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Fixed Point Theorem for Ɓ-type Contraction in Partial Metric Spaces

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Abstract

The objective of this work is to study Ɓ-types of contraction mappings in the settings of partial metric space and establish fixed point results. As a result, a fixed point theorem has been established for a pair of Ɓ-type contraction mappings with a unique common fixed point. The study's main findings, in particular, expand and extend a fixed point theorem first proposed by Bijender et. al. in 2021.

Keywords: Ɓ-contraction, contraction mappings, fixed point, common fixed point, complete metric space

1. Introduction

According to Browder, the study of fixed points is one of the most significant advancements in the field of mathematics, which, through the introduction of non-linear functional analysis, such as dynamic and vital mathematics, has given a fresh push to classical fixed point theory. This approach is used in a variety of contemporary fields of analysis, with topological aspects playing an important role, such as the link with degree theory. In metric fixed point theory, we investigate conclusions involving mainly isometric features. The difference between metric fixed point

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theory and seemingly broader topological theories is frequently blurred. The cornerstone of the metric theory is that of iterative estimations to demonstrate the existence and uniqueness of solutions.

However, Banach, a Polish mathematician, is recognized for integrating the fundamental concepts into an abbreviated structure that can be applied to a wide range of problems, well beyond basic differential and integral equations. Banach's profound fixed point hypothesis developed the metric fixed point theory for contraction mappings on complete metric spaces. The fixed point theory of a few essential single-valued maps is fascinating in itself, with structural demonstrations and applicability in diverse disciplines like image processing engineering, economics, physics, telecommunications, and computer sciences. There are many generalisations and extensions of the Banach contraction principle in existing literature [1, 4, 5] etc.

During his study on the denotational semantics of data flow networks, Matthews [2] established the concept of partial metric space by substituting conventional metric with a partial metric, which has the intriguing characteristic that the self-distance of any point in space may not be zero. The Banach contraction principle holds in partial metric space, as demonstrated by Matthews [2], and may be utilized in programme verification.

This article is organized as; section 2 contains some basic definitions of partial metric theory from existing literature. In section 3, we proved some fixed point theorems for Ɓ-contraction mapping and a pair of Ɓ-contraction mappings. And finally, in section 4, we conclude our work and provide an idea that some further improvement may be possible.

2. Preliminaries

Definition 2.1. Let E be a non-empty set. A partial metric space is a pair (E,σ), where σ is a function $\sigma: E\times E\rightarrow \mathbb{R}^+$ is called the partial metric space, such that for all $\zeta,\xi,\omega\in E$ the following axioms holds [2]:

(p_1) $\zeta = \xi \Leftrightarrow \sigma(\zeta,\xi) = \sigma(\zeta,\zeta) = \sigma(\xi,\xi)$;

(p_2) $\sigma(\zeta,\zeta) \leq \sigma(\zeta,\xi)$;

(p_3) $\sigma(\zeta,\xi) = \sigma(\xi,\zeta)$;

(p_4) $\sigma(\zeta,\xi) \leq \sigma(\zeta,\omega) + \sigma(\omega,\xi) - \sigma(\omega,\omega)$.

From (p_1) and (p_2), it is true that if $\sigma(\zeta,\xi) = 0$, then $\zeta = \xi$ but the converse need not be true $i.e.$ if $\zeta = \xi$ then $\sigma(\zeta,\xi)$ may be not equal to zero. For an example partial metric space from the computation view of the point may be found in [2, 6].

Definition 2.2. Let $E \neq \emptyset$ be a set and the pair (E,σ) be a partial metric space [2]. Then

1. a sequence $\{\zeta_n\}$ in (E,σ) is convergent to $\zeta \in E \Leftrightarrow$ $\sigma(\zeta,\zeta) = \lim_{n\to\infty}\sigma(\zeta,\zeta_n)$;

2. a sequence $\{\zeta_n\}$ in (E,σ) is a Cauchy sequence $\Leftrightarrow$ $\lim_{n,m\to\infty}\sigma(\zeta_n,\zeta_m)$ exists and is finite.

3. If every Cauchy sequence ζ_n in E converges with regard to the topology τ_p to $\zeta \in E$ such that $\sigma(\zeta,\zeta) = \lim_{n,m\to\infty}\sigma(\zeta_n,\zeta_m)$, then partial metric space is called complete.

Lemma 2.3. Let $E \neq \emptyset$ be a set and the pair (E,σ) be a partial metric space [2]. The function $\sigma^s : E \times E \to [0,\infty)$ given by $\sigma^s(\zeta,\xi) = 2\sigma(\zeta,\xi) - \sigma(\zeta,\zeta) - \sigma(\xi,\xi)$, $\forall \zeta,\xi \in E$ is a metric.

Lemma 2.4. Let $E \neq \emptyset$ be a set and the pair (E,σ) be a partial metric space [2]. Then

1. a sequence ζ_n is a cauchy sequence in $(E,\sigma) \Leftrightarrow$ it is a cauchy sequence in the metric space (E,σ^s);

2. a partial metric space (E,σ) is complete $\Leftrightarrow$ the metric space (E,σ^s) is complete.

Mathews [2] proved the following fixed point result on complete partial metric spaces.

Theorem 2.5. Let T be a self map on complete partial metric space (E,σ). If there is a real number c with $0 \leq c < 1$ such that

$$\sigma(T\zeta, T\xi) \leq c\sigma(\zeta,\xi), \forall_{\zeta,\xi \in E},$$

then T has a unique fixed point.

3. Main Results

The notion of Ƀ-contraction was introduced by Bijender et al. [5] is a generalization of Banach contraction. As a consequence, several fixed point findings for Ƀ-contraction mappings, such as extension and generalisation, have been discovered. Bijender et al. [5] defined it as follows;

Definition 3.1. Let Ƀ be a set of mappings $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfying [5]:

(m_1) for all $a, b \in \mathbb{R}^+$ such that $a < b,\ \phi(a) < \phi(b)$ $i.e.$ strictly increasing;

$(m_2)\lim_{n\to\infty} a_n = 0 \Leftrightarrow$ if $\lim_{n\to\infty}\phi(a_n) = 0$;$\lim_{n\to\infty}\phi(a_n) = 0$; where $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of positive numbers; $(m_3)\phi$ is continuous on $(0,\infty)$.

Let (E,σ) be a partial metric space. A self-map $T: E \to E$ is said to be an Ƀ-contraction if there exists $\alpha \in (0,1)$ and $\phi \in \mathcal{B}$ such that Ƀ

$$\forall_{\zeta,\xi} \in E,\ d(T\zeta, T\xi) > 0 \Rightarrow \phi(d(T\zeta,T\xi)) \leq \alpha\phi(d(\zeta,\xi)). \quad (1)$$

Remark 3.2. Let $\phi \in \mathcal{B}$ be an identity mapping $i.e \phi(\zeta) = \zeta.$ It's a simple task to verify that ϕ satisfies (m_1)-(m_3) and the contractive condition (1) reduces to usual contraction [5]. $i.e.,$

$$\sigma(T\zeta, T\xi) \leq \alpha\sigma(\zeta,\xi) \text{ for all} \zeta,\xi \in E, T\zeta \neq T\xi.$$

Remark 3.3. But the case when ϕ is not identity mapping, still usual contraction is a particular case of Ɓ-contraction [5].

Example 3.4. Consider $\phi(\zeta) = \sqrt{\zeta}, \zeta > 0$ then ϕ satisfies (m_1)-(m_3) and the contractive condition (1) takes the form [5]

$$\sigma(T\zeta, T\xi) \leq \alpha^2 \sigma(\zeta, \xi) \text{ for all} \zeta, \xi \in E, T\zeta \neq T\xi.$$

Example 3.5. Let $\phi(\zeta) = \zeta^n, \zeta > 0$, where ϕ satisfy (m_1)-$(m_3)$and the contractive condition (1) for the ϕ-contraction T, the following condition holds [5]

$$\sigma(T\zeta, T\xi) \leq \alpha^{\frac{1}{n}} \sigma(\zeta, \xi) \;\; for\;\; all\; \zeta, \xi \in E, T\zeta \neq T\xi.$$

Example 3.6. Let $\phi(\zeta) = \frac{\zeta}{\zeta+1}, \zeta > 0.$ Then ϕ satisfy (m_1)-$(m_3)$and the contractive condition (1) for the Ɓ-contraction T is reduces to [5]

$$\sigma(T\zeta, T\xi) \leq \alpha \frac{(\sigma(T\zeta, T\xi)+1)}{(\sigma(\zeta, \xi)+1)} \sigma(\zeta, \xi), \forall \zeta, \xi \in E, T\zeta \neq T\xi.$$

Remark 3.7. We observe that in the above examples, the contraction condition remains satisfied even if [5] $T\zeta = T\xi, \forall\, \zeta, \xi \in E$.

Remark 3.8. From the condition m_1 and the definition of Ɓ-contraction T, one can easily conclude that, that every $\mathcal{B}$-contraction T is a contraction mapping [5], $i.e.$

$$\sigma(T\zeta, T\xi) \leq \lambda \sigma(\zeta, \xi) \;\; for\;\; all\; \zeta, \xi \in E, T\zeta \neq T\xi, \lambda \in (0,1).$$

Thus every Ɓ-contraction T is continuous mapping.

Remark 3.9. Let ϕ_1 and ϕ_2 be two mappings satisfying(m_1)-(m_3). If$\phi_1(\zeta) \leq \phi_2(\zeta)$for all$\zeta > 0$and mapping $G = \phi_2 - \phi_1$ is non-decreasing then every ϕ_1-contraction T is ϕ_2- contraction [5]. Form, remark (3.8), we have

$$G(\sigma(T\zeta, T\xi)) \leq \alpha G(\sigma(\zeta, \xi)) \;\; for\;\; all\; \zeta, \xi \in E, T\zeta \neq T\xi.$$

Thus

$$\phi_2(\sigma(T\zeta, T\xi)) = \phi_1(\sigma(T\zeta, T\xi)) + G(\sigma(T\zeta, T\xi))$$
$$\leq \alpha\{\phi_1(\sigma(\zeta, \xi)) + G(\sigma(\zeta, \xi))\}$$

$$\leq \alpha\phi_2(\sigma(\zeta,\xi)),\ for\ \ all\ \zeta,\xi \in E, T\zeta \neq T\xi.$$

Theorem 3.10. Let (E,σ) be a complete partial metric space and let self-map $T: E \to E$ be a Ƀ-contraction. Then T has unique fixed point $\zeta^* \in E$ such that $\sigma(\zeta^*,\zeta^*) = 0$

Proof: Let ζ_0 be an arbitrary point in E. Define a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ such that

$$\zeta_{n+1} = T\zeta_n, n = 0,1,2,\ldots \tag{2}$$

If there exist $n \in \mathbb{N}$ such that $\sigma(\zeta_n, T\zeta_n) = 0,$ then proof is complete. So suppose that

$$0 < \sigma(\zeta_n, T\zeta_n) = \sigma(T\zeta_{n-1}, T\zeta_n) \text{ for all } n \in \mathbb{N}. \tag{3}$$

For any $n \in \mathbb{N}$, we have

$$\phi(\sigma(T\zeta_{n-1}, T\zeta_n)) \leq \alpha\phi(\sigma(\zeta_{n-1},\zeta_n)).$$

After repeating the same process, we have

$$\phi(\sigma(T\zeta_{n-1}, T\zeta_n)) \leq \alpha^n\phi(\sigma(\zeta_0,\zeta_1)) \to 0\ \ as\, n \to \infty. \tag{4}$$

Which together with (m_2), provide

$$\lim_{n\to\infty}\phi(\sigma(T\zeta_{n-1}, T\zeta_n)) = 0$$

i.e. $\lim_{n\to\infty}\sigma(\zeta_n, T\zeta_n) = 0$. (5)

Now, claim that $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Contrary, suppose that there exist $\epsilon > 0$ and the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of natural numbers such that

$$r_n > s_n > n,\ \sigma(\zeta_{r_n},\zeta_{s_n}) \geq \epsilon,\ \sigma(\zeta_{r_{n-1}},\zeta_{s_n}) < \epsilon,\ \forall n \in \mathbb{N}, \tag{6}$$

thus

$$\epsilon \leq \sigma(\zeta_{r_n},\zeta_{s_n}) \leq \sigma(\zeta_{r_n},\zeta_{r_{n-1}}) + \sigma(\zeta_{r_{n-1}},\zeta_{s_n}) - \sigma(\zeta_{r_{n-1}},\zeta_{r_{n-1}})$$
$$\leq \sigma(\zeta_{r_n},\zeta_{r_{n-1}}) + \sigma(\zeta_{r_{n-1}},\zeta_{s_n})$$

$$\leq \sigma(\zeta_{r_{n-1}}, T\zeta_{r_{n-1}}) + \epsilon.$$

By using (4) and the above inequality, we get

$$\lim_{n\to\infty} \sigma(\zeta_{r_n}, \zeta_{s_n}) = \epsilon. \tag{7}$$

Therefore, from (5), there exists $n \in \mathbb{N}$ such that

$$\sigma\big(\zeta_{r_m}, T\zeta_{r_m}\big) < \tfrac{\epsilon}{3} \text{ and } \sigma\big(\zeta_{s_m}, T\zeta_{s_m}\big) < \tfrac{\epsilon}{3},\ \forall m \geq n. \tag{8}$$

Further, we shall prove that

$$\sigma(T\zeta_{r_m}, T\zeta_{s_m}) = \sigma(\zeta_{r_{m+1}}, \zeta_{s_{m+1}}) > 0,\ \forall m \geq n. \tag{9}$$

For this, suppose that there exists $p \geq np \geq n$such that

$$\sigma(\zeta_{r_{p+1}}, \zeta_{s_{p+1}}) = 0. \tag{10}$$

using (6), (8), and (10), we have

$$\begin{aligned}
\epsilon \leq \sigma(\zeta_{r_p}, \zeta_{s_p}) &\leq \sigma(\zeta_{r_p}, \zeta_{r_{p+1}}) + \sigma(\zeta_{r_{p+1}}, \zeta_{s_p}) - \sigma(\zeta_{r_{p+1}}, \zeta_{r_{p+1}}) \\
&\leq \sigma(\zeta_{r_p}, \zeta_{r_{p+1}}) + \sigma(\zeta_{r_{p+1}}, \zeta_{s_{p+1}}) + \sigma(\zeta_{s_{p+1}}, \zeta_{s_p}) \\
&\quad - \sigma(\zeta_{r_{p+1}}, \zeta_{r_{p+1}}) - \sigma(\zeta_{s_{p+1}}, \zeta_{s_{p+1}}) \\
&\leq \sigma(\zeta_{r_p}, T\zeta_{r_p}) + \sigma(\zeta_{r_{p+1}}, \zeta_{s_{p+1}}) + \sigma(\zeta_{s_p}, T\zeta_{s_p}) \\
&< \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.
\end{aligned}$$

It is a contradiction, so (9) is true. Therefore

$$\phi\left(\sigma(Tx_{r_m}, Tx_{s_m})\right) \leq \alpha\phi\left(\sigma(x_{r_m}, x_{s_m})\right). \tag{11}$$

By (m_3), (7) and (11), we get

$$\phi(\epsilon) \leq \alpha\phi(\epsilon).$$

It is a contraction, shows that the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and by the completeness of (E, σ), the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is convergent to some point $\zeta^* \in E$, and

$$\sigma(T\zeta^*,\zeta^*) = \lim_{n\to\infty}\sigma(T\zeta_n,\zeta_n)$$
$$= \lim_{n\to\infty}\sigma(\zeta_{n+1},\zeta_n)$$
$$= \sigma(\zeta^*,\zeta^*) = 0.$$

For uniqueness, suppose that there exists $\zeta(\neq \zeta^*) \in E$ such that $T(\zeta) = \zeta$, therefore

$$\sigma(T\zeta,T\zeta^*) = \sigma(\zeta,\zeta^*) > 0,$$

then

$$\phi(\sigma(\zeta,\zeta^*)) = \phi(\sigma(T\zeta,T\zeta^*)) \leq \alpha\phi(\sigma(\zeta,\zeta^*)),$$

which is a contradiction. Hence the fixed point is unique.

Theorem 3.11. Let (E,σ) be a complete partial metric space and $f,g: E \to E$ be a pair of two self maps of Ɓ-contraction, such that for all $\zeta,\xi \in E$, we have

$$\sigma(f\zeta,g\xi)) > 0 \Rightarrow \phi(\sigma(f\zeta,g\xi)) \leq \alpha(\sigma(\mathbb{C}(\zeta,\xi))) \qquad (12)$$

where,

$$\mathbb{C}(\zeta,\xi) = \max\left\{\sigma(\zeta,\xi),\sigma(\zeta,f\zeta),\sigma(\xi,g\xi),\frac{\sigma(\zeta,g\xi)+\sigma(\xi,f\zeta)}{2}\right\},$$

then there's a unique fixed point that both f and g have in common.

Proof: Let $\zeta_0 \in E$ be any arbitrary point and define a sequence $\{\zeta_n\} \in E$, for all $n \in \mathbb{N}$ such that $\zeta_{n+1} = f_{\zeta_n}$ and $\zeta_{n+2} = g\zeta_{n+1}$ for $n = 0,1,2,\ldots$

Let $\sigma(\zeta_{n+1},\zeta_{n+2}) > 0,$ for all $n \in \mathbb{N} \cup \{0\}$ with $\zeta_{n+1} \neq \zeta_{n+2}$. Thus from equation (12), we have

$$\phi(\sigma(\zeta_{n+1},\zeta_{n+2})) = \phi(\sigma(f\zeta_n,g\zeta_{n+1})) \leq \alpha\eta(\mathbb{C}(\zeta_n,\zeta_{n+1}))$$
$$= \alpha\phi(\max\{\sigma(\zeta_n,\zeta_{n+1}),\sigma(\zeta_n,f\zeta_n),\sigma(\zeta_{n+1},g\zeta_{n+1}),$$
$$\frac{\sigma(\zeta_n,g\zeta_{n+1}) + \sigma(\zeta_{n+1},f\zeta_n)}{2}\})$$

$$= \alpha\phi(\max\{\sigma(\zeta_n,\zeta_{n+1}), \sigma(\zeta_n,\zeta_{n+1}), \sigma(\zeta_{n+1},\zeta_{n+2}), \frac{\sigma(\zeta_n,\zeta_{n+2}) + \sigma(\zeta_{n+1},\zeta_{n+1})}{2}\})$$

$$= \alpha\phi(\max\{\sigma(\zeta_n,\zeta_{n+1}), \sigma(\zeta_{n+1},\zeta_{n+2}), \frac{\sigma(\zeta_n,\zeta_{n+1}) + \sigma(\zeta_{n+1},\zeta_{n+2})}{2}\})$$

$$= \alpha\phi(\max\{\sigma(\zeta_n,\zeta_{n+1}), \sigma(\zeta_{n+1},\zeta_{n+2})\}).$$

Suppose that $\max\{\sigma(\zeta_n,\zeta_{n+1}), \sigma(\zeta_{n+1},\zeta_{n+2})\} = \sigma(\zeta_{n+1},\zeta_{n+2})$. Then

$$\phi(\sigma(\zeta_{n+1},\zeta_{n+2})) \leq \alpha\phi(\sigma(\zeta_{n+1},\zeta_{n+2})),$$

which is not possible. So $\max\{\sigma(\zeta_n,\zeta_{n+1}), \sigma(\zeta_{n+1},\zeta_{n+2})\} = \sigma(\zeta_n,\zeta_{n+1})$, thus we can write

$$\phi\big(\sigma(\zeta_{n+1},\zeta_{n+2})\big) \leq \alpha\phi\big(\sigma(\zeta_n,\zeta_{n+1})\big), \ for \ all \ n \in \mathbb{N} \cup \{0\}. \tag{13}$$

After repeating same process, we have

$$\phi(\sigma(\zeta_n,\zeta_{n+1})) \leq \alpha^n\phi(\sigma(\zeta_0,\zeta_1)). \tag{14}$$

$\to 0 \ \ as \ n \to \infty$. Which together with ($m_2$), provide

$$\lim_{n\to\infty} \sigma(\zeta_n,\zeta_{n+1}) = 0. \tag{15}$$

Now, claim that $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Contrary, suppose that there exist $\epsilon > 0$ and the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of natural numbers such that

$$r_n > s_n > n, \ \ \sigma(\zeta_{r_n},\zeta_{s_n}) \geq \epsilon, \ \ \sigma(\zeta_{r_{n-1}},\zeta_{s_n}) < \epsilon, \ \ \forall n \in \mathbb{N} \tag{16}$$

Thus

$$\epsilon \leq \sigma(\zeta_{r_n},\zeta_{s_n}) \leq \sigma(\zeta_{r_n},\zeta_{r_{n-1}}) + \sigma(\zeta_{r_{n-1}},\zeta_{s_n}) - \sigma(\zeta_{r_{n-1}},\zeta_{r_{n-1}})$$
$$\leq \sigma(\zeta_{r_n},\zeta_{r_{n-1}}) + \sigma(\zeta_{r_{n-1}},\zeta_{s_n})$$

$$\leq \sigma(\zeta_{r_{n-1}}, \zeta_{r_n}) + \epsilon \leq \sigma(\zeta_{r_{n-1}}, \zeta_{r_n}) + \epsilon$$

By using (15) and the above inequality, we get

$$\lim_{n\to\infty} \sigma(\zeta_{r_n}, \zeta_{s_n}) = \epsilon. \tag{17}$$

Therefore, from (15), there exists $n \in \mathbb{N}$ such that

$$\sigma(\zeta_{r_m}, \zeta_{r_{m+1}}) < \frac{\epsilon}{3} \text{ and } \sigma(\zeta_{s_m}, \zeta_{s_{m+1}}) < \frac{\epsilon}{3}, \ \forall m \geq n \tag{18}$$

Further, we shall prove that

$$\sigma(f\zeta_{r_m}, g\zeta_{s_m}) = \sigma(\zeta_{r_{m+1}}, \zeta_{s_{m+1}}) > 0, \ \forall m \geq n. \tag{19}$$

For this suppose that, there exists $p \geq np \geq n$ such that

$$\sigma(\zeta_{r_{p+1}}, \zeta_{s_{p+1}}) = 0. \tag{20}$$

using (16), (18) and (20), we have

$$\begin{aligned}
\epsilon \leq \sigma(\zeta_{r_p}, \zeta_{s_p}) &\leq \sigma(\zeta_{r_p}, \zeta_{r_{p+1}}) + \sigma(\zeta_{r_{p+1}}, \zeta_{s_p}) - \sigma(\zeta_{r_{p+1}}, \zeta_{r_{p+1}}) \\
&\leq \sigma(\zeta_{r_p}, \zeta_{r_{p+1}}) + \sigma(\zeta_{r_{p+1}}, \zeta_{s_{p+1}}) + \sigma(\zeta_{s_{p+1}}, \zeta_{s_p}) \\
&-\sigma(\zeta_{r_{p+1}}, \zeta_{r_{p+1}}) - \sigma(\zeta_{s_{p+1}}, \zeta_{s_{p+1}}) \\
&\leq \sigma(\zeta_{r_p} \zeta_{r_{p+1}}) + \sigma(\zeta_{r_{p+1}}, \zeta_{s_{p+1}}) + \sigma(\zeta_{s_{p+1}}, \zeta_{s_p}) \\
< \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} = \frac{2\epsilon}{3}.
\end{aligned}$$

It is contradiction, so (19) is true. Therefore

$$\phi(\sigma(fx_{r_m}, gx_{s_m})) \leq \alpha\phi(\sigma(x_{r_m}, x_{s_m})) \tag{21}$$

By (m_3), (18) and (20), we get

$$\phi(\epsilon) \leq \alpha\phi(\epsilon).$$

It is a contraction, shows that the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and by the completeness of (E, σ), sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is

convergent to some point $\zeta^* \in E$. It follows that $\zeta_{n+1} \to \zeta^*$ and $\zeta_{n+2} \to \zeta^*$ as $n \to \infty$. Hence by the continuity of g it is implies that $\zeta^* = \lim_{n\to\infty}\zeta_n = \lim_{n\to\infty}\zeta_{n+1} = \lim_{n\to\infty}\zeta_{n+2} = \lim_{n\to\infty}g\zeta_{n+1} = g\lim_{n\to\infty}\zeta_{n+1} = g\zeta^*$
Hence from 12, we have

$$\phi(\sigma(\zeta^*,f\zeta^*)) = \phi(\sigma(f\zeta^*,g\zeta^*)) \leq \alpha\phi(\mathbb{C}(\zeta^*,\zeta^*)) = \alpha\phi(\sigma(\zeta^*,\zeta^*)),$$

implies that $\sigma(\zeta^*,f\zeta^*) = 0$ and by (p_1)and (p_2), we obtain that $\zeta^* = f\zeta^*$. Thus $f\zeta^* = g\zeta^* = \zeta^*$. Hence (f,g) has a common fixed point $\zeta^* \in E$. Suppose that there exists $\zeta \in E$ such that $\zeta \neq \zeta^*$ and $\zeta = g\zeta$. From the contractive condition (12), we have

$$\phi(\sigma(f\zeta^*,g\zeta)) \leq \alpha\phi(\mathbb{C}(\zeta^*,\zeta)),$$

where

$$\begin{aligned}\mathbb{C}(\zeta^*,\zeta) &= \max\{\sigma(\zeta^*,\zeta),\sigma(\zeta^*,f\zeta^*),\sigma(\zeta,g\zeta),\tfrac{\sigma(\zeta^*,g\zeta)+\sigma(\zeta,f\zeta^*)}{2}\}\\ &= \max\{\sigma(\zeta^*,\zeta),\sigma(\zeta,\zeta),\sigma(\zeta^*,\zeta^*),\tfrac{\sigma(\zeta^*,\zeta)+\sigma(\zeta,\zeta^*)}{2}\}\\ &= \max\{\sigma(\zeta^*,\zeta),\tfrac{\sigma(\zeta^*,\zeta)+\sigma(\zeta,\zeta^*)}{2}\}\\ &= \sigma(\zeta^*,\zeta).\end{aligned}$$

Thus, we have $\phi(\sigma(f\zeta^*,g\zeta)) \leq \alpha\phi(\sigma(\zeta^*,\zeta))$ and we obtain that $\sigma(\zeta^*,\zeta) < \sigma(f\zeta^*,g\zeta)$, which is a contradiction. Hence $\zeta^* = \zeta$ and ζ^* is a unique common fixed point of (f,g).

4. Conclusion

The objective of this work is to study Ƀ-types of contraction mappings in the settings of partial metric space and establish fixed point results. As a result, a fixed point theorem has been established for a pair of Ƀ-type contraction mappings with a unique common fixed point. The study's main findings, in particular, expand and extend a fixed point theorem first proposed by Bijender in 2021. We hope that the findings investigated in this paper provide an important and technically sound contribution to the field and will be useful to researchers for further promotion and enhancement of their theoretical work in the field of partial metric spaces. For the purpose of future scope, some further generalization can be made through Ɓ-contraction in the setting of partial metric spaces, metric spaces, metric-like spaces.

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