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Correlation of Altering JS-metric with Dislocated Metric

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Abstract

The JS-metric space is a generalized metric space that was first established by Jleli and Samet in the year 2015. We have extended this metric with the aid of altering distance functions and commenced the concept of Altering JS-metric space. Hitzler and Seda introduced the idea of dislocated metric space in the year 2000. In this article, we have examined certain properties of the Altering JS-metric spaces and have discussed the interrelation between the dislocated metric space and the Altering JS-metric space.

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1. Introduction

Metric space is one of the important concepts in the area of analysis which has significance in diverse disciplines of science and humanities. The inquiry of metric spaces was initiated as a distance functional by a French mathematician Frechet [10] in the year 1906 in his doctoral thesis. Later Hausdorff [9] coined the term metric space in 1914. Thereby, a lot of generalisations of the metric spaces started to emerge in the domain of research over the years. In the early stages, generalization was done by excluding some conditions of a metric and in the latter stages the distance values were altered and

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in recent times, the conditions of a metric were remodified to obtain new sort of Generalized metric spaces. Dislocated metric space was first established by Hitzler and Seda [4] in the year 2000. Dislocated metric space is a restricted metric space which may have a non-zero self-distance. The dislocated metric space has its own importance in several areas of logic programming and electrical engineering. Jleli and Samet [1] analysed a generalised metric known to be the JS-metric in 2015. This metric revives a lot of other metric variants. The Altering JS-metric space [5] is a variant which amalgamated the JS-metric and altering distance function. The concept of altering distance function was initiated by Delbosco [2] and Scof [3] in their fixed-point theorems in complete metric spaces. In 1984, Khan [6] used the notion of altering distance function and proved some contractive fixed-point theorems. A huge number of researchers studied the Altering distance function and investigated their role in fixed point theorems [7, 8].

In this article, we have investigated the Altering JS-metric space which is an extension of the JS-metric metric space with the aid of altering distance function. Examples are furnished for a deeper understanding of the definition and rudimentary properties of this space are explored. We have also discussed certain properties of this extended metric and have given the interrelation between the dislocated metric space and the Altering JS-metric space.

2. Preliminaries

The metric due to Jleli and Samet [1] is given as follows:

Consider a non-empty set $\mathbb{A}$ and a mapping $\theta: \mathbb{A} \times \mathbb{A} \to [0, \infty]$. For every $a \in \mathbb{A}$, let us define the set

$S(\theta, \mathbb{A}, a) = \{\{a\ n\} \subset \mathbb{A}: \lim_{n\to\infty}\theta(a_n,a)=0\}$

Definition 2.1. Let $\mathbb{A}$ be a non-empty set and $\theta: \mathbb{A} \times \mathbb{A} \to [0, \infty]$ be a function satisfying the conditions:

(D1) $\theta(a,b) = 0 \Longrightarrow a=b, \forall\ a,b\in\mathbb{A}$,

(D2) $\theta(a,b)=\theta(b,a), \forall\ a,b\in\mathbb{A}$,

(D3) there exists $\kappa > 0$ such that

$a,b\in A, \{a_n\}\in S(\theta, A, a) \Longrightarrow \theta(a,b)\leq \kappa \limsup_{n\to\infty}\theta(a_n,b)$

Then θ is called a JS-metric and (A,θ) is called JS-metric space.

Definition 2.2. [2,3] A non-negative function $\Omega: [0,\infty) \to [0,\infty)$ is an altering distance function if

(1) Ω is continuous and monotonically non-decreasing,

(2) $\Omega(t) = 0$ iff $t = 0$,

(3) $h.t^r \leq \Omega(t)$ for all $t > 0$ and $h,r > 0$ are constants.

Let us denote the set of all altering distance functions by Σ. It is obvious that Σ is non-empty.

Example 2.3. All the functions $\Omega: [0,\infty) \to [0,\infty)$ defined below, satisfies (1)-(3)

and hence will be an altering distance function.

(i) The identity function defined by, $\Omega(t)=t$

(ii) $\Omega(t)=te^t$

(iii) $\Omega(t)=t.2^{t+1}$

It can be observed that, the altering distance function is from set of non-negative reals to itself and hence the adjunct of the above two concepts is corroborated since a metric serves this purpose.

Definition 2.4. [5] Let A be a non-empty set and $\theta: A \times A \to [0, \infty]$ be a given function. We say that θ is an Altering JS-metric on A if it satisfies the following axioms:

(J1) $\theta(a,b) = 0 \Longrightarrow a=b, \forall\, a,b \in A$,

(J2) $\theta(a,b) = \theta(b, a), \forall\, a,b \in A$,

(J3) there exists an altering distance function Ω and a constant κ > 0 such that

$a,b\in A, \{a_n\}\in S(\theta, A, a) \Longrightarrow \Omega[\theta(a,b)]\leq \limsup_{n\to\infty}\Omega[\kappa.\theta(a_n,b)]$

The pair (A, θ) is said to be an Altering JS-metric space.

The axiom (J3) is trivially true if the set S (θ, A, *a*) is empty for every element of A. It can be observed that, the Altering JS-metric space becomes a JS-metric space if $\Omega(t)=t$

Example 2.5. Let the set A=ℝ A = ℝ and θ: ℝ × ℝ → [0, ∞] be defined by $\theta(a,b) = |a-b|^3$

(J1) $\theta(a,b) = |a-b|^3 = 0 \Rightarrow a-b = 0 \Rightarrow a = b$

(J2) $\theta(a,b) = |a-b|^3 = |b-a|^3 = \theta(b,a)$

(J3) Let $a,b \in A, \{a_m\} \in S(\theta, A, a)$. Then, $\lim_{m\to\infty}\theta(a_m,a)=0 \Rightarrow \lim_{m\to\infty}|a_m-a|^3=0$.

Now for any $b \in A$, we have, $|b-b|^3=0 \Rightarrow \lim_{m\to\infty}|b-b|^3=0$.

Thus, $a_m \longrightarrow a$ and $b \longrightarrow b$ implies $|a_m-b|^3 \longrightarrow |a-b|^3$.

Now, $\theta(a,b) = \lim_{m\to\infty}|a_m-b|^3 = \limsup_{m\to\infty}|a_m-b|^3$

Let $\Omega \in \Sigma$, and since it is continuous and monotonically non-decreasing,

$\Omega[\theta(a,b)] = \Omega[\limsup_{m\to\infty}|a_m-b|^3] \le \limsup_{m\to\infty}\Omega[|a_m-b|^3]$

Hence, the pair (ℝ, θ) is said to be an Altering JS-metric space.

Similarly, the usual metric is an Altering JS-metric.

In the above example, for every $a \in A$, the collection S(θ, A, *a*) is non-void. Also, the set S(θ, A, *a*) is identical to that of the set S(d, A,*a*) where d is the usual metric defined on the set of positive reals.

Example 2.6. Let A be any arbitrary set with distance function θ (*a*,*b*) = c, c is

non-negative. Then (A, θ) is an Altering JS-metric space.

(J1) Since c is non-negative, Axiom (J1) holds trivially.

(J2) θ (*a*,*b*) = c = θ (*b*,*a*). Thus, Symmetricity holds.

The set S (θ, A, *a*) will be empty for every element of A and so the axiom (J3) is trivially true.

Now, as for the topological concept of convergence is defined in the conventional way as follows.

Definition 2.7. [5] Let (A, θ) be an Altering JS-metric space. Let $\{a_n\}$ be a sequence in A and $a \in$ A. We say that the sequence, $\{a_n\}$ is θ-convergent to a if,

$\lim_{n\to\infty}\theta(a_n, a) = 0$

(i.e.) $\{a_n\} \in S(\theta, A, a)$

In this case, a is called the θ-limit of $\{a_n\}$ and we write $a_n \xrightarrow{\theta} a$.

Example 2.8. Let (A ,θ) be defined as in Example 2.5. Consider the sequence $\left\{r + \frac{1}{n}\right\}_{n \in I}$ in $\mathbb{R}$, where r is any positive real number. Then the sequence $\left\{r + \frac{1}{n}\right\}$ is θ-convergent to r.

Definition 2.9. [4] Consider a non-empty set A. The function d: A × A → [0, ∞) is said to be a dislocated metric if it satisfies the conditions:

(i) $d(a,b) = 0 \Longrightarrow a=b, \forall\, a,b \in$ A,

(ii) $d(a,b) = d(b,a) \forall\, a,b \in$ A,

(iii) $d(a,b) \leq d(a,c) + d(c,b)$, ∀ a,b,c∈A.

Then (A,d) is the dislocated metric space.

Example 2.10. Consider the non-empty set A = R and the function d: A × A → [0, ∞) defined by, $d(a,b) = \begin{cases} r & if\ a = b \\ r+1 & if\ a \neq b \end{cases}$

where r is some positive real number. Axioms (i) and (ii) are trivial. Now, to check the triangle inequality, consider the below cases.

Case 1: If $a=b$, then $d(a,b) = r$

For any c≠a, we have, $d(a,c)+d(c,b)=2(r+1)$

For any $c = a = b$, we get, $d(a,c)+d(c,b)=2r$

Case 2: If $a \neq b$, then $d(a,b) = r+1$

For any c not equal to both a and b, we have, $d(a,c) + d(c,b)=2(r + 1)$

For any $c = a$ or $c = b$, we get, $d(a,c) + d(c,b)=2r + 1$

Thus, in all the cases $d(a,b) \leq d(a,c) + d(c,b)$. Hence $(\mathbb{R},d)$ is a dislocated metric space.

Note that, for $r = 0$ the above metric becomes a discrete metric.

Definition 2.11. [4] Let (A,d) be an Altering JS-metric space. Let $\{a_n\}$ be a sequence in A and $a \in A$. We say that the sequence, $\{a_n\}$ is d-convergent to a if,

$d(a_n,a) \rightarrow 0$ as $n \rightarrow \infty$

a is called the d-limit of $\{a_n\}$.

3. Properties of Altering JS-Metric Space

This section discusses some properties of the Altering JS-metric space.

Proposition 3.1. If $\theta_1,\theta_2, \ldots, \theta_n$ are Altering JS-metrics then $\max\{\theta_1,\theta_2,\ldots, \theta_n\}$ is also an Altering JS-metric for any finite n.

Proof.

Let A be a non-empty set and $\theta: A \times A \rightarrow [0, \infty]$ be defined by

$\theta(a,b) = \max\{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\}$

(i) $\theta(a,b) = 0 \Rightarrow max\{\theta_1(a,b),\theta_2(a,b),\ldots,\theta_n(a,b)\}= 0$

$\Rightarrow \theta_1(a,b) = \theta_2(a,b) = \ldots = \theta_n(a,b)=0$

Since $\theta_1, \theta_2, \ldots, \theta_n$ are Altering JS-metrics we get, $a = b$

(ii) $\theta(a,b)=max\{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\}$

$= max\{\theta_1(b,a), \theta_2(b,a), \ldots,\theta_n(b,a)\}$

$=\theta(b,a)$

(iii) Let $\{a_n\}$ be a sequence in A converging to a and Ω be an Altering distance function.

Now, $\Omega[\theta(a,b)] = \Omega[max\ \{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\}]$

Without loss of generality let,

$max\{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\} = \theta_i(a,b)$ for some $1 \le i \le$ n.

Hence, we have $\Omega[\theta(a,b)] = \Omega[\theta_i(a,b)]$

Since θ_i is an Altering JS-metric, by Axiom (J3), there exists $\Omega \in \Sigma$ and $\kappa > 0$ such that,

$$\Omega\ [\theta_i(a,b)] \ \le \ \lim_{n\to\infty} sup\ [\ \Omega(\ \kappa.\theta_i(a_n,b)]$$

Thus, we get,

$$\Omega[\theta(a,b)] \ \le\ lim_{n\to\infty}\ sup\ [\ \Omega(\ \kappa.\theta_i(a_n,b)]$$

$$=\ lim_{n\to\infty}\ sup\ [\ \Omega(\kappa.max\{\ \theta_1(a_n,b),\theta_2(a_n,b),\ldots,\theta_n(a_n,b)\})]$$

$$\Omega[\theta(a,b)] \ \le\ \lim_{n\to\infty} sup\ [\ \Omega(\ \kappa.\theta(a_n,b)]$$

Hence $max\ \{\theta_1,\theta_2,\ldots,\theta_n\}$ is an Altering JS-metric.

Proposition 3.2. If $\theta_1,\theta_2 \ldots, \theta_n$ are Altering JS-metrics then $min\{\theta_1,\theta_2,\ldots,\theta_n\}$ is also an Altering JS-metric for any finite n.

Proof.

Let A be a non-empty set and θ: A × A ⟶ [0, ∞] be defined by

$\theta(a,b) = min\{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\}$

(i) $\theta(a,b) = 0 \Rightarrow min\{\theta_1(a,b),\theta_2(a,b),\ldots,\theta_n(a,b)\} = 0$

$\Rightarrow \theta_i(a,b) = 0$ for some $1 \le i \le n$.

Since θ_i is an Altering JS-metric we get, $a = b$.

(ii) $\theta(a,b)=min\{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\}$

$= min\{\theta_1(b,a), \theta_2(b,a), \ldots,\theta_n(b,a)\}$

$=\theta(b,a)$

(iii) Let $\{a_n\}$ be a sequence in A converging to a and Ω be an Altering distance function.

Now, $\Omega[\theta(a,b)] = \Omega[min\{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\}]$

Without loss of generality let,

$\min\{\theta_1(a,b), \theta_2(a,b),\ldots,\theta_n(a,b)\}= \theta_i(a,b)$ for some $1 \le i \le n$.

Hence, we have $\Omega[\theta(a,b)] = \Omega[\theta_i(a,b)]$

Since θ_i is an Altering JS-metric, by Axiom (J3), there exists $\Omega \in \Sigma$ and $\kappa > 0$ such that,

$$\Omega[\theta_i(a,b)] \le \lim_{n\to\infty} sup\ [\Omega(\kappa.\theta_i(a_n,b)]$$

Thus, we get,

$$\Omega[\theta(a,b)] \le lim_{n\to\infty} sup\ [\Omega(\kappa.\theta_i(a_n,b)]$$

$$= lim_{n\to\infty} sup\ [\Omega(\kappa.min\{\theta_1(a_n,b), \theta_2(a_n,b), \ldots, \theta_n(a_n,b)\})]$$

$$\Omega[\theta(a,b)] \le \lim_{n\to\infty} sup\ [\Omega(\kappa.\theta(a_n,b)]$$

Hence $min\{\theta_1, \theta_2, \ldots, \theta_n\}$ is an Altering JS-metric.

Proposition 3.3. If θ_1: A × A → [0,∞] and θ_2: A × A → [0,∞]are Altering JS-metrics such that the sets S (θ_1, A,a) and S (θ_2, A,a) are empty for every element of A , then $\theta_1 + \theta_2$ is also an Altering JS-metric.

Proof.

Since the sets S (θ_1, A, a) and S (θ_2, A, a) are empty for every element of A, it is enough to check the axioms (J1) and (J2).

Now, let θ: A × A → [0, ∞] be given by

θ (a,b)=θ_1 (a,b)+θ_2 (a,b)

(i) $\theta(a,b)=0 \Rightarrow \theta_1 (a,b)+\theta_2 (a,b)=0$

$\Rightarrow \theta_1 (a,b)=\theta_2 (a,b)=0$

Since θ_1 and θ_2 are Altering JS-metric spaces we get, $a = b$

(ii) $\theta(a,b)=\theta_1 (a,b)+\theta_2 (a,b)$

$= \theta_1 (b,a)+\theta_2 (b,a)$

$= \theta(b,a)$

Hence $\theta_1 + \theta_2$ is an Altering JS-metric.

Proposition 3.4. If θ_1: $A \times A \to [0,\infty]$ and θ_2: $A \times A \to [0,\infty]$ are Altering JS-metrics such that the sets $S(\theta_1, A, a)$ and $S(\theta_2, A, a)$ are empty for every element of A, then $\theta_1 \theta_2$ is also an Altering JS-metric.

Proof.

(i) $\theta(a,b)=0 \Rightarrow \theta_1 (a,b)\, \theta_2 (a,b)=0$

$\Rightarrow$ Either $\theta_1 (a,b)=0$ or $\theta_2 (a,b)=0$

Again, since θ_1 and θ_2 are Altering JS-metric spaces we get, $a = b$.

(ii) $\theta(a,b)=0 = \theta_1 (a,b)\, \theta_2 (a,b)$

$=\theta_1 (b,a)\, \theta_2(b,a)$

$\theta(b,a)$

Hence $\theta_1 \theta_2$ is an Altering JS-metric.

Observations:

1. The above two propositions also hold true if the sets $S(\theta_1, A, a)$ and $S(\theta_2, A, a)$ are non-empty. (i.e.) the sum and product of two Altering JS-metrics is an Altering JS-metric.
2. In general, $\theta_1+ \theta_2+ \theta_3+...+ \theta_n$ and $\theta_1\theta_2\theta_3... \theta_n$ are Altering JS-metrics provided $\theta_1,\theta_2, \ldots, \theta_n$ are Altering JS-metrics such that the sets $S(\theta_1, A, a)$, $S(\theta_2, A, a)$, $S(\theta_3, A, a)$,… $S(\theta_n, A,a)$ may or may not be empty.

4. Comparison of Altering JS-Metric Space and Dislocated Metric Space

In this section we have discussed the relationship between the Altering JS-metric space and the dislocated metric space.

Proposition 4.1. Every dislocated metric space is an Altering JS-metric space.

Proof.

Let (A, d) be a dislocated metric space.

Axioms (J1) and (J2) are trivially true. Hence, to prove that (A, d) is an altering JS-metric space, it is enough to prove that d satisfies the axiom (J3).

Let $\{a_n\}$ be the sequence converging to a in A. By the triangle inequality, we get,

$d(a,b) \leq d(a,a_n)+d(a_n,b)$

Taking $\lim_{n\to\infty} sup$ on both sides,

$$d(a,b) \leq \limsup_{n\to\infty} d(a_n,b)$$

Let $\Omega \in \Sigma$, then, $\Omega(d(a,b)) \leq \Omega(\limsup_{n\to\infty} d(a_n,b))$

Since Ω is continuous and monotonically non decreasing,

$$\Omega(d(a,b)) \leq \limsup_{n\to\infty} \Omega(d(a_n,b)\,)\,)$$

Thus (A, d) ia an Altering JS-metric space.

The converse need not be true. (i.e.) An Altering JS-metric space need not be a dislocated metric space.

Example 4.2. Consider the example defined in Example 2.5. with A = $\mathbb{R}$ and the metric function $\theta: \mathbb{R} \times \mathbb{R} \to [0,\infty]$ given by $\theta(a,b) = |a-b|^3$. Then ($\mathbb{R}$, θ) is an Altering JS-metric space.

If we consider, $a=-1$, $b = 1$ and c = $-1/2$, then

$|-1 -1|^3 = 8$ and $|-1+1/2|^3+|-1/2-1|^3= 28/8$

$$\Longrightarrow \theta(-1,1) \nleq \theta(-1,-1/2) + \theta(-1/2,1)$$

And thus ($\mathbb{R}$, θ) fails to satisfy the triangle inequality which implies that it is not a dislocated metric space.

Observations:

A constant sequence need not be convergent in either of the Altering JS-metric space or the dislocated metric space. This is because both the spaces allow the dislocation of the points. For a constant sequence $\{a_n\}$ in A, $a_n=a\ \forall\ n$ and hence $\lim_{n\to\infty}\theta(a_n,a) = 0 \Longrightarrow \lim_{n\to\infty}\theta(a,a) = 0$. It can be noted that the axiom $\theta(a, a) = 0$ ensures the convergence of constant sequence.

In [5] it has been proved that the constant sequence need not be convergent in an Altering JS-metric space. Here the Example 2.8, contradicts the convergence of constant sequence in a dislocated metric space and the Example 2.6, abdicates this convergence in an Altering JS-metric space.

5. Conclusion

Thus, we have investigated the Altering JS-metric space and the dislocated metric space. Also, we have analysed the interrelation between the Altering JS-metric space and the dislocated metric space. It has also been concluded that the existence of the zero self-distance contributes to the convergence of a constant sequence in a metric space. The authors may carry out the study of incorporating an Altering JS-metric space over a dislocated metric space in the upcoming works.

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