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# Fixed Point Theorem for Fuzzy $\mathcal{B}$-type Contractions in Fuzzy Metric Spaces

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**Abstract**

Recently, in 2021, Bijender et al. proposed the establishment of $\mathcal{B}$-contraction. Such contraction is a genuine generalisation of the standard contraction in the study of metric fixed point theory. The aim of the present study is the establishment of the novel concept of the fuzzy $\mathcal{B}$-type contraction in the settings of fuzzy metric space, and such contractions are also used to establish a few fixed point theorems.

**Keywords:** Fuzzy metric space, fixed point, complete metric space, fuzzy $\mathcal{B}$-contraction

**MSC.** 47H10, 54H25.

## 1. Introduction and Preliminaries

Zadeh [1] proposed the notion of fuzzy sets in 1965 as a novel means of representing ambiguity in regular activities. Following Zadeh's ground-breaking discovery, there has been a concerted attempt to find fuzzy equivalents of classical ideas. Fuzzy topology,

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among other subjects, is experiencing steady growth. Obtaining an acceptable idea of fuzzy metric space is one of the most difficult challenges in fuzzy topology. Several authors have looked into this issue from various perspectives. George and Veeramani [2], in particular have developed and investigated the concept of fuzzy metric space.

Menger first proposed the notion of a statistical measure in 1951. Kramosil and Michalek proposed the concept of a fuzzy metric based on the concept of a statistical metric. We term it a KM-fuzzy metric here. Although a KM-fuzzy metric is similar to a statistical metric in some ways, its definitions and meanings are fundamentally different. The initial notion of a KM-fuzzy metric was somewhat changed by George and Veeramani in 1994, and we name this variation the GV-fuzzy metric. Many natural instances of fuzzy measurements are now possible with this update, especially fuzzy metrics derived from metrics.

Recently Bijender et al. introduced the concept of $B$-contraction and proved fixed point theorems, which provide a new generalization of the Banach contraction principle in a different way than in the known results from the existing literature. In this article, we introduced a new notion of fuzzy $B$-contraction and established some fixed-point theorems in complete fuzzy metric spaces.

## 2. Preliminaries

**Definition 2.1** [6]A continuous $t$-norm is a binary operation $\circledast: [0,1]^2 \to [0,1]$, if the following axioms are satisfied:

1. $\circledast$ satisfy the associative and commutative properties;

2. $\circledast$ satisfy the continuity;

3. $\alpha \circledast 1 = \alpha, 0 \leq \alpha \leq 1$;

4. $\alpha \circledast \beta \leq \gamma \circledast \delta$, whenever $\alpha \leq \gamma$ and $\beta \leq \delta$, for every $0 \leq \alpha, \beta, \gamma, \delta \leq 1$,

these are some common examples of continuous $t$-norms:

1. $\alpha \circledast_1 \beta = min\{\alpha, \beta\}$;
2. $\alpha \circledast_2 \beta = \frac{\alpha\beta}{max\{\alpha,\beta,\lambda\}}$ for $0 < \lambda < 1$;
3. $\alpha \circledast_3 \beta = \alpha\beta$;
4. $\alpha \circledast_4 \beta = max\{\alpha + \beta - 1, 0\}$.

**Definition 2.2** [2] Consider$E$ be a non-empty set, $\circledast$is a continuous $t$-norm, and $W$ is a fuzzy set on $E^2 \times [0, \infty)$, then the triplate $(E, W, \circledast)$is called a fuzzy metric space if for each$\varsigma, \vartheta, z \in E$,the aforementioned requirements are met:

1. $W(\varsigma, \vartheta, t) > 0$,
2. $W(\varsigma, \vartheta, t) = 1 \Leftrightarrow \varsigma = \vartheta$,
3. $W(\varsigma, \vartheta, t) = W(\vartheta, \vartheta, t)$,
4. $W(\varsigma, \vartheta, t) * W(y, z, s) \leq W(\vartheta, z, t + s)$,
5. $W(\varsigma, \vartheta, .): (0, \infty) \to [0,1]$ satisfy the continuity axiom.

Let $(E, W, \circledast)$ be a fuzzy metric space. For $t > 0$, the open ball $B(\varsigma, r, t)$ with centre $\varsigma \in E$ and a radius $0 < r < 1$ is defined by

$$B(\varsigma, r, t) = \{\vartheta \in E: W(\varsigma, \vartheta, t) > 1 - r\}$$

**Example 2.3** [7] Let$E = R$, denote$a \circledast b = ab$for all$a, b \in [0,1]$. For each$t \in (0, \infty)$, define

$$W(\varsigma, \vartheta, t) = \frac{t}{t + |\varsigma - \vartheta|},$$

for all $\varsigma, \vartheta \in E$, then it is easy to see that $(E, W, \circledast)$ is a fuzzy metric space.

**Example 2.4** [7]Let$(E, d)$be a metric space and$\varpi$be an increasing and continuous function from $R^+$ into$(0,1)$such that$lim_{n\to\infty}\varpi(t) = 1$, for these some common functions

1. $\varpi(\varsigma) = \frac{\varsigma}{\varsigma+1}$,
2. $\varpi(\varsigma) = sin(\frac{\pi\varsigma}{2\varsigma+1})$
3. $\varpi(\varsigma) = 1 - e^{-\varsigma}$
4. $\varpi(\varsigma) = e^{\frac{-1}{\varsigma}}$,

assume $\alpha \circledast \beta \leq \alpha\beta, \forall \alpha, \beta \in [0,1]$ and each $t \in (0, \infty)$, define

$W(\varsigma, \vartheta, t) = [\varpi(t)]^{d(\varsigma,\vartheta)}$,

for all $\varsigma, \vartheta \in E$, then triplate $(E, W, \circledast)$ is called fuzzy metric space.

**Definition 2.5** [2]Assume that the triplate $(E, W, \circledast)$ be a fuzzy metric space,

1. The sequence $\{\varsigma_n\} \in E$ converges to a point $\varsigma \in E$ if $lim_{n\to\infty}W(\varsigma_n, \varsigma, t) = 1 \forall t > 0$;
2. If for each $0 < \epsilon < 1$ and $t > 1$, $\exists n_0 \in N$ such that $W(\varsigma_n, \varsigma_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$, then the sequence $\{\varsigma_n\}$ in $E$ is a Cauchy sequence;
3. If every Cauchy sequence is convergent in set $E$ then triplate $(E, W, \circledast)$ called complete fuzzy metric space.

**Lemma 2.6** [7] For all $\varsigma, \vartheta \in E$, $W(\varsigma, \vartheta, .)$ is nondecreasing function.

**Definition 2.7** [7]Assume that the triplate $(E, W, \circledast)$ is a fuzzy metric space then$W$is continuous on$E^2 \times (0, \infty)$if

$$\lim_{n\to\infty} W(\varsigma_n, \vartheta_n, t_n) = W(\varsigma, \vartheta, t),$$

whenever $\{(\varsigma_n, \vartheta_n, t_n)\}$ is a sequence in $E^2 \times (0, \infty)$, that is

$$\lim_{n\to\infty} W(\varsigma_n, \varsigma, t) = \lim_{n\to\infty} W(\vartheta_n, \vartheta, t) = 1$$

$$\lim_{n\to\infty} W(\varsigma, \vartheta, t_n) = \lim_{n\to\infty} W(\varsigma, \vartheta, t).$$

## 2. Main Results

First and foremost, in the best interests of the reader, we recollect the definition from [5] as follows:

**Definition 3.1** [5] Let $B$ be the collection of mappings$\phi: R^+ \to R$holds the following axioms:

1. for all $a, b \in R^+$ such that $a < b, \phi(a) < \phi(b) i.e.$ strictly increasing;
2. $lim_{n\to\infty} a_n = 0 \Leftrightarrow$ if $lim_{n\to\infty} \phi(a_n) = 0$; where $\{a_n\}_{n\in N}$ is sequence of positive numbers;
3. $\phi$ is continuous on $(0, \infty)$.

Let $(E, d)$ be a metric space. A self-map $f: E \to E$ is said to be an $B$-contraction if there exists $\alpha \in (0,1)$ such that

$$\forall_{\varsigma,\vartheta} \in E,\ d(f\varsigma, f\vartheta) > 0 \Rightarrow \phi(d(f\varsigma, f\vartheta)) \leq \lambda \phi(d(\varsigma, \vartheta)).$$
(3.1)

Here, we introduce the concept of fuzzy $B$-contraction and prove several fixed-point theorems in fuzzy metric spaces.

**Definition 3.2** Assume that the triplate $(E, W, \circledast)$is a fuzzy metric space and take$\phi \in B$ then the self-map$f: E \to E$is said to be a fuzzy$B$-contraction if$\exists \lambda \in (0,1)$ such that

$$\phi(W(f\varsigma, f\vartheta, t)) \leq \lambda\phi(W(\varsigma, \vartheta, t))$$

where $\varsigma, \vartheta \in E$.

**Theorem 3.3** Consider that the triplate $(E, W, \circledast)$ be an $W$-complete fuzzy metric space and a self-map $f: E \to E$, then suppose that $\phi: R^+ \to R^+$ a function satisfies the properties (M-1),(M-2) and (M-3). Furthermore, let $\lambda \in (0,1)$. If for any $t > 0$, $f$ satisfies the following condition

$$\phi(W(f\varsigma, f\vartheta, t)) \leq \lambda\phi(W(\varsigma, \vartheta, t)) \qquad (3.2)$$

where $\varsigma, \vartheta \in E$ and $\varsigma \neq \vartheta$, then $f$ possesses a unique fixed point.

**Proof:** Let $\varsigma_0$ in $E$ be an arbitrary point and fixed. The sequence $\{\varsigma_n\}_{n\in N}$ satisfy

$$\varsigma_{n+1} = f\varsigma_n, n = 0,1,2, \dots, \qquad (3.3)$$

if there exists $n \in N$ such that $W(\varsigma_n, f\varsigma_n, t) = 1,$ then proof is complete.

So suppose that

$$0 < W(\varsigma_n, f\varsigma_n, t) = W(f\varsigma_{n-1}, f\varsigma_n, t) < 1 \ for \ all \, n \in N. \qquad (3.4)$$

For any $n \in N$, we have

$$\phi(W(f\varsigma_{n-1}, f\varsigma_n, t)) \leq \alpha\phi(W(\varsigma_{n-1}, \varsigma_n, t)).$$

After repeating same process, we have

$$\phi\big(W(f\varsigma_{n-1}, f\varsigma_n, t)\big) \le \alpha^n \phi\big(W(\varsigma_0, f\varsigma_0, t)\big). \qquad (3.5)$$

$\to 0 \ as\, n \to \infty$. Which together with (M-2), provide

$\lim_{n\to\infty} \phi(W(f\varsigma_{n-1}, f\varsigma_n, t)) = 0$

i.e.

$$\lim_{n\to\infty} W(f\varsigma_{n-1}, f\varsigma_n, t) = 1. \qquad (3.6)$$

Now, claim that $\{\varsigma_n\}_{n\in N}$ is satisfy the $W$-Cauchy sequence axioms. Assume that $\exists \epsilon > 0$ and the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of natural numbers such that

$$r_n > s_n > n,\ W(\varsigma_{r_n}, \varsigma_{s_n}, t) \le 1-\epsilon,\ M(\varsigma_{r_{n-1}}, \varsigma_{s_{n-1}}, t) > 1-\epsilon, W(\varsigma_{s_{n-1}}, \varsigma_{s_n}, t) > 1-\epsilon \qquad (3.7)$$

$\forall n \in N \cup \{0\}$. Thus

$$1-\epsilon \ge W(\varsigma_{r_n}, \varsigma_{s_n}, t) \ge W(\varsigma_{r_{n-1}}, \varsigma_{r_n}, \tfrac{t}{2}) \circledast W(\varsigma_{r_n}, \varsigma_{s_n} \tfrac{t}{2}) \le W(\varsigma_{r_{n-1}}, \varsigma_{r_n}, \tfrac{t}{2}) \circledast (1-\epsilon)$$

By using (3.6) and the above inequality, we get

$$W(\varsigma_{r_n}, \varsigma_{s_n}, t) = 1-\epsilon, for\ every\ t. \qquad (3.8)$$

Moreover from (3.2)

$$\phi(W(\varsigma_{r_n}, \varsigma_{s_n}, t)) \le \lambda\phi(W(\varsigma_{r_{n-1}}, \varsigma_{s_{n-1}}, t)) < W(\varsigma_{r_{n-1}}, \varsigma_{s_{n-1}}, t)$$

Therefore, from (3.7)

$$1-\epsilon \ge W(\varsigma_{r_n}, \varsigma_{s_n}, t) < W(\varsigma_{r_{n-1}}, \varsigma_{s_{n-1}}, t) > 1-\epsilon.$$

It is a contraction, shows that the sequence $\{\varsigma_n\}_{n\in N}$ is a $W$-Cauchy sequence in $W$-complete fuzzy metric space $(E, W, \circledast)$. Thus, we observe that $\exists\varsigma \in E$ satisfying $lim_{n\to\infty}\varsigma_n = \varsigma$. Further, we will prove that $\varsigma$ fixed point of

$$W(f\varsigma, \varsigma, t) = \lim_{n\to\infty} W(f\varsigma_n, \varsigma_n, t)$$

$$= \lim_{n\to\infty} W(\varsigma_{n+1}, \varsigma_n, t)$$

$$= 1.$$

For uniqueness, suppose that there exists $\vartheta(\neq \varsigma) \in E$ such that $f(\vartheta) = \vartheta$, therefore

$$0 < W(f\varsigma, f\vartheta, t) = M(\varsigma, \vartheta, t) < 1,$$

then

$$\phi(W(\varsigma, \vartheta, t)) = \phi(W(f\varsigma, f\vartheta, t)) \leq \alpha\phi(W(\varsigma, \vartheta, t)),$$

which is a contradiction, hence fixed point is unique.

**Theorem 3.4** Let $(E, W, \circledast)$ be an $W$-complete fuzzy metric space and $f: E \to E$ be a self-map on $E$ and assume that $\phi: R^+ \to R^+$ a function satisfies the properties (M-1), (M-2) and (M-3). Furthermore, let $\lambda \in (0,1)$. If for any $t > 0$, $f$ meets the following condition

$$\frac{1}{2} W(\varsigma, f\varsigma, t) < W(\varsigma, \vartheta, t) \Rightarrow \phi\big(W(f\varsigma, f\vartheta, t)\big) \leq \alpha\phi\big(W(\varsigma, \vartheta, t)\big), \quad (3.9)$$

where $\varsigma, \vartheta \in E$ and $\varsigma \neq \vartheta$, then $f$ has a unique fixed point.

**Proof:** Let $\varsigma_0 \in E$ and define a sequence $\{\varsigma_n\}_{n=1}^{\infty}$ by

$$\vartheta_{n+1} = f\varsigma_n, \quad n = 0,1,2,\ldots \qquad (3.10)$$

If there exists $n \in N$ such that $W(\varsigma_n, f\varsigma_n, t) = 1$, then proof is complete. So, suppose that $0 < W(\varsigma_n, f\varsigma_n, t) < 1, \forall n \in N$. Therefore

$$\frac{1}{2}W(\varsigma_n, f\varsigma_n, t) < W(\varsigma_n, f\varsigma_n, t),\ \forall n \in N. \qquad (3.11)$$

For any $n \in N$, we have

$$\phi(W(f\varsigma_n, f^2\varsigma_n, t)) \leq \alpha\phi(W(\varsigma_n, f\varsigma_n, t))\ \ i.e.\ \phi(W(\varsigma_{n+1}, f\varsigma_{n+1}, t)) \leq \alpha\phi(W(\varsigma_n, f\varsigma_n, t)),$$

by repeating this process, we get

$$\phi(W(\varsigma_n, f\varsigma_n, t)) \leq \alpha^n\phi(M(\varsigma_0, f\varsigma_0, t)), \qquad (3.12)$$

as $n \to \infty$ we get, $lim_{n\to\infty}\phi(W(\varsigma_n, f\varsigma_n, t)) = 0$, which together (M-2) gives

$$\lim_{n\to\infty} W(\varsigma_n, f\varsigma_n, t) = 1. \qquad (3.13)$$

Now, claim that $\{\varsigma_n\}_{n\in N}$ is a Cauchy sequence. Contrary, suppose that there exist $\epsilon > 0$ and the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of natural numbers such that

$$r_n > s_n > n,\ W(\varsigma_{r_n}, \varsigma_{s_n}, t) \leq 1-\epsilon,\ W(\varsigma_{r_{n-1}}, \varsigma_{s_{n-1}}, t) > 1-\epsilon, W(\varsigma_{s_{n-1}}, \varsigma_{s_n}, t) > 1-\epsilon \qquad (3.14)$$

$\forall n \in N \cup \{0\}$. Thus

$$1-\epsilon \geq W(\varsigma_{r_n}, \varsigma_{s_n}, t) \geq W(\varsigma_{r_{n-1}}, \varsigma_{r_n}, \frac{t}{2}) \circledast W(\varsigma_{r_n}, \varsigma_{s_n} \frac{t}{2}) \leq W(\varsigma_{r_{n-1}}, \varsigma_{r_n}, \frac{t}{2}) \circledast (1-\epsilon)$$

By using (3.5) and the above inequality, we get

$$\lim_{n\to\infty} W(\varsigma_{r_n}, \varsigma_{s_n}, t) = 1-\epsilon, for\ \ every\ \ t. \qquad (3.15)$$

Moreover from (3.1)

$$\phi(W(\varsigma_{r_n},\varsigma_{s_n},t)) \leq \lambda\phi(W(\varsigma_{r_{n-1}},\varsigma_{s_{n-1}},t)) < W(\varsigma_{r_{n-1}},\varsigma_{s_{n-1}},t)$$

Therefore, from (3.6)

$$1-\epsilon \geq W(\varsigma_{r_n},\varsigma_{s_n},t) < W(\varsigma_{r_{n-1}},\varsigma_{s_{n-1}},t) > 1-\epsilon.$$

It is a contraction, shows that the sequence $\{\varsigma_n\}_{n\in N}$ is a $W$-Cauchy sequence in $W$-complete fuzzy metric space $(E,W,\circledast)$. Thus, we observe that $\exists\varsigma \in E$ satisfying $lim_{n\to\infty}\varsigma_n = \varsigma$, i.e.

$\lim_{n\to\infty} W(f\varsigma_n, f\varsigma, t) = 1$

therefore

$$1 = \lim_{n\to\infty} W(f\varsigma_n, f\varsigma, t) = \lim_{n\to\infty} W(\varsigma_{n+1}, f\varsigma, t) = W(\varsigma, f\varsigma, t).$$

For uniqueness, let us suppose that $\vartheta(\neq\varsigma) \in E$ such that $f\varsigma = \varsigma \neq \vartheta = f\vartheta,$

So, we have $\frac{1}{2} = \frac{1}{2}W(\varsigma, f\varsigma, t) < W(\varsigma,\vartheta,t)$ and from the definition we obtain

$$\phi(W(\varsigma,\vartheta,t)) = \phi(W(f\varsigma,f\vartheta,t)) \leq \alpha\phi(W(\varsigma,\vartheta,t)),$$

which is a contradiction. Thus, the fixed point is unique.

## 4. Conclusion

The objective of this work is to study fuzzy $B$-types of contraction and mappings in the settings of fuzzy metric space and establish fixed point results. The study's main findings, in particular, expand and extend a fixed-point theorem first proposed by Bijender in 2021. We hope that the findings investigated in this paper provide an important and technically sound contribution to the field and will be useful to researchers for further promotion and enhancement of their theoretical work in the field of partial metric spaces. For the purpose of future scope, some further generalization can be made through $B$-contraction in the setting of partial metric spaces, metric spaces, and metric-like spaces.

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