Mapana-Journal of Sciences
2021, Vol. 20, No. 3, 43-48
ISSN 0975-3303 | https://doi.org/10.12723/mjs.58.5

Quantum States of a Relativistic Charged Particle in Crossed Electric and Magnetic Fields

Basavaraj Kagali* and Tavarekere Shivalingaswamy†

Abstract

The exact eigenvalues and eigenfunctions of a spinless relativistic charged particle are obtained here. The results are compared with those from classical mechanics.

Keywords: Eigenenergies, Eigenfunctions, Boundstates, Crossed Electric and Magnetic Fields, Landau Levels, Klein Gordon equation, Landau gauge

1. Introduction

The motion of relativistic charged particles in mutually perpendicular uniform electric and magnetic fields is well-known. Two distinct cases are recognised. In the case of a strong electric field $(E > BC)$. The motion of a charged particle will be unbounded, essentially accelerated motion along the electric field direction. On the other hand, for strong magnetic fields $(BC > E)$, the motion turns out be a drifting motion along a direction perpendicular to both electric and magnetic fields with a looping motion around the magnetic field direction, hence semi bounded. In this article we discuss the motion of the charged particle quantum mechanically-arriving at semi-quantised states.

* Department of Physics, Bangalore University, Bengaluru, Karnataka, India; bakagali@gmail.com
† Department of Physics, Government College (Autonomous), Mandya, Karnataka, India; tssphy1@gmail.com

2. The Relativistic Wave Equation and its Solution

Let us assume that the uniform electric field E_0 is along the y axis. Then the corresponding electrostatic potential may be expressed as

$$\Phi = -E_0 y. \tag{1}$$

Let the magnetic field B_0 be along the z −axis. Then the corresponding vector potential can be written as

$$\vec{A} = -yB_0\hat{i}. \tag{2}$$

If $\vec{P}$ and $\in$ are the relativistic momentum and Energy of a charged particle having charge q in the above mentioned electric and magnetic fields, we have the equation

$$(p_x - qA_x)^2c^2 + p_y^2c^2 + p_z^2c^2 + m_0^2c^4 = (\in +qE_0 y)^2. \tag{3}$$

Since the two potentials do not depend on z, the motion along z direction corresponds to that of a free particle. Hence, for convenience, we may take $p_z = 0$ and consider the motion in the $x - y$ plane only.

The quantum states of a charged particle can be obtained by solving the wave equation corresponding to equation (3). It can be written as

$$\left(-i\hbar\frac{d}{dx} + qB_0 y\right)^2 c^2\Psi_\in(x,y) - \hbar^2c^2\frac{d^2}{dy^2}\Psi_\in(x,y) + m_0^2c^4\Psi_\in(x,y)$$
$$= (\in +qE_0 y)^2\Psi_\in(x,y). \tag{4}$$

Noticing that there is no x variable in the equation we can put

$$\Psi_\in(x,y) = e^{\frac{i\alpha x}{\hbar}}\; Y_\in(y), \tag{5}$$

where α is a constant. Then we get the following equation for $Y_\in(y)$:

$$(\alpha + qB_0 y)^2c^2\; Y_\in(y) - \hbar^2c^2\frac{d^2}{dy^2}\; Y_\in(y) + m_0^2c^4\; Y_\in(y) = (\in +qE_0 y)^2\; Y_\in(y). \tag{6}$$

Simplifying, we get

$$(p_y)^2_{op}\ Y_\in(y) + c^2(\alpha^2 + 2\alpha q B_0 y + q^2 B_0^2 y)\ Y_\in(y)$$

$$= (\in^2 + q^2 E_0^2 y^2 + 2q E_0 y \in)\ Y_\in(y) \qquad (7)$$

Where $(p_y)_{op} = -i\hbar \frac{d}{dy}$.

Rearranging the terms, we get

$$(p_y)^2_{op}\ Y_\in(y) + q^2(c^2 B_0^2 - E_0^2)y^2\ Y_\in(y) + y(2\alpha q B_0 c^2 - 2q E_0 \in)\ Y_\in(y)$$

$$= (\in^2 - \alpha^2 c^2 - m_0^2 c^4)\ Y_\in(y) \qquad (8)$$

Let us put $\beta^2 = q^2\left(B_0^2 - \frac{E_0^2}{c^2}\right)$

And $\gamma^2 = (\epsilon^2 - \alpha^2 c^2 - m_0^2 c^4)/c^2$

Then equation (8) can be written as

$$(p_y)^2_{op}\ Y_\in(y) + \beta^2 y^2\ Y_\in(y) + \left(2\alpha q B_0 - 2\frac{\in}{c^2} q E_0\right) y\ Y_\in(y) = \gamma^2\ Y_\in(y) \qquad (9)$$

For the strong magnetic case of $B_0 > \frac{E_0}{c}$ which corresponds to $\beta^2 > 0$, we see the possibility of bounded motion in the y direction. Equation (9) can be rewritten as

$$(p_y)^2_{op}\ Y_\in(y) + \beta^2\left[y^2 + \frac{2q\left(\alpha B_0 - \frac{\in}{c^2}E_0\right) y}{\beta^2}\right] Y_\in(y) = \gamma^2\ Y_\in(y) \qquad (10)$$

or

$$(p_y)^2_{op}\ Y_\in(\bar{y}) + \beta^2 \bar{y}^2\ Y_\in(\bar{y}) = \left[\gamma^2 + \frac{q^2\left(\alpha B_0 - \frac{\in}{c^2}E_0\right)^2}{\beta^2}\right] Y_\in(\bar{y}) = \bar{\in}\ Y_\in(\bar{y})$$

(11)

Where

$\bar{y} = y + q\frac{\left(\alpha B_0 - \frac{\epsilon}{c^2}E_0\right)}{\beta^2}$ and $\bar{\epsilon} = \gamma^2 + \frac{q^2\left(\alpha B_0 - \frac{\epsilon}{c^2}E_0\right)^2}{\beta^2}$.

Equation (11) can be compared with that for a simple harmonic oscillator moving around the point $\bar{y} = 0$. Therefore, the eigenenergy values of the charge under consideration can be obtained from the eigenvalue equation:

$$2\hbar\beta\left(n + \frac{1}{2}\right) = \bar{\epsilon} = \gamma^2 + \frac{q^2\left(\alpha B_0 - \frac{\epsilon}{c^2}E_0\right)^2}{\beta^2} \qquad (12)$$

where $n = 0,1,2 \ldots$

Equation (12) is a quadratic equation that can be solved for the relativistic energy ϵ. From physical considerations only the positive root for ϵ is to be accepted.

The eigenfunctions for the y part of the motion can be written as

$$Y_\epsilon(\bar{y}) = N_n e^{-\frac{1}{2}\frac{\beta^2}{\hbar^2}\bar{y}^2} H_n\left(\frac{\beta\bar{y}}{\hbar}\right) \qquad (13)$$

Where $H_n\left(\frac{\beta\bar{y}}{\hbar}\right)$ are Hermite polynomials of n^{th} order in $\bar{y}$ with normalisation constant.

$$N_n = \sqrt{\left(\frac{\beta/\hbar}{\sqrt{\pi}2^n n!}\right)}$$

The total wave function for the system can be written as

$$\psi_\epsilon(x, y) = e^{\frac{i\alpha x}{\hbar}}\, Y_\epsilon(\bar{y}).$$

The non-relativistic approximation to the energy eigenvalues can be obtained from the equation (12) ignoring the second term on the right hand side. That is

$$2\hbar\beta\left(n+\frac{1}{2}\right) \simeq \gamma^2 = \frac{(\epsilon^2 - \alpha^2 c^2 - m_0^2 c^4)}{c^2}$$

Hence,

$$\epsilon \simeq \sqrt{m_0^2 c^4 + \alpha^2 c^2 + 2\hbar\beta\left(n+\frac{1}{2}\right)c^2} \quad (14)$$

$$\epsilon \simeq m_0 c^2 \left(1 + \frac{1}{2}\frac{\alpha^2 c^2}{m_0^2 c^4} + \frac{2\hbar\beta\left(n+\frac{1}{2}\right)c^2}{2m_0^2 c^4}\right) \quad (15)$$

$$Therefore, \quad \epsilon \simeq m_0 c^2 + \frac{1}{2}\frac{\alpha^2}{m_0} + \frac{\hbar\beta}{m_0}\left(n+\frac{1}{2}\right). \quad (16)$$

Here, the first term is the rest energy of the particle, the second term is the kinetic energy of motion along the $x-$ direction and the third term corresponds to oscillatory motion with frequency $\frac{\beta}{m_0} = \frac{q}{m_0}\left(B_0^2 - \frac{E_0^2}{c^2}\right)^{\frac{1}{2}}$. Interestingly the last term reproduces Landau levels if $E_0 = 0$, as it should.

3. Results and Discussion

We have deduced the exact eigenvalues and eigenfunctions of a spinless relativistic charged particle in crossed electric and magnetic fields. The eigenvalues have a quantized part and a continuous part corresponding to linear uniform motion along the $x-$ direction that is perpendicular to both electric and magnetic fields. The nonrelativistic eigenvalues correspond to the sum of a free particle energies and quantized energy levels similar to those of Landau levels. For $E > BC$ we do not get bounded motion and hence no quantised levels. Our results agree with nonrelativistic results under appropriate limits.

Acknowledgement

The authors would like to thank Prof. K. S. Mallesh for useful discussions.

References

John David Jackson, Classical Electrodynamics, Third Edition, Wiley Student Edition, India, 2007.

L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Third Revised English Edition, Pergamon Press, New York, 1971.

David J. Griffiths, Introduction to Electrodynamics, Third Edition, Prentice Hall, New Jersey, 1999.

Shankar R., Principles of Quantum Mechanics, Second Edition, 14th Printing, Springer India, 2014.

Phillips A. C., Introduction to Quantum Mechanics, John Wiley and Sons Ltd. New York, USA, 2003.

David A. B. Miller, Quantum Mechanics for Scientists and Engineers, Cambridge University Press, New York, USA, 2008.

Efstratios Manousakis, Practical Quantum Mechanics, Oxford Univ. Press, 2016.