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Eigenstates of a Charged Simple Harmonic Oscillator in a Uniform Magnetic Field

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Abstract

The exact eigenstates of a simple harmonic oscillator and those of a charged particle in a uniform magnetic field are well known and frequently used. In this short article, we work out the exact eigenstates of a charged harmonic oscillator placed in a uniform magnetic field using a novel gauge choice for the vector potential.

Keywords: Simple Harmonic Oscillator, Uniform Magnetic field, Landau levels, Landau gauge, Energy eigenvalues and eigenfunctions

1. Introduction

The harmonic oscillator potential is of great physical interest as it finds a wide range of applications in a variety of problems in classical and quantum mechanics, such as vibrations of atoms and molecules in a crystal lattice etc. The harmonic oscillator potential is used in describing nuclear forces and nuclear structures, quark model of hadrons etc. In the theory of electromagnetic fields, the quanta of radiation are considered as excitations of the collection of an independent oscillator. The harmonic oscillator problem is considered one of the important problems in quantum theory as it can be solved exactly. The eigenstates of a free charged particle subjected to a uniform magnetic field have been obtained and used

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to explain several effects in magnetic materials. However, we can consider electrons in certain types of media as weakly bound harmonic oscillators. Hence, it is worthwhile working out the exact eigenstates of charged harmonic oscillators subjected to uniform magnetic fields. A similar experiment is conducted in this short article using a proper choice of the magnetic vector potential. We may find specific applications of our results in some modern dimensional solids etc.

The Schrodinger Equation

The Hamiltonian of a Simple Harmonic Oscillator may be written as

$$\boldsymbol{H = \frac{1}{2m} p_y^2 + \frac{1}{2} K\, y^2} \tag{1}$$

Let the uniform magnetic field be along $\boldsymbol{z}$ direction. Hence,

$$\vec{B} = B_0 \hat{k} \tag{2}$$

This field can be produced by the vector potential

$$\vec{A} = \frac{1}{2} \overrightarrow{B_0} \times \vec{r} \tag{3}$$

We can choose $\vec{\boldsymbol{A}} = -\boldsymbol{B_0 y \hat{i}}$ to generate the chosen magnetic field. It is a particular choice of gauge (called the Landau gauge). This choice simplifies the calculations.

In the presence of this potential, the minimum coupling rule leads to the Hamiltonian

$$H = \frac{1}{2m} p_y^2 + \frac{1}{2m} (p_x + qB_0 y)^2 + \frac{1}{2} m\omega_0^2 y^2 \tag{4}$$

The Schrodinger equation for the eigenfunction $\boldsymbol{\psi(x, y)}$ and the energy eigenvalue $\boldsymbol{E}$ can be written as

$$\left\{ -\frac{1}{2m} \left(\hbar^2 \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial x} + qB_0 y \right)^2 + \frac{1}{2} m\omega_0^2 y^2 \right\} \psi(x, y) = E\psi(x, y) \tag{5}$$

Where $\boldsymbol{K = m\omega_0^2}$ and $\boldsymbol{q}$ is the charge on the oscillator.

Since equation (5) is independent of $\boldsymbol{x}$, we can put $\boldsymbol{\psi(x,y) = X(x)Y(y)}$ for the wave function. In fact, we can take $\boldsymbol{X(x) = e^{-i\alpha x}}/\hbar$. Where $\boldsymbol{\alpha}$ is a constant that will be equal to the $\boldsymbol{x}$ component of the momentum. Removing the $\boldsymbol{x}$ dependence from equation (5) by the separation of variables method, we easily get the following equation for y-dependent part of the wave function:

$$\left\{-\frac{1}{2m}\frac{d^2}{dy^2} + \frac{1}{2m}(\alpha + qB_0 y)^2 + \frac{1}{2}m\omega_0^2 y^2\right\}Y(y) = EY(y) \tag{6}$$

Rearranging the terms, we get

$$\left\{-\frac{1}{2m}\frac{d^2}{dy^2} + \left(\frac{1}{2}m\omega_0^2 y^2 + \frac{q^2B_0^2}{2m}y^2\right) + \frac{\alpha q B_0}{m}y\right\}Y(y) = \left(E - \frac{\alpha^2}{2m}\right)Y(y) \tag{7}$$

This can be further simplified to

$$\left\{-\frac{1}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\left(\omega_0^2 + \frac{q^2B_0^2}{m^2}\right)y^2 + \frac{\alpha q B_0}{m}y\right\}Y(y) = \left(E - \frac{\alpha^2}{2m}\right)Y(y) \tag{8}$$

But, we can write

$$\frac{1}{2}m\left(\omega_0^2 + \frac{q^2B_0^2}{m^2}\right)y^2 + \frac{\alpha q B_0}{m}y = \frac{1}{2}m\left(\omega_0^2 + \frac{q^2B_0^2}{m^2}\right)\left[y^2 + \frac{2\alpha q B_0 y}{m^2\left(\omega_0^2 + \frac{q^2B_0^2}{m^2}\right)}\right]$$
$$= \frac{1}{2}m\omega^2\{[y+\beta]^2 - \beta^2\}$$

where $\beta = \dfrac{\alpha q B_0}{m^2\omega^2}$ and $\boldsymbol{\omega^2 = \omega_0^2 + \dfrac{q^2B_0^2}{m^2}}$

Then, we can rewrite equation (8) in the familiar form:

$$-\frac{1}{2m}\frac{d^2}{dy^2}Y(\bar{y}) + \frac{1}{2}m\omega^2 Y(\bar{y}) - \frac{\alpha^2q^2B_0^2}{2m^3\omega^2}Y(\bar{y}) = \left(E - \frac{\alpha^2}{2m}\right)Y(\bar{y}) \tag{9}$$

Where we put $\boldsymbol{\bar{y}} = y + \beta$.

This can be seen as a simple harmonic oscillator equation with frequency $\boldsymbol{\omega}$ about $\boldsymbol{y = -\beta}$ with modified energy.

Hence, taking over the well-known results for the simple harmonic oscillator, we can deduce immediately that the eigenvalues for the present system as:

$$E(\alpha, n) = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{\alpha^2 q^2 B_0^2}{2m^3\omega^2} + \frac{\alpha^2}{2m} \tag{10}$$

These are the quantized energy levels with frequency $\boldsymbol{\omega = \sqrt{\omega_0^2 + \frac{q^2 B_0^2}{m^2}}}$ and $\boldsymbol{Y(\bar{y})}$ will be simple harmonic oscillator wave functions around $\boldsymbol{y = -\beta}$. The wave functions are given by

$$\boldsymbol{Y_n(\bar{y}) = N_n e^{-\frac{1}{2}\left(\frac{m\omega}{\hbar}\right)(\bar{y})^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,\bar{y}\right)}$$

where $\boldsymbol{\bar{y} = y + \beta}$ and $\boldsymbol{N_n = \left(\frac{\sqrt{m\omega/\hbar}}{\sqrt{\pi}\,2^n n!}\right)^{\frac{1}{2}}}$

$\boldsymbol{H_n\left(\sqrt{m\omega/\hbar}\;\bar{y}\right)}$ are the Hermite polynomials

Conclusions

Here we have obtained the exact eigenstates of a charged harmonic oscillator placed in a uniform magnetic field. The solutions reduce to harmonic oscillator values if $\boldsymbol{B_0 = 0}$ and the well-known Landau levels if $\boldsymbol{\omega_0 = 0}$ as they should. $\boldsymbol{\alpha}$ plays the role of constant momentum in the $\boldsymbol{x}$ direction. When there is momentum in the $\boldsymbol{x}$ direction, we simply get a simple harmonic oscillator with a modified frequency $\boldsymbol{\omega}$.

This short article shows that a simple harmonic oscillator in a constant magnetic field behaves like another oscillator with a modified frequency. This exact result may find application in modern two-dimensional solid systems.

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