A Comparison on the Bounds of Chromatic Preserving Number and Dom-Chromatic Number of Cartesian Product and Kronecker Product of Paths

T N Janakiraman[*] and M Poobalaranjani[†]

Abstract

Let G be a simple graph with vertex set V and edge set E. A Set $S \subseteq V$ is said to be a chromatic preserving set or a cp-set if $\chi(<S>) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the chromatic preserving number or cp-number of G and is denoted by cpn(G). A cp-set of cardinality cpn(G) is called a cpn-set. A subset S of V is said to be a dom- chromatic set (or a dc-set) if S is a dominating set and $\chi(<S>) = \chi(G)$. The minimum cardinality of a dom-chromatic set in a graph G is called the dom-chromatic number (or dc- number) of G and is denoted by $\gamma_{ch}(G)$. The Kronecker product $G_1 \wedge G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph G with vertex set $V_1 \times V_2$ and any two distinct vertices (u_1, v_1) and (u_2, v_2) of G are adjacent if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$. The Cartesian product $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$ where any two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever (i) $u_1 = u_2$ and $v_1v_2 \in E_2$ or (ii) $u_1u_2 \in E_1$ and $v_1 = v_2$. These two products have no common edges. They are almost like complements but not exactly. In this paper, we discuss the behavior of the cp-number and dc-number and their bounds for product of paths in the two cases. A detailed comparative study is also done.

[*] Department of Mathematics, National Institute of Technology, Tiruchirapalli-620015, India, tnjraman2000@yahoo.com; janaki@nitt.edu

[†] Seethalakshmi Ramaswami College, Tiruchirapalli-620002, India; mpranjani@hotmail.com

1. Introduction

A *dominating set* S is a subset of vertices in a graph G = (V, E) such that every vertex in G either belongs to S or has a neighbor in S. The minimum cardinality of a dominating set of G is called the domination number of G, denoted by γ(G). A *proper coloring* of a graph G is a function from the vertices of G to a set of colors such that any two adjacent vertices have distinct colors. The chromatic number of a graph G is the minimum number of colors needed for a proper coloring of G and is denoted by χ(G). If χ(G) = k, then G is said to be *k-chromatic.* Both coloring and domination problems have many applications in the real life situations. Though the two problems are studied independently and extensively, not much research has been done on combining domination and coloring. We made an attempt in combining these two parameters. A dominating set S is said to be a conditional dominating set if the subgraph induced by S satisfies a given condition. Combining the concepts of domination and coloring, we define a new conditional dominating set in [1] and [2], termed as dom-chromatic set. A set S $\subseteq$ V(G) is said to be a *dom-chromatic set (or a dc-set)* of G, if S is a dominating set of G and χ(<S>) = χ(G). The minimum cardinality of a dom-chromatic set of G is called the *dom-chromatic number* (or *dc- number*) of G and is denoted by γ_{ch}(G).

In general, a dc-set can be extracted in two ways: (1) Find a minimum dominating set S. If χ(<S>) ≠ χ(G), add minimum number of vertices to S until the chromatic number of the subgraph induced by these vertices is same as the chromatic number of the graph (2) Find a smallest subset S of V such that χ(<S>) = χ(G). Add minimum number of vertices to S until S dominates G. Though a dc-set can be obtained for a given graph in these two ways, second method is more effective as it involves the concept of critical graphs. The concept of critical graphs was introduced by Dirac in [3]. He defined a graph G to be a critical graph if χ(<G>) < χ(G – v) for each vertex v of G. The complete graph K_2 is an example of a

2-critical graph and odd cycles are 3-critical graphs. Thus, to find a minimum dc-set of a given graph G, first it is necessary to find a smallest critical subgraph whose chromatic number is the same as that of G. This motivated us to define a new set called a chromatic preserving set or a cp-set [4]. A Set S of vertices is said to be a *chromatic preserving set* or a *cp-set* if $\chi(<S>) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the *chromatic preserving number* or *cp-number* of G and is denoted by cpn(G).

For the graph in figure 1, the shaded vertices form both a dc-set and a cp-set. It can be verified that the dc-number and cp-number of the graph is 3.
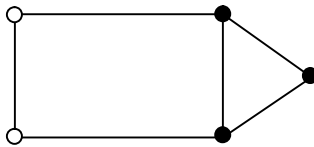

Figure 1

In this paper, we compare the cp-number, dc-number and their bounds for Cartesian product and Kronecker product of paths. Generally, graph products lead to varieties of new graphs. Imrich and Izbicki [5] have shown that there exist exactly twenty products of graphs defined on the Cartesian product of the vertex sets of the two factor graphs, where the adjacencies of the vertices in the product, depend only on the adjacencies in the two factor graphs. Weichsel [6] defined the *Kronecker product* $G_1 \wedge G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ as the graph with vertex set $V_1 \times V_2$ in which any two distinct vertices (u_1, v_1) and (u_2, v_2) of G are adjacent if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$. It has been shown that the Kronecker product is a good choice to construct lager networks and has received much research attention recently. The *Cartesian product* $G_1 \times G_2$ of two graphs G_1 and G_2 was first defined by Shapiro [7] as the graph with vertex set $V_1 \times V_2$ and any two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent only if either $u_1 = u_2$ and $v_1v_2 \in E_2$ or $u_1u_2 \in E_1$ and $v_1 = v_2$. Sabidussi [8] and Teh and Yap [9] also independently studied this product. Domination number of the Cartesian product of two graphs has received much attention, with a main objective to prove the well-known Vizing's conjecture. The

conjecture states that the domination number of the Cartesian product of two graphs is at least as large as the product of the respective domination numbers. It remains an open problem. The conjectured bound has been verified for many large classes of graphs. Vizing's conjecture has inspired many researchers to investigate the possibility of similar bounds for other domination parameters as well as other graph products.

From the definitions of the products, it is clear that any edge of a Kronecker product cannot be in Cartesian product and vice versa. i.e., these two products have no common edges. This leads to the fact that the two products have different structures. But in the case of paths, both products are bipartite. The graph $P_m \times P_n$ is also known as an m x n grid graph.

In this paper, we compare the cp-number and dc-number of $P_m \wedge P_n$ and $P_m \times P_n$ for m = 2, 3 and 4. The graphs $P_4 \times P_n$ and $P_4 \wedge P_n$ are shown in figure 2(a) and 2(b) respectively.
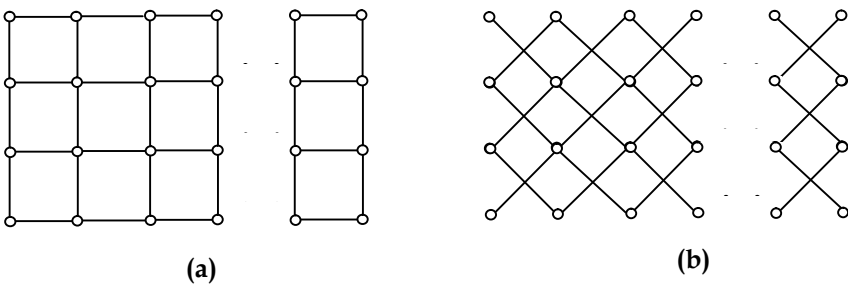

(a)   (b)

Figure 2

2. Prior Results

Conjecture 2.1 (Vizing's conjecture): For any two graphs G and H, $\gamma(G \times H) \geq \gamma(G) \gamma(H)$

Observation 2.2 [1]: For any bipartite graph G, cpn(G) = 2.

Proposition 2.3 [10]: For n ≥ 3,

$$\gamma(P_n) = \begin{cases} \dfrac{n}{3}, & \text{if } n \equiv 0 \ (\text{mod } 3) \\ \dfrac{(n+2)}{3}, & \text{if } n \equiv 1 \ (\text{mod } 3) \\ \dfrac{(n+1)}{3}, & \text{if } n \equiv 2 \ (\text{mod } 3) \end{cases}$$

Proposition 2.4 [1]: For $n \geq 3$,

$$\gamma_{ch}(P_n) = \begin{cases} \dfrac{(n+3)}{3}, & \text{if } n \equiv 0 \ (\text{mod } 3) \\ \dfrac{(n+2)}{3}, & \text{if } n \equiv 1 \ (\text{mod } 3) \\ \dfrac{(n+4)}{3}, & \text{if } n \equiv 2 \ (\text{mod } 3) \end{cases}$$

Theorem 2.5 [11]: If G_1 and G_2 are connected graphs having no odd cycle then $G_1 \wedge G_2$ has exactly two connected components.

Theorem 2.6 [12]: For $n \geq 2$,

$$\gamma(P_2 \times P_n) = \begin{cases} \dfrac{(n+1)}{2}, & \text{if } n \text{ is odd} \\ \dfrac{(n+2)}{2}, & \text{if } n \text{ is even} \end{cases}$$

3. CP-Number of Kronecker Product and Cartesian Product

Proposition 3.1: For $n, m \geq 2$,

  (i) $cpn(P_m \wedge P_n) = 2$

  (ii) $cpn(P_m \times P_n) = 2$

Proof: The proof follows from the fact that both $P_m \wedge P_n$ and $P_m \times P_n$ are bipartite.

4. Dom-Chromatic Number in Kronecker Product

Proposition 4.1: For $n \geq 3$,

$$\gamma_{ch}(P_2 \wedge P_n) = \begin{cases} \gamma(P_2 \times P_n) + 1 = \dfrac{2n}{3} + 1, & \text{if } n \equiv 0 \pmod 3 \\[2ex] \gamma(P_2 \times P_n) = \dfrac{2(n-1)}{3} + 2, & \text{if } n \equiv 1 \pmod 3 \\[2ex] \gamma(P_2 \times P_n) + 1 = \dfrac{2(n-2)}{3} + 3, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof: It is evident from figure 3 that $P_2 \wedge P_n = 2P_n$ and hence the result follows.
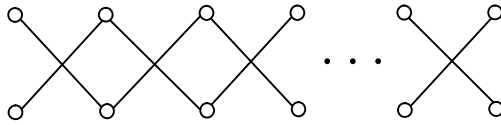
$P_2 \wedge P_n$:

Figure 3

Proposition 4.2: For $n \geq 3$, $\gamma_{ch}(P_3 \wedge P_n) = \gamma(P_3 \wedge P_n) = n$

Proof: We show the result for $n = 3, 4, 5, 6, 7$ and generalize the result for $n \geq 8$.

(a)   (b)   (c)

(d)   (e)
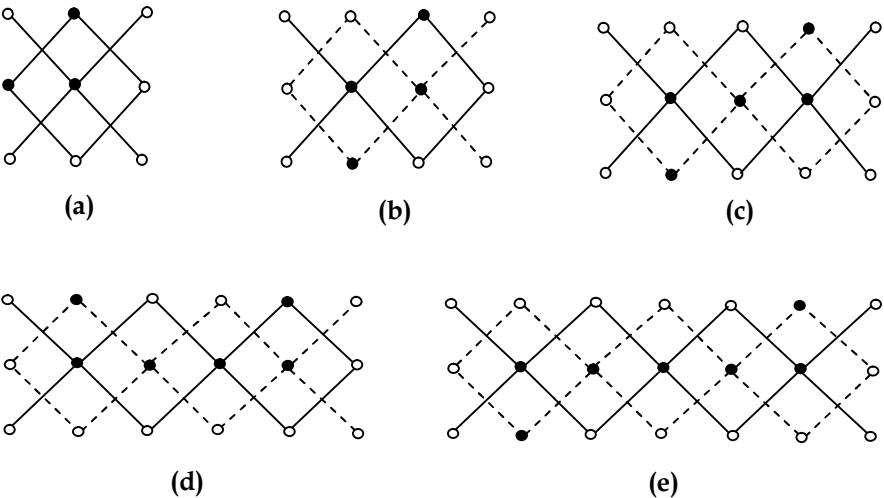
Figure 4

If n = 3, then $P_3 \wedge P_3$ is $K_{1,4} \cup C_4$ as shown in figure 4(a). Thus, it can be seen that $\gamma_{ch}(P_3 \wedge P_3) = \gamma(P_3 \wedge P_3) = 3$.

If n = 4, then the two components of $P_3 \wedge P_4$ are shown in figure 4(b). The copies are identical and disjoint. For each copy, the domination number as well as the dc-number is 2 and hence, $\gamma_{ch}(P_3 \wedge P_4) = 4$.

If n = 5, then the two components of $P_3 \wedge P_5$ are shown in figure 4(c). For one component the domination number is 2 and for the other component, the domination number and the dc-number are the same as 3. Hence $\gamma(P_3 \wedge P_5) = \gamma_{ch}(P_3 \wedge P_5) = 5$.

To see the pattern we take n = 6 and 7.

Let n = 6. The graph $P_3 \wedge P_6$ contains two identical components as shown in figure 3(d). For each component, the domination number and the dc-number is 3 and hence, $\gamma_{ch}(P_3 \wedge P_4) = \gamma_{ch}(P_3 \wedge P_6) = 6$.

Let n = 7. The graph $P_3 \wedge P_7$ contains two components as shown in figure 4(d). For one component, the domination number is 3 and for the other component, the domination number and the dc-number are the same as 4. Hence, $\gamma(P_3 \wedge P_7) = \gamma_{ch}(P_3 \wedge P_7) = 7$. Hence $\gamma_{ch}(P_3 \wedge P_7) = \gamma(P_3 \wedge P_7) = 7$.

In general, when n is even, $P_3 \wedge P_n$ has two identical components as given in figure 5.
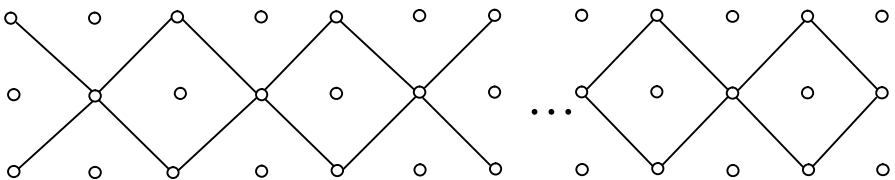

Figure 5

The domination number of the components can be obtained by taking the vertices in the middle row. (Refer figure 6). The number of vertices in the middle row equals $\left(\dfrac{n}{2}\right)$.
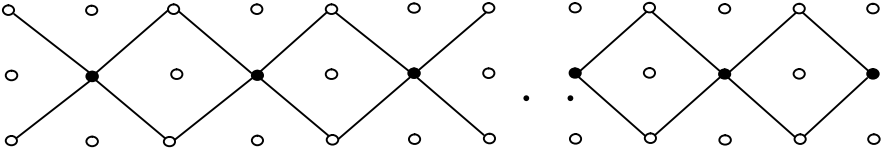
Figure 6

Now in one component, change the dominating vertex in the last column to the column before the last (Refer figure 7).
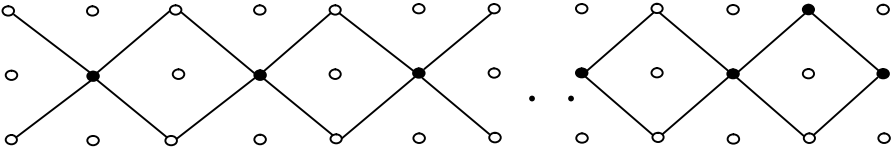

Figure 7

This rearrangement of the dominating vertex yields a dc-set of the component and of cardinality equals the domination number of the component. Taking the union of this set together with a minimum dominating set of the other component gives a minimum dc-set. Thus, $\gamma(P_3 \wedge P_n) = \gamma(P_3 \wedge P_n) = 2\left(\dfrac{n}{2}\right) = n$.

Suppose that n is odd. The two components are given in figure 8 and 9.
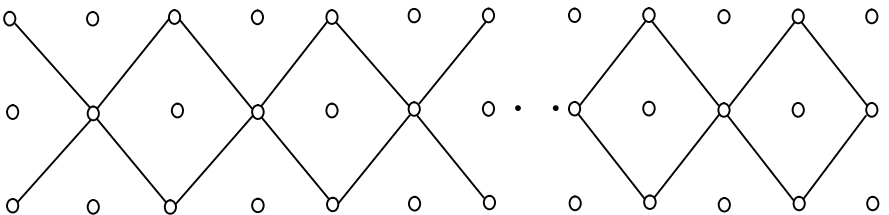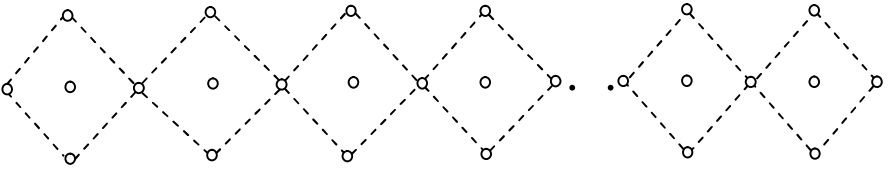

Figure 8

Figure 9

As before, the vertices in the middle row will form a γ-set. The domination number of the component in figure 8 is $\left(\dfrac{n-1}{2}\right)$ and for the component in figure 8 is $\left(\dfrac{n+1}{2}\right)$. In the γ-set, replace vertex in the last column of figure 9 by a vertex adjacent to it, then the γ-set becomes a γ$_{ch}$-set.

Thus, γ$_{ch}$(P$_3$ ∧ P$_n$) = n.

Proposition 4.3: For n ≥ 4,

$$\gamma_{ch}(P_4 \wedge P_n) = \begin{cases} \gamma(P_4 \wedge P_n) = n, & \text{if } n \equiv 0 \ (\text{mod } 4) \\ \gamma(P_4 \wedge P_n) = n+1, & \text{if } n \equiv 1, 3 \ (\text{mod } 4) \\ \gamma(P_4 \wedge P_n) = n+2, & \text{if } n \equiv 2 \ (\text{mod } 4) \end{cases}$$

Proof: The graph P$_4$ ∧ P$_4$ is given in figure 10(a). It has two identical components and one of its components is given in figure 10(b). The domination number as well as the dc-number of the component is 2.

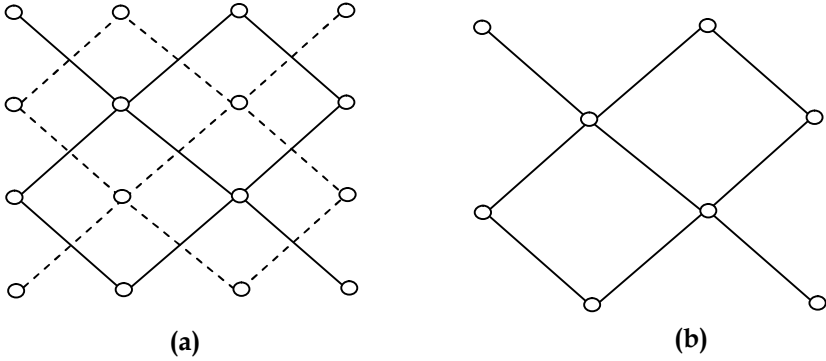(a)   (b)

Figure 10

In figure 11(a), (b) and (c), one of the components of $P_4 \wedge P_5$, $P_4 \wedge P_6$, $P_4 \wedge P_7$ are shown.   We partition the graphs into 4 x 4 blocks. The graph $P_4 \wedge P_5$ it is partitioned into a 4 x 4 and a 4 x 1 block, the graph $P_4 \wedge P_6$ is partitioned into 4 x 4 and a 4 x 2 block and the graph $P_4 \wedge P_7$ is partitioned into 4 x 4 and a 4 x 3 block.

Consider $P_4 \wedge P_5$. The first block is dominated by 2 adjacent vertices. The second block has two isolated vertices and it can be dominated by a vertex in the first block.

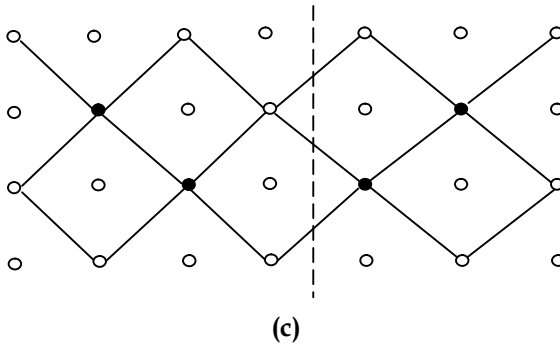Thus, $\gamma_{ch}(P_4 \wedge P_5) = 2 \times 3 = 6$.

(a)   (b

(c)
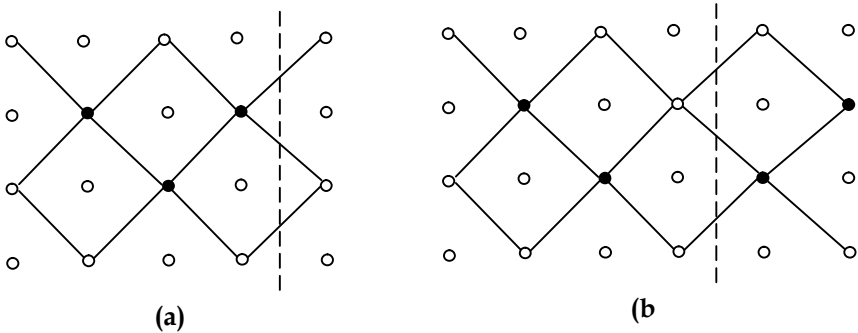
Figure 11

Consider $P_4 \wedge P_6$. The first block is dominated by 2 adjacent vertices. The second block has an induced path P_4 and is dominated by 2 vertices in the second block. Thus $\gamma_{ch}(P_4 \wedge P_6) = 2 \times 4 = 8$.

Consider $P_4 \wedge P_7$. The first block is dominated by 2 adjacent vertices and the second block is also dominated by 2 adjacent vertices. Thus $\gamma_{ch}(P_4 \wedge P_7) = 2 \times 4 = 8$.

Let B_1, B_2 and B_3 represent the blocks with 1 column, 2 columns and 3 columns respectively.

In general, when a component is partitioned in to 4 x 4 blocks, if $n \equiv 0 \pmod 4$, then using the fact that each block is dominated by 2 adjacent vertices, dc-number of the component = domination number $= 2\left(\dfrac{n}{4}\right)$.

Therefore, $\gamma_{ch}(P_4 \wedge P_n) = 2\left(2\left(\dfrac{n}{4}\right)\right) = n$.

Suppose that $n \equiv 1 \pmod 4$. Then the number of 4 x 4 blocks in each component $= \left(\dfrac{n-1}{4}\right)$. Each component has one B_1 block.

Hence, dc-number of each component = domination number = $2\left(\dfrac{n-1}{4}\right) + 1$.

Thus, $\gamma_{ch}(P_4 \wedge P_n) = 2\,(2\left(\dfrac{n-1}{4}\right) + 1) = n + 1$.

When $n \equiv 2 \pmod 4$, the component has a B_2 block and when $n \equiv 3 \pmod 4$, the component has a B_3 block, and in both cases the blocks are dominated by two vertices. Hence, if $n \equiv 2 \pmod 4$, then

dc-number of each component = domination number = $2\left(\dfrac{n-2}{4}\right) + 2$.

Thus, $\gamma_{ch}(P_4 \wedge P_n) = 2\,(2\left(\dfrac{n-2}{4}\right) + 2) = n + 2$.

If $n \equiv 3 \pmod 4$, then

dc-number of each component = domination number = $2\left(\dfrac{n-3}{4}\right) + 2$.

Thus, $\gamma_{ch}(P_4 \wedge P_n) = 2\,(2\left(\dfrac{n-3}{4}\right) + 2) = n + 1$.

5. Dom-Chromatic Number in Cartesian Product

For Cartesian product, the dominating set structure is based on the result given in [12].

Proposition 5.1: For $n \geq 2$,

$$\gamma_{ch}(P_2 \times P_n) = \begin{cases} \gamma(P_2 \times P_n) = \dfrac{n+2}{2}, & \text{if } n \text{ is even} \\[2mm] \gamma(P_2 \times P_n) + 1 = \dfrac{n+3}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Proof: The minimum dominating set is formed by choosing the vertices diagonally in the alternate columns and beginning from the first column. If n is odd, then we have one of the following minimum domination structures as given in figure 12 (a), when $n \equiv 3 \pmod 4$ and figure 12 (b), when $n \equiv 1 \pmod 4$.
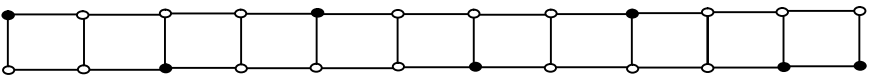
(a) (n = 11 ≡ 3 (mod 4))
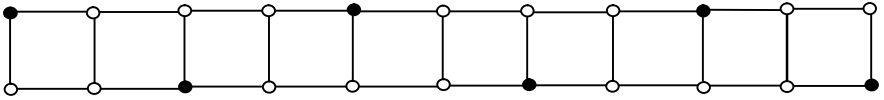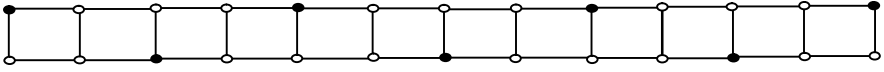
(b) (n = 13 ≡ 1 (mod 4))

Figure 12

A dc-set can be obtained by choosing any vertex from the remaining vertices. In other words,
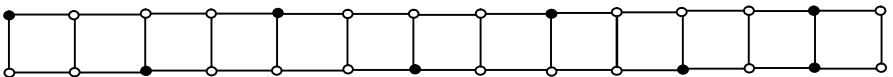
dc-number = domination number + 1.

Thus, $\gamma_{ch}(P_2 \times P_n) = \gamma(P_2 \times P_n) + 1$

$$= \left(\frac{n+1}{2}\right) + 1 = \left(\frac{n+3}{2}\right)$$

If n is even, then we have one of the following minimum domination structures given in figure 13 (a), when n ≡ 0 (mod 4) and figure 13 (b), when n ≡ 2 (mod 4).

(a) (n = 12 ≡ 0 (mod 4))

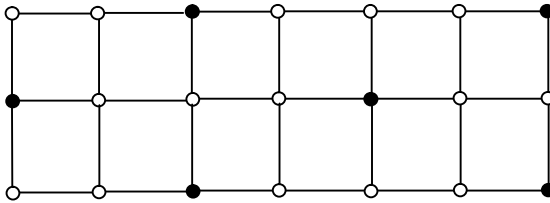**(b)** (n = 14 ≡ 2 (mod 4))

Figure 13

Clearly the γ-set show   ure 13 is a dc-set.

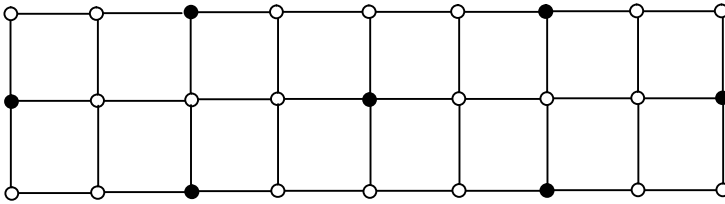Hence, $\gamma_{ch}(P_2 \times P_n) = \gamma(P_2 \times P_n) = \left(\frac{n+2}{2}\right)$.

Proposition 5.2: For n ≥ 3,

$$\gamma_{ch}(P_3 \times P_n)= \begin{cases} \gamma(P_3 \times P_n) + 1= \left\lceil \dfrac{(3n+4)}{4} \right\rceil + 1, \text{ if n is odd} \\[3mm] \gamma(P_3 \times P_n) = \left\lfloor \dfrac{(3n+4)}{4} \right\rfloor, \qquad \text{if n is even} \end{cases}$$

Proof: When n is odd, the domination structure is shown in figure 14 (a), when $n \equiv 3 \pmod 4$ figure 14 (b), when $n \equiv 1 \pmod 4$.

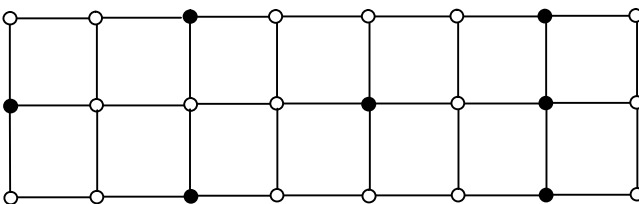**(a)** ($n = 7 \equiv 3 \pmod 4$)

(b) ($n = 9 \equiv 1 \pmod 4$)

Figure 14

By adding any vertex to the given dominating set we get a dc-set. Hence, if n is odd, $\gamma_{ch}(P_3 \times P_n) = \gamma(P_3 \times P_n) + 1 = \left\lceil \dfrac{3n+4}{4} \right\rceil + 1$.

If n is even, the domination structure is shown in figure 15 (a), when $n \equiv 0 \pmod 4$ and figure 15 (b), when $n \equiv 2 \pmod 4$.

(a) ($n = 8 \equiv 0 \pmod 4$)
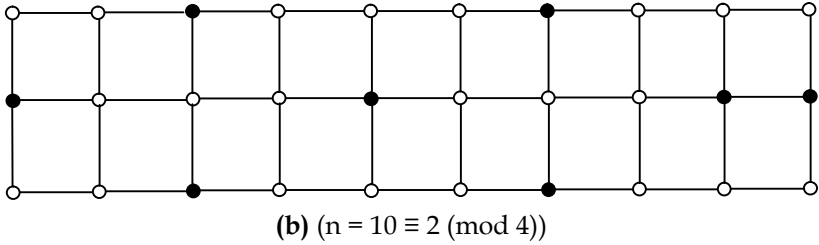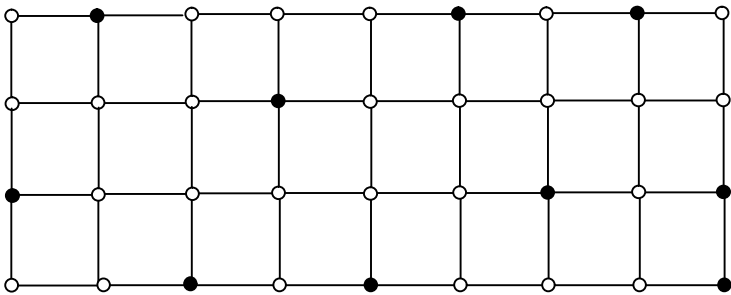
(b) (n = 10 ≡ 2 (mod 4))

Figure 15

The minimum dominating set itself is a dc-set. Hence, if n is even,

$$\gamma_{ch}(P_3 \times P_n) = \gamma(P_3 \times P_n) = \left\lfloor \frac{3n + 4}{4} \right\rfloor.$$

Proposition 5.3: For $n \geq 4$,

$$\gamma_{ch}(P_4 \times P_n) = \begin{cases} \gamma(P_4 \times P_n) = n + 1, & \text{if } n = 5, 6, 9 \\ \gamma(P_4 \times P_n) + 1 = n + 1, & \text{otherwise} \end{cases}$$

Proof: The proof is trivial. The domination structure is given in figure 16.

6. Comparison of DC-N Figure 16 **Cartesian Product and Kronecker Product**

For any bipartite graph, the dc-number is either the domination number or greater than the domination number by one. Since both the products are bipartite graph, this is applicable to these products

also. Though their cp-numbers are the same, the products cannot be totally classified as graphs whose domination number and the dc-number are the same or the dc-number is greater than the domination number by 1.

References

[1] T N Janakiraman and M Poobalaranjani, Dom-Chromatic sets in bipartite graphs. *Int. J. Engg. Sci. Advanced Computing and Bio-technology,* vol. 1**,** pp. 80-95, 2010.

[2] T N Janakiraman and M Poobalaranjani, Dom-Chromatic sets of graphs. *Int. J. Engg. Sci. Advanced Computing and Bio-technology,* vol. 2**,** pp. 88-103, 2011.

[3] G A Dirac, A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.,* vol. 27**,** pp. 85-92, 1952.

[4] T N Janakiraman and M Poobalaranjani, On the chromatic preserving sets. *Int. J. Engg. Sci. Advanced Computing and Bio-technology,* vol. 1**,** pp. 29- 42, 2010.

[5] W Imrich and H Izbicki, Associative products of graphs, *Monatsh. Math.,* vol. 80**,** pp. 277–281, 1975.

[6] P M Weishsel, The Kronecker product of graphs. *Proc. Amer. Math. Soc.,* vol. 13**,** pp. 470-52, 1962.

[7] S H Shapiro, The embedding of graphs in cubes and design of sequential relay circuits. *Bell. Telephone Lab. Memorandum* (unpublished report), 1953.

[8] G Sabidussi. Graph multiplication. *Math. Z.,* vol. 72**,** pp. 446-457, 1960.

[9] H H Teh and H P Yap, Some construction problems of homogeneous graphs. *Bull. Math. Soc., Nanyang Univ.,* pp. 164-196, 1964.

[10] T W Haynes, S T Hedetniemi and P J Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[11] T Sitthiwirattham, Independednt and vertex covering number of Kronecker product of $K_{m, n}$, *Applied Mathematical Sciences*, vol. 6, pp. 1403-1408, 2012.

[12] T Y Chang, W E Clark and E O Hare, Domination numbers of complete grid graphs-I, Dec. 19, 1995.