\ (\mathrm{m}od\ 3); \\ 1, & if\ i \equiv 2\ (\mathrm{m}od\ 3). \end{cases}$

Case 2. Let $n = 3k + 1 \equiv 1\ (\mathrm{m}od\ 3)$.

The labelling scheme of $v_0', v_1', \ldots, v_{n-5}'$ are same as the vertices v_i'; $i = 0,1,\ldots,n-1$ as in case 1. Now for the last four pendent vertices $v_{n-4}', v_{n-3}', v_{n-2}', v_{n-1}'$ the f are defined as $f(v_{n-4}') = 2$, $i = n-4, n-1$ and $f(v_i') = 5$, for $i = n-3, n-2$ respectively.

Case 3. Let $n = 3k + 2 \equiv 2\ (\mathrm{m}od\ 3)$.

Here we label the vertices $v_3', v_4', \ldots, v_{n-3}'$ as

$$f(v_i') = \begin{cases} 3, & if\ i \equiv 0\ (\mathrm{m}od\ 3); \\ 5, & if\ i \equiv 1\ (\mathrm{m}od\ 3); \\ 0, & if\ i \equiv 2\ (\mathrm{m}od\ 3). \end{cases}$$

The f values of the vertices v_0', v_1', v_2', v_{n-2}' and v_{n-1}' are $f(v_0') = 4$, $f(v_2') = 1$ and $f(v_i') = 5$, for $i = 1, n-2, n-1$ respectively.

Hence $\lambda(S_{2n}) = 5 = \Delta + 2$.

Corollary 1 A graph G contains a cycle C_n of length n. If two adjacent vertices v_0, v_{n-1} of C_n have two edges then for $n \equiv 0\ (\mathrm{m}od\ 3)$ and $n \equiv 1\ (\mathrm{m}od\ 3)$, $\lambda(G) = \Delta + 1$.

Lemma 9 Let G be a graph obtained from S_{2n} by adding an edge to each of the pendent vertex of S_{2n}, then $\lambda(S_{2n}) = \lambda(G) = \Delta + 2 = 5$.

Proof. According to the previous lemma let the graph be obtained by joining an edge (v_i', v_i'') to each of the pendent vertex v_i'. Then

v_i'''s are the pendent vertices of the new graph. Here we have to label only the pendent vertices. To label the vertices v_i'''s we consider the following three cases.

Case 1. For $n \equiv 0 \,(\mathrm{m}od\ 3)$.

$f(v''_{3i+2}) = 3$, for $i = 0,1,\ldots,\frac{n-3}{3}$ and $f(v_i'') = 1$, for other vertices.

Case 2. For $n \equiv 1 \,(\mathrm{m}od\ 3)$.

$f(v''_{3i+2}) = 3$, for $i = 0,1,\ldots,\frac{n-7}{3}$, $f(v''_{n-4}) = 4$, $f(v_i'') = 0$, for $i = n-3, n-2, n-1$ and $f(v_i'') = 1$, for other vertices.

Case 3. For $n \equiv 2 \,(\mathrm{m}od\ 3)$.

$f(v''_{3i+2}) = 3$, for $i = 0,1,\ldots,\frac{n-5}{3}$, $f(v_i'') = 0$, for $i = n-2$, $n-1$ and $f(v_i'') = 1$, for other vertices.

So, $\lambda(G) = 5 = \Delta + 2$.

Corollary 2 Let G be a graph obtained from C_n by adding P_i, $i = 1,2,3,\ldots$ to one or more vertices of C_n, then $\lambda(G) = \lambda(S_{2n}) = \Delta + 2 = 5$.

Lemma 10 Let $G = C_n \bigcup_{v_0} C_m$ then $\lambda(C_n \bigcup_{v_0} C_m) = 5 = \Delta + 1$, where Δ is the degree of the common vertex v_0.

Proof. Let C_n and C_m be two cycles of G. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n such that v_i is adjacent to v_{i+1}, $0 \le i \le n-2$ and v_0 is adjacent to v_{n-1}. The label $f(v_i)$ of $v_i \in C_n$ is assign as in Lemma 6.

Again let $v_0, v_1', \ldots, v_{m-1}'$ be the vertices of C_m such that v_0 is adjacent to v_1' and v_{m-1}'. Also, v_i' is adjacent to v_{i+1}', $0 \le i \le m-2$.

At the time of labelling of the vertices of C_n, we assign the label of v_0 to 0, i.e., $f(v_0)=0$.

Now, we assign the labels to the vertices of C_m by considering three cases, viz., $m=3k$, $m=3k+1$ and $m=3k+2$.

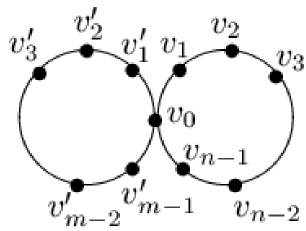

Figure 6: Illustration of Lemma 10.

Case 1. If $n \equiv 0$ (mod 3), $m \equiv 0\ (mod\ 3)$.

Here $f(v_0)=0$. Then the labelling of other vertices of C_m is as follows:

$$f(v'_1)=3,\ f(v'_2)=5 \text{ and } f(v'_i)=\begin{cases}0, & if\ i \equiv 0\ (mod\ 3);\\ 3, & if\ i \equiv 1\ (mod\ 3);\\ 5, & if\ i \equiv 2\ (mod\ 3).\end{cases}$$

If $n \equiv 0\ (mod\ 3)$, $m \equiv 1\ (mod\ 3)$.

Then the label of the vertices $v_0, v'_1, \ldots, v'_{m-5}$ are same as given in the above subcase. Now the label of the last four vertices $v'_{m-4}, v'_{m-3}, v'_{m-2}, v'_{m-1}$ are

$f(v'_{m-4})=0$, $f(v'_{m-3})=3$, $f(v'_{m-2})=1$, and $f(v'_{m-1})=5$ respectively.

When $m=4$ then,

$f(v'_1)=3$, $f(v'_2)=1$ and $f(v'_3)=5$.

If $n \equiv 0$ (mod 3), $m \equiv 2$ (mod 3).

We label the vertices v_1', v_2', v_3' and v_4' as

$f(v_1')=3$, $f(v_2')=5$, $f(v_3')=1$ and $f(v_4')=2$.

And other vertices as same process as in above subcase (for $m \equiv 0$ (mod 3)).

When $m=5$, then we label the graph as

$$f(v_i)=\begin{cases}0, & if\ i \equiv 0 (mod\ 3);\\ 3, & if\ i \equiv 1 (mod\ 3);\\ 5, & if\ i \equiv 2 (mod\ 3);\end{cases} \text{ and } f(v_i')=\begin{cases}2, & if\ i=1;\\ 5, & if\ i=2;\\ 1, & if\ i=3;\\ 4, & if\ i=4.\end{cases}$$

Case 2. *If* $n \equiv 1$ *(mod 3),* $m \equiv 1$ *(mod 3).*

The labelling scheme of the vertices of C_m are same as in case 1 (for $m \equiv 1$ (mod 3)).

When $n=4$ and $m=4$, then

$$f(v_i)=\begin{cases}0, & if\ i=0;\\ 3, & if\ i=1;\\ 5, & if\ i=2;\\ 2, & if\ i=3;\end{cases} \text{ and } f(v_i')=\begin{cases}4, & if\ i=1;\\ 1, & if\ i=2;\\ 5, & if\ i=3.\end{cases}$$

If $n \equiv 1$ *(mod 3),* $m \equiv 2$ *(mod 3).*

When $m>5$, the labels of C_m are same as in case 1.

When $m=5$, then label the vertices of C_m as same as in case 1. Now we label the vertices of C_n as same as the labelling of C_m in case 1 (for $m \equiv 1$ (mod 3)), except the labelling of the vertices v_{n-2} and v_{n-1}. We label these vertices as

$f(v_{n-2})=1$ and $f(v_{n-1})=5$ respectively.

Case 3. *If* $n \equiv 2$ *(mod 3),* $m \equiv 0$ *(mod 3).*

For the vertices $v'_4, v'_5, \ldots, v'_{m-3}$ the labelling procedure is

$$f(v'_i) = \begin{cases} 0, & if\ i \equiv 1(\mathrm{m}od\ 3); \\ 3, & if\ i \equiv 2(\mathrm{m}od\ 3); \\ 5, & if\ i \equiv 0(\mathrm{m}od\ 3); \end{cases}$$

Then we label other vertices as

$f(v'_1) = 4$, $f(v'_2) = 1$, $f(v'_3) = 5$, $f(v_{m-2}) = 1$ and $f(v_{m-1}) = 5$.

When $n = 5$ and $m = 3$, then we label the vertices as

$f(v_0) = 0$, $f(v_1) = 2$, $f(v_2) = 4$, $f(v_3) = 1$, $f(v_4) = 5$,

and $f(v'_1) = 3$, $f(v'_2) = 5$, $f(v'_3) = 1$, $f(v'_4) = 4$.

Therefore, $\lambda(G) = \lambda(C_n \bigcup_{v_0} C_m) = 5 = \Delta + 1$.

Some times a cycle C_3 of length 3 is called a triangle. A triangle may be a subgraph of a cactus graph. Also, a triangle shape star, (i.e., all the triangles have a common cutvertex) be a subgraph of a cactus graph. Let $T_0, T_1, \ldots, T_{n-1}$ be the n triangles meet at a common cutvertex v_0 and we denote this graph by G, which is equivalent to $\bigcup_{v_0} T_i$. The number of vertices and edges of G are $2n+1$ and $3n$ respectively.

Lemma 11 Let a graph G contains finite number of cycles of finite lengths with a common cutvertex. Then $\lambda(G) = \Delta + 2$, when G contains odd number triangles and $\Delta + 1$, for other cases, where Δ is the degree of the cutvertex.

Proof. First we prove that if the graph contains n triangles then $\lambda(G) = \Delta + 1\ or\ \Delta + 2$ according as n is even or odd. Let us denote the n such triangles by $T_0, T_1, T_2, \ldots, T_{n-1}$ (shown in Figure 6). Let v_0 be the common cutvertex. If Δ be its degree, then $\Delta = 2n$. Let v_0, v_{i1} and v_{i2} be the vertices of T_i. We label v_0 by 0.

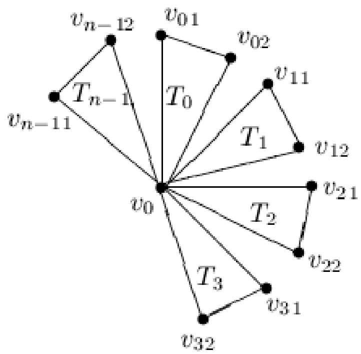

Figure 7: A graph contains n triangles

Then the labels of T_i's are as follows:

If i is even then, $f(v_{ij}) = \begin{cases} 2i+2, & for\ j=1; \\ 2i+4, & for\ j=2; \end{cases}$

and if i is odd then, $f(v_{ij}) = \begin{cases} 2i+1, & for\ j=1; \\ 2i+3, & for\ j=2. \end{cases}$

If n is even then let $n=2k$ for some k. The label of the vertex $v_{2k-1,2}$ of the cycle T_{2k-1} is $f(v_{2k-1,2}) = 4k+1 = 2n+1 = \Delta+1$.

And if n is odd then $n=2k+1$. Then the labels of the vertex $v_{2k,2}$ of the cycle T_{2k} is $f(v_{2k,2}) = 4k+4 = 2n+2 = \Delta+2$.

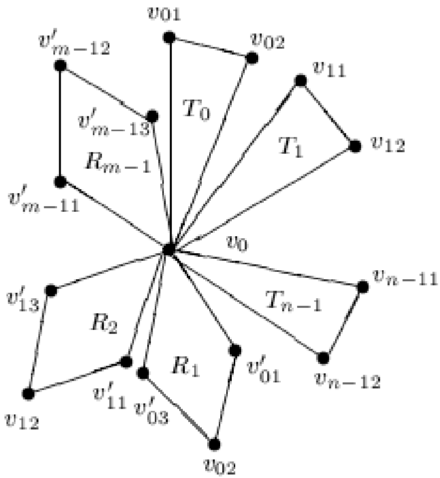

Figure 8: The graph contains n triangles and m cycles of lengths 4

Now if we prove for a graph which contains n number of cycles of length 3 and m number of cycles of length 4, $\lambda(G)=\Delta+1$, then the result is true for general case. Let the graph contains T_0, T_1, …, T_{n-1} be the n number of cycles of length 3 and R_0, R_1, …, R_{m-1} be the m number of cycles of length 4 (shown in Figure 7). All cycles are joined with v_0. Then $\Delta = 2n+2m$. Again let v_0, v_{ij}; $j=1,2$ be the vertices of T_i's and v_0, v'_{jk}; $k=1,2,3$ be the vertices of R_j's. Using the above rule we label the vertices of all T_i's. Here $f(v_0)=0$. Then we label vertices of R_j's are as follows.

If n is even the labelling of the adjacent vertices of v_0 of R_j's are as

$$\text{for } j=0,1,\dots,m-1,\ f(v'_{jk})=\begin{cases}2n+2j+2, & \text{for } k=1;\\ 2n+2j+3, & \text{for } k=3.\end{cases}$$

If n is odd then we label the adjacent vertices of v_0 of R_j's as for $j = 1,2,\ldots,m-1$, $f(v'_{jk}) = \begin{cases} 2n+2j+2, & for\, k=1; \\ 2n+2j+3, & for\, k=3; \end{cases}$ and

$$f(v'_{0k}) = \begin{cases} 2n+1, & for\, k=1; \\ 2n+3, & for\, k=3. \end{cases}$$

Other vertices of R_j's as $f(v'_{j2}) = 1$, for $j = 0,1,\ldots,m-1$.

So we get $f(v'_{m-1,3}) = 2n+2(m-1)+3 = 2n+2m+1 = \Delta+1$.

Hence the lemma is proved.

Corollary 3 If a graph G contains finite number of cycles of any length (except odd number of cycles of length three) with a common cutvertex, then $\lambda(G) = \Delta+1$, where Δ be the degree of the cutvertex.

Lemma 12 Let G be a graph which contains finite number of cycles of any length and finite number of edges. If v_0 be the common cutvertex with degree Δ then $\lambda(G) = \Delta+1$.

Proof. If we prove that for a graph G contains n number of cycles of length 3, m number of cycles of length 4 and p number of edges, $\lambda(G) = \Delta+1$, then generally we can say that the above statement is true.

According to the previous lemma let v'_k; $k = 0,1,\ldots,p-1$ be the end vertices of the edges joined with the common cutvertex v_0. Here $\Delta = 2n+2m+p$. Now we label the end vertices by $L(2,1)$-labelling as follows.

$f(v'_k) = 2n+2m+2+k$, for $k = 0,1,\ldots,p-1$.

Here $f(v'_{p-1}) = 2n+2m+2+p-1 = 2n+2m+p = \Delta+1$.

So, $\lambda(G) = \Delta+1$.

Corollary 4 When the end vertices of the edges of the graph G have another edges and each vertices of the cycles have another edges, then the value of $\lambda(G)$ remains unchanged.

Lemma 13 Let G be a graph, contains a cycle of any length and finite number of edges, they have a common cutvertex v_0. If Δ be the degree of the cutvertex then, $\lambda(G)=\Delta+1$.

Proof. Let G be a graph which contains a cycle C_n of length n and p number of edges. Again let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n and $v'_0, v'_1; v'_2, \ldots; v'_{p-1}$ are the end vertices of all edges. Here v_0 is the common cutvertex with degree $\Delta(=2+p)$. Then we label the graph as follows.

Case 1. For $n \equiv 0$ (mod 3) and $n \equiv 1$ (mod 3).

We first label the vertex v_0 by 0. That is, $f(v_0)=0$. Then we label the vertex v'_0 by 3, i.e., $f(v'_0)=3$. And then we label the end vertices of edges as

$f(v'_i)=4+i$, $i=1;2,\ldots;p-1$.

Now $f(v'_{p-1})=4+(p-1)=3+p=\Delta+1$.

Case 2. For $n \equiv 3$ (mod 3).

Here we label the first end vertex of the edge v'_0 by 4, i.e., $f(v'_0)=4$. Then we label the remaining end vertices of the edges as same as given in the above case.

From two cases we see that, $\lambda(G)=\Delta+1$.

Hence the result.

Lemma 14 Let G be a graph contains a cycle of any length and each vertex of the cycle has another cycle of length three. If Δ is the degree of G then $\lambda(G)=\Delta+3$.

Proof. Let G be a graph which contains a cycle C_n of length n and each vertex of C_n connect with another cycle of length three. Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of C_n and $v_{01}, v_{02}; v_{11}, v_{12}; \ldots; v_{n-11}, v_{n-12}$ are the vertices of all C_3's. That is

$v_0, v_1, \ldots, v_{n-1}$ are all cut vertices of the graph G. First we label the vertices of C_n by using Lemma 6.

We label the graph G as follows.

Case 1. For $n \equiv 0\,(mod\ 3)$.

If $i \equiv 0$ (mod 3), then $f(v_{ij}) = \begin{cases} 3, & for\ j = 1; \\ 5, & for\ j = 2. \end{cases}$

If $i \equiv 1$ (mod 3), then $f(v_{ij}) = \begin{cases} 5, & for\ j = 1; \\ 7, & for\ j = 2. \end{cases}$

If $i \equiv 2$ (mod 3), then $f(v_{ij}) = \begin{cases} 1, & for\ j = 1; \\ 6, & for\ j = 2. \end{cases}$

Case 2. For $n \equiv 1\,(mod\ 3)$.

Here we label the vertices of different cycles of lengths 3 as same as in the above case except the vertices which are adjacent to v_{n-4}, v_{n-3}, v_{n-2} and v_{n-1}. Now we label those vertices as

$$f(v_{n-4,j}) = \begin{cases} 2, & for\ j = 1; \\ 5, & for\ j = 2; \end{cases} \qquad f(v_{n-3,j}) = \begin{cases} 5, & for\ j = 1; \\ 7, & for\ j = 2; \end{cases}$$

$$f(v_{n-2,j}) = \begin{cases} 5, & for\ j = 1; \\ 7, & for\ j = 2; \end{cases}$$

$$f(v_{n-1,j}) = \begin{cases} 2, & for\ j = 1; \\ 6, & for\ j = 2. \end{cases}$$

Case 3. For $n \equiv 2 \ (mod\ 3)$.

Now we label the vertices $v_{11}, v_{12}; v_{21}, v_{22}; \ldots; v_{n-41}, v_{n-42}$ as per rule given in case 1. The label of the vertices $v_{01},\ v_{02};\ v_{n-31},\ v_{n-32};\ v_{n-21},\ v_{n-22}; v_{n-11},\ v_{n-12}$ are as follows.

$$f(v_{0j}) = \begin{cases} 4, & for\ j = 1, \\ 6, & for\ j = 2, \end{cases} \qquad f(v_{n-3,j}) = \begin{cases} 0, & for\ j = 1, \\ 6, & for\ j = 2, \end{cases}$$

$$f(v_{n-2,j}) = \begin{cases} 5, & for\ j = 1, \\ 7, & for\ j = 2, \end{cases}$$

$$f(v_{n-1,j}) = \begin{cases} 5, & for\ j = 1, \\ 7, & for\ j = 2, \end{cases}$$

Thus, from all the above cases, it follows that $\lambda(G) = \Delta + 3$.
W

Corollary 5 Let G be a graph contains a cycle of any length and each vertex of the cycle has two or many cycles of length three. If Δ be the degree of G then $\lambda(G) = \Delta + 3$.

Corollary 6 Let G be a graph contains a cycle of any length and each vertex of the cycle has another cycle of any length (> 3). If Δ be the degree of G then $\lambda(G) = \Delta + 2$.

Lemma 15 Let G_1 and G_2 be two cactus graphs. If $\Delta_1 + 1 \leq \lambda(G_1) \leq \Delta_1 + 3$ and $\Delta_2 + 1 \leq \lambda(G_2) \leq \Delta_2 + 3$, then, $\Delta + 1 \leq \lambda(G) \leq \Delta + 3$, where $G = G_1 \bigcup_v G_2$.

Proof. Let G_1 and G_2 be two cactus graphs and Δ_1, Δ_2 be the degrees of them. Again let u and v be two vertices of that graphs and x, y be the labels of u and v respectively. If we merge two cactus graphs G_1 and G_2 with the vertex v then we get a new cactus graphs G $(= G_1 \bigcup_v G_2)$. Let Δ be the degree of new cactus graph G, where $\max\{\Delta_1, \Delta_2\} \leq \Delta \leq \Delta_1 + \Delta_2$. For the graph G_1, $\Delta_1 + 1 \leq \lambda(G_1) \leq \Delta_1 + 3$ and G_2, $\Delta_2 + 1 \leq \lambda(G_2) \leq \Delta_2 + 3$. Now we have to prove that the lower and upper bounds of λ will preserve for the new cactus graph G.

Assume that the label of u be fixed and it be 0, i.e., $x = 0$, and the label y of v lies between 0 to $\Delta_2 + 3$. That is, the label difference between x and y will be $0, 1, \ldots, \Delta_2 + 3$.

Let the label of the vertices u and v be same, i.e., $x = y$. If we merge the two cactus graphs, then label of v remains unchanged and the labels of adjacent vertices of v will changed as per the rule of $L(2,1)$-labelling, i.e., the label difference between any two adjacent vertices is at least 2 and any two vertices which are at distance two is at least 1. If we increase the label numbers by 1 of all vertices of G_2 except v then there are at least one vertex in which we adjust the labelling to preserve the lower and upper bounds of λ.

When the label difference between x and y is 1, i.e., $y = x + 1$, then without loss of generality we assume that the label numbers of adjacent vertices of u are $x + 2$ and $x + 3$. And the label numbers of adjacent vertices of v are $x + 3$ and $x + 4$ respectively. If we merge the two cactus graphs then the labels of u and v will be same. Then we change the label number $x + 3$ to $x + 5$ to the graph G_2. If we increase the label numbers of all vertices of G_2 by 1 except v then we get at least one vertex in which we adjust the labelling to preserve the lower and upper bounds of λ, i.e., the λ value of new cactus graph can't be less than $\Delta + 1$ and greater than $\Delta + 3$.

Similarly, for the label differences $2, 3, \ldots, \Delta_2 + 3$, the lower and upper bounds of λ for the new cactus graph will preserve.

Hence the proof.

The $L(2,1)$-labelling of all subgraphs of cactus graphs and their combinations are discussed in the previous lemmas. From these results we conclude that the λ-value of any cactus graph can not be more than $\Delta+3$. Hence we have the following theorem.

Theorem 1 If Δ is the degree of a cactus graph G, then $\Delta+1 \le \lambda(G) \le \Delta+3$.

Proof. The $L(2,1)$-labelling of all possible subgraphs of cactus graph are discussed and have shown that $\Delta+1 \le \lambda \le \Delta+3$. Let G be obtained by v-union of two cactus graphs then G becomes a cactus graph and it is proved that $\lambda(G)$ should satisfy the inequality $\Delta+1 \le \lambda(G) \le \Delta+3$ (Lemma 15).

Hence the theorem.

5. The Algorithm and Time Complexity

5.1 Construction of an equivalent graph G' of G

Using DFS we obtain all blocks and cutvertices of a cactus graph $G=(V,E)$. Let the blocks be B_0, B_1, $B_3, \ldots,$ B_{N-1} and the cut vertices be C_0, C_1, $\ldots$, C_{R-1} where N is the total number of blocks and R is the total number of cut vertices.

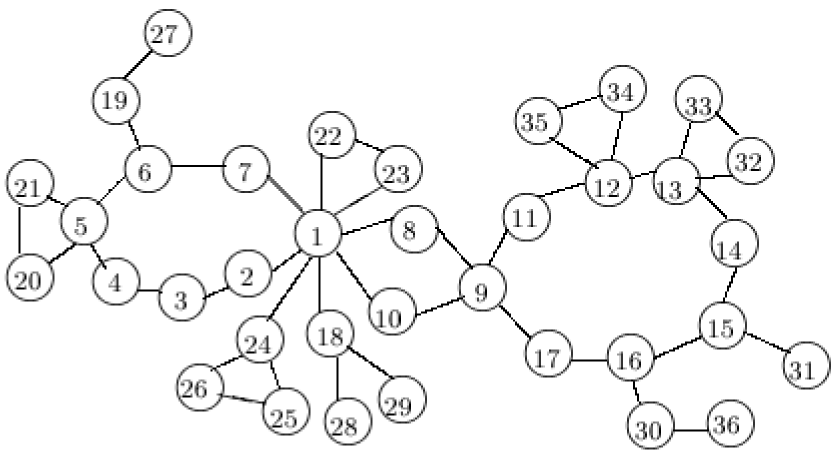

Figure 9: A cactus graph G.

The blocks of the cactus graph shown in Figure 9 are

$\{B_0 = (1,2,3,4,5,6,7),\ B_1 = (1,24),\ B_2 = (1,18),\ B_3 = (1,8,9,10),\ B_4 = (1,22,23),$
$B_5 = (6,19), B_6 = (5,20,21),\ B_7 = (24,25,26),\ B_8 = (18,28),\ B_9 = (18,29),$
$B_{10} = (9,11,12,13,14,15,16,17),\ B_{11} = (19,27),\ B_{12} = (16,30),\ B_{13} = (15,31),$
$B_{14} = (13,32,33),\ B_{15} = (12,34,35),\ B_{16} = (30,36)\}$

and the cutvertices are $C = \{1,5,9,12,13,15,16,18,19,24,30\}$.

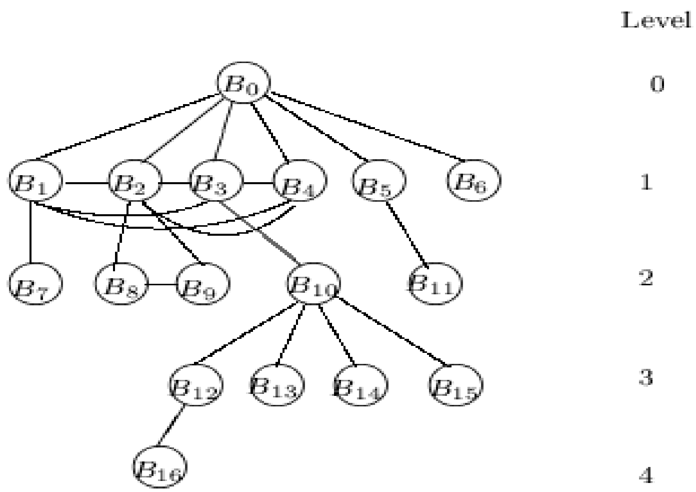

Figure 10: The equivalent graph G' of G.

Now we have in a position to construct an *equivalent graph* G' of G whose vertices are the blocks of G and an edge is defined between two blocks if they are adjacent blocks of G.

i.e., $G' = (V', E')$ where $V' = \{B_0, B_1, \ldots, B_{N-1}\}$

and $E' = \{(B_i, B_j) : i \neq j, i, j = 0,1,\ldots,N-1,\ B_i$ and B_j are adjacent blocks $\}$.

The graph G' for the graph G of Figure 8 is shown in Figure 9.

5.2 Labelling of vertices

To start the algorithm of $L(2,1)$-labelling of a cactus graph, we first construct a graph G' which is equivalent to the given graph G. Next we take any arbitrary finite length cycle of any length of G as starting block. The block is so chosen that the degree of the cutvertex is maximum. We denote the start block as B_0 at level 0. Now we label the block B_0 using the Lemma 6.

Then we label the blocks of level 1 from left to right. The blocks of level 1 say $B_{11}, B_{12}, B_{13}, \ldots,$ are either edges or triangles or cycles of finite lengths. We consider the first block B_{11} of level 1 which is adjacent to B_0. If the block is an edge then label the block by using Lemma 8. If it is cycle then we label it by using Lemma 10. The blocks which are adjacent to B_0 only at level 1 we label them according to the rule of the block B_{11}. Next we consider the second block B_{12}. If it is not adjacent to B_{11}, then we label the vertices of B_{12} using Lemma 8 or 10. If it is adjacent to B_{11}, then we label the vertices of B_{12} by using the Lemmas 11, 12, 13 and 15. Now we consider the block B_{1i}. If it is not adjacent with any block of level 1, then we label it by Lemmas 8 and 10. But, if it is adjacent at least one block of level 1 then we follow the rules of Lemmas 11, 12, 13 and 15.

Now we label the the vertices of the blocks of level 2 then level 3 and so on as per the procedure mentioned above.

Suppose a block of level l, say B_{lj} is an edge and it is adjacent to a block say $B_{l+1,k}$ of level $l+1$ which is also an edge. Then we label the block by using the Lemma 5.

Suppose a block, say $B_{k-1,i}$ of level $k-1$ is a triangle, its adjacent block at level k, say B_{kj}, is also a cycle of finite length. If each vertices of the block B_{kj} contains triangles, one of them say $B_{k+1,p}$ of level $k+1$, then we label the block by using the Lemma 14.

Algorithm MINLV

Input: The cactus graph $G = (V, E)$.

Output: Label of its vertices.

Step 1: Compute the blocks and cut vertices of G and construct an equivalent graph G' of G.

Step 2: Take any arbitrary cycle of any length, say B_0 as starting block, where the degree of the cut vertex of B_0 is maximum.

Step 3: We label the block B_0 using Lemma 6.

Step 4: Consider the blocks B_{1j}, $j = 1, 2, 3, \ldots$, of level 1. Label the blocks from left to right as follows.

(i) Take the first block B_{11} which is adjacent to B_0. If it is an edge then we label B_{11} by using Lemma 8 and if it is cycle then we label it by using Lemma 10.

(ii) Next we consider the second block B_{12}. If it is not adjacent to B_{11}, then label it by using Lemma 8 or 10. If B_{12} is adjacent to B_{11}, then label it by using Lemmas 11, 12, 13 and 15.

(iii) Consider the block B_{1i}. If it is not adjacent to any block of level 1, then label it by Lemma 8 or 10. But if it is adjacent to at least one block of level 1 then follow the rules of Lemmas 11, 12, 13 and 15.

(iv) The blocks which are adjacent to B_0 only then label them by the process similar to B_{11}.

Step 5: Consider the blocks of subsequent and so on. We label them as per the procedure of step 4, step 6 and step 7.

Step 6: Suppose a block of level l, say B_{lj} is an edge and another block $B_{l+1,k}$ of level $l+1$ adjacent to B_{lj}, which is also an edge. Then we label them by using Lemma 5.

Step 7: If a block $B_{k-1,i}$ at level $k-1$ is a triangle, let its adjacent block at level k be B_{kj} and assume that it is also a cycle of finite length. If each vertex of B_{kj} connected with a cycle triangle, one of them say $B_{k+1,p}$ at level $k+1$, then label the block by using Lemma 14.

end MINLV

5.3 Time complexity

The correctness of the algorithm follows from the lemmas proved in the paper.

Theorem 2 The time complexity of the algorithm MINLV is $O(n)$.

Proof. The blocks and cutvertices of any graph can be computed in $O(m+n)$ time [19]. For the cactus graph $m = O(n)$. Hence step 1 of algorithm MINLV takes $O(n)$ time.

The time complexity to label the vertices of a block of size m_1 is $O(m_1)$. Step 4 labels the vertices of the blocks which are at level 1 of G'. If the number of vertices of all blocks of this level is m_2, then the time complexity for step 4 is $O(m_2)$. That is the time complexity depends upon the number of vertices of the whole graph. Since the number of vertices of the entire graph is $O(n)$, the time complexity of the algorithm is $O(n)$.

Thus the $L(2,1)$-labelling of any cactus graph can be done using $O(n)$ time.

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