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Topologies Induced on Vertex Set of Graphs

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Abstract

In this paper, subbasis for different topologies on the vertex set of a simple graph without isolated vertices is introduced. Some properties of these topologies are investigated. The interior, closure, exterior, and boundary of a vertex induced subgraph were defined, and some basic properties were studied.

Keywords: subbasis for a topology, discrete topology, vertex induced subgraph, interior, closure, exterior, boundary
AMS Subject classification: 57M15, 54A10, 54H99

1. Introduction

A link between graph theory and topology can be made by defining a relation on the graph. Graphs can be regarded as a one-dimensional topological space. While discussing connected graphs or homeomorphic graphs, the adjectives have the same meaning as in topology. So, graph theory can be regarded as a subset of the topology of, say, one-dimensional simplicial complexes. A connected graph has a natural distance function. So it can be viewed as a kind of discrete metric space. So graph theory can be regarded

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as a subset of the topology of metric spaces. A topological space is defined by points and open sets. It could be constructed as a bipartite graph. The points are vertices in one partite set, the open sets are vertices in the other partite set, and each open set is joined by edges to its elements. In the strict definitional sense, it is probably possible to get all concepts of graph theory expressed in the language of topology. A relation on graph represents a key for bridging graph theory and topological structures. The relation induces new types of topological structures in the graph. In 1967, J.W. Evans et al. [7] showed that there is a one-to-one correspondence between the labeled transitive directed graph with n points and the labeled topologies on n points. In 1967, S.S. Anderson et al. [1] investigated the lattice-graph of the topologies of transitive directed graphs presented by J.W. Evans et al.[7]. In 2010, C. Marijuan [11] studied the relationship between directed graphs and finite topologies. In 2013, M. Amiri et.al. [4] induced topology on the vertex set of an undirected graph. In 2018, Kilieman and Abdulkalex [10] associated incidence topology with vertex set of simple graphs without isolated vertices. In this paper, a different family of subbasis for topology is defined, and the type of topology generated by subbasis via some particular graphs is discussed. Also, some properties of interior, closure, exterior, and boundary of vertex induced subgraphs of a graph are explored.

2. Preliminaries

Fundamental definitions and preliminaries of graph theory and topological spaces can be found in the sources [5], [6], [9].

Shokry Nada et al.[12] defined a relation R on $V(G)$ by $R = \{((2m_x + n_x)_x, (2m_y + n_y)_y): x, y \in V(G)\}$,where m_x and m_y are the number of loops of vertices x and y, respectively and n_x and n_y are the number of multiple edges of vertices x and y, respectively. Then he defined the post class for each v_i as the open neighbourhood of v_i in R which is denoted as v_iR and constructed a subbase for a topology by $S_G = \cup \{v_iR: v_i \in V(G)\}$. In this paper, the relations adjacency, non-adjacency, incidence, non-incidence on the vertex set of a graph are used to generate subbasis for topologies on the vertex set of graphs.

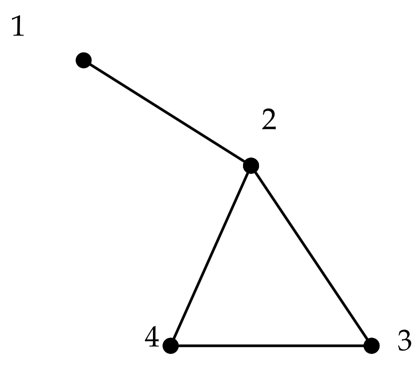
Throughout the paper, the graph under discussion is the simple undirected graph which is not star graph.

Definition 2.1 Let $G = (V(G), X(G))$ be a graph. For $v \in V(G)$, the neighbourhood set N_v of v is defined as $N_v = \{u \in V(G): uv \in X(G)\}$ and the non-neighbourhood set NA_v of v as $NA_v = \{u \in V(G): uv \notin X(G)\}$. For $e \in X(G)$, define $I(e)$ as the set of all vertices incident with e and $Ni(e)$ as the set of all vertices not incident with e.

Definition 2.2 Let $G = (V(G), X(G))$ be a graph without isolated vertices. Define S_N as the family of N_v for all $v \in V(G)$, (i.e.) $S_N = \{N_v: v \in V(G)\}$. Then S_N forms a subbase for a topology T_A on $V(G)$ and the pair $(V(G), T_A)$ is called graph adjacency topological space. Define S_I as the family of $I(e)$ for all $e \in X(G)$, (i.e.) $S_I = \{I(e): e \in X(G)\}$. Then S_I forms a subbase for a topology T_I on $V(G)$. The pair $(V(G), T_I)$ is called graph incidence topological space. For $|X(G)| > 2$, define S_{Ni} as the family of $Ni(e)$ for all $e \in X(G)$, (i.e.) $S_{Ni} = \{Ni(e): e \in X(G)\}$. Then S_{Ni} forms a subbase for a topology T_{Ni} on $V(G)$. The pair $(V(G), T_{Ni})$ is called graph non-incidence topological space. If $|V(G)| = n$ and $0 \leq d(v) \leq n - 2$ for all $v \in V(G)$, define S_{NA} as the family of NA_v for all $v \in V(G)$, (i.e.) $S_{NA} = \{NA_v: v \in V(G)\}$. Then S_{NA} forms a subbase for a topology T_{NA} on $V(G)$ and the pair $(V(G), T_{NA})$ is called graph non-adjacency topological space.

The sets in the topologies are called open sets, and the complement of open sets is called closed sets.

Example 2.3: In this example, topologies using the relations adjacency, incidence, and non-incidence are generated; but a topology using the relation non-adjacency cannot be generated.



$S_N = \{\{2\},\{1,3,4\},\{2,4\},\{2,3\}\}$, $\boldsymbol{B} = \{\varphi,\{2\},\{1,3,4\},\{2,4\},\{2,3\},\{4\},\{3\}\}$

$$T_A = \{\varphi,\{2\},\{1,3,4\},\{2,4\},\{2,3\},\{4\},\{3\},\{3,4\},\{2,3,4\},\{1,2,3,4\}\}$$

$S_I = \{\{1,2\},\{2,3\},\{3,4\},\{2,4\}\}$, $\boldsymbol{B} = \{\varphi,\{2\},\{1,2\},\{2,3\},\{3,4\},\{2,4\},\{4\},\{3\}\}$

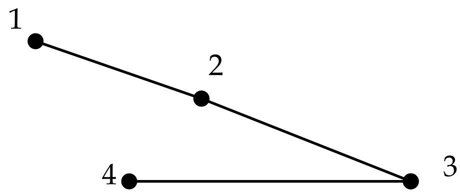
$$T_I = \{\varphi,\{2\},\{3\},\{4\},\{1,2\},\{2,3\},\{3,4\},\{2,4\},\{2,3,4\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}$$

$S_{Ni} = \{\{1,2\},\{3,4\},\{1,4\},\{1,3\}\}$,

$\boldsymbol{B} = \{\varphi,\{1\},\{1,2\},\{1,3\},\{1,4\},\{3,4\},\{4\},\{3\}\}$

$T_{Ni} = \{\varphi,\{1\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{1,2,3\},\{1,2,3,4\}\}$.

Example 2.4 In this example, topology using the relation non-adjacency is generated.



$S_{NA} = \{\{3,4\},\{4\},\{1\},\{1,2\}\}$, $\mathbf{B} = \{\varphi,\{3,4\},\{4\},\{1\},\{1,2\}\}$

$$T_{NA} = \{\varphi,\{3,4\},\{4\},\{1\},\{1,2\},\{1,4\},\{1,3,4\},\{1,2,4\},\{1,2,3,4\}\}$$

3. Properties of Topologies on Vertex Set of a Graph

In this section, the nature, and properties of topologies generated by subbasis on vertex set of a graph are presented.

Theorem 3.1 Let $G = (V(G), X(G))$ be a graph without isolated vertices.

(a) If $|X(G)| > 1$, then

1. If $v \in V(G)$ is an end vertex then $\{v\} \notin T_A$ and $\{v\} \notin T_I$.
2. If $deg(v) \geq 2$ for all $v \in V(G)$, then T_I is a discrete topology on $V(G)$.

(b) If $|X(G)| > 2$ and $v \in V(G)$ is an end vertex then $\{v\} \in T_{Ni}$.

Proof: (a)1. If $|X(G)| > 1$ and $v \in V(G)$ is an end vertex, then$v$belongs to neighbourhood set of exactly one vertex and$v$belongs to I_e for exactly one edge in G. So $\{v\} \notin T_A$ and $\{v\} \notin T_I$.

2. If $|X(G)| > 1$ and $deg(v) \geq 2$ for all $v \in V(G)$, then at least two distinct edges, say, e_i and $e_j, i \neq j$, are incident with$v$so that $I(e_i) \cap I(e_j) = \{v\}$. Thus for all $v \in V(G)$, $\{v\}$ belongs to the basis of T_I so that T_I is a discrete topology on $V(G)$.
(b) If $|X(G)| > 2$ and v belongs to $V(G)$ is an end vertex, then only one edge, say e_1 is incident with v. All other edges are not incident with v. Hence $\{v\}$ belongs to the basis of T_{Ni} and $\{v\} \in T_{Ni}$.
Theorem 3.2 Let $P_n = v_1 e_1 v_2 e_2 v_3 \ldots . e_{n-2} v_{n-1} e_{n-1} v_n$ be a path of length n. Then $\{v_2\}$ and $\{v_{n-1}\}$ do not belong to T_{Ni}.
Proof: InP_n,$v_2 \notin Ni(e_1)$and$Ni(e_2)$;$v_{n-1} \notin Ni(e_{n-2})$ and $Ni(e_{n-1})$ and also $v_2 \in Ni(e_3), Ni(e_4), \ldots . Ni(e_{n-1})$ along with v_n. Thus $\{v_2\}$ and $\{v_{n-1}\}$ do not belong to basis of T_{Ni} and $\{v_2\}$ and $\{v_{n-1}\}$ do not belong to T_{Ni}.
From the proof of above theorems, the following observations can be made.

1. On the vertex set of K_n, for $n \geq 3$, T_I, T_{Ni}, T_A are discrete topologies.
2. On the vertex set of a non-trivial tree, T_A, T_{NA} and T_I are not discrete topologies.
3. On the vertex set of a connected Eulerian graph, T_I is a discrete topology.
4. On the vertex set of C_n, for $n \geq 3$, T_I, T_{Ni}, T_{NA} are discrete topologies.
5. For a cutvertex v of a graph G, $\{v\} \in T_I$ and $\{v\} \notin T_{NA}$.
6. On the vertex set of $K_{m,n}$, for $m, n > 1$, T_{Ni} and T_{NA} are discrete topologies.
7. If $G_1 = (V(G_1), X(G_1))$ and $G_2 = (V(G_2), X(G_2))$ are two graphs with $S_N(G_1)(S_{NA}(G_1) respy.) = \{B_1, B_2, \ldots, B_n\}$ and $S_N(G_2)(S_{NA}(G_2) = \{A_1, A_2, \ldots, A_m\}$ as a subbasis for $T_A(G_1)$ ($T_{NA}(G_1)$ and $T_A(G_2)$ ($T_{NA}(G_2)$) respectively, then $\{B_1 \cup V(G_2), \ldots, B_n \cup V(G_2), A_1 \cup V(G_1), \ldots, A_m \cup V(G_1)\}$ is a subbasis for T_A of G_1+G_2 (T_{NA} of $G_1 \cup G_2$) and $\{B_1, B_2, \ldots, B_n, A_1, A_2, \ldots, A_m\}$ is a subbasis for T_A of $G_1 \cup G_2$ (T_{NA} of G_1+G_2).

If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ and $S_{N_i}, i = 1,2, \ldots k$ is a subbasis for T_A of G_i, and $S_{I_i}, i = 1,2, \ldots k$ is a subbasis for T_I of G_i, then the family of all elements of all S_{N_i} forms a subbasis for T_A of G and the family of all elements of all S_{I_i} form a subbasis for T_I of G

Let G be a disconnected graph with components $G_1, G_2, \ldots., G_k$. For $i = 1,2, \ldots. k$, if $\{B_{i1}, B_{i2}, \ldots., B_{im}\}$ is a subbasis for T_{Ni} (T_{NA} respy.) of G_i, then

$\{B_{11} \cup V(G_2) \cup \ldots.\cup V(G_{i-1}) \cup V(G_i) \cup V(G_{i+1}) \cup \ldots.\cup V(G_k)$,

$B_{12} \cup V(G_2) \cup \ldots.\cup V(G_{i-1}) \cup V(G_i) \cup V(G_{i+1}) \cup \ldots.\cup V(G_k)$, ...,

$B_{1m} \cup V(G_2) \cup \ldots.\cup V(G_{i-1}) \cup V(G_i) \cup V(G_{i+1}) \cup \ldots.\cup V(G_k)$,

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$B_{i1} \cup V(G_1) \cup V(G_2) \cup \ldots.\cup V(G_{i-1}) \cup V(G_{i+1}) \cup \ldots.\cup V(G_k)$,

$B_{i2} \cup V(G_1) \cup V(G_2) \cup \ldots.\cup V(G_{i-1}) \cup V(G_{i+1}) \cup \ldots.\cup V(G_k)$, ...,

$B_{im} \cup V(G_1) \cup V(G_2) \cup \ldots.\cup V(G_{i-1}) \cup V(G_{i+1}) \cup \ldots.\cup V(G_k)$,

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$B_{k1} \cup V(G_1) \cup \ldots.\cup V(G_{i-1}) \cup V(G_i) \cup V(G_{i+1}) \cup \ldots.\cup V(G_{k-1})$,

$B_{k2} \cup V(G_1) \cup \ldots.\cup V(G_{i-1}) \cup V(G_i) \cup V(G_{i+1}) \cup \ldots.\cup V(G_{k-1})$, ...,

$B_{km} \cup V(G_1) \cup \ldots.\cup V(G_{i-1}) \cup V(G_i) \cup V(G_{i+1}) \cup \ldots.\cup V(G_{k-1})\}$

is a subbasis for $T_{Ni}(G)$ ($T_{NA}(G)$).

1. If $G_1 = (V(G_1), X(G_1))$ and $G_2 = (V(G_2), X(G_2))$ are two graphs without isolated vertices and $|X(G_1)| > 2, |X(G_2)| > 2$ with $S_{Ni}(G_1) = \{B_1, B_2, \ldots, B_n\}$ and $S_{Ni}(G_2) = \{A_1, A_2, \ldots A_m\}$ as subbasis for $T_{Ni}(G_1)$ and $T_{Ni}(G_2)$ respectively, then $\{B_1 \cup V(G_2), \ldots., B_n \cup V(G_2), A_1 \cup V(G_1), A_2 \cup V(G_1), \ldots.., A_m \cup V(G_1)\}$ is a subbasis for $T_{Ni}(G_1 \cup G_2)$.
2. If $v \in V(G)$ is an end vertex then $\{v\} \in T_{NA}$.
3. If Pn = $v_1e_1v_2e_2v_3 \ldots. e_{n-2}v_{n-1}e_{n-1}v_n$ is a path of length n, then $\{v_2\}$ and $\{v_{n-1}\}$ do not belong to T_{NA}.

4. For any graph G with $d(v) \leq n-2$ for all $v \in V(G)$,T_A on $V(G)$=T_{NA} on $V(G^c)$ and T_A on $V(G^c)$=T_{NA} on V(G).
5. If G is a k-regular graph with $k \leq n-2$, then T_{NA} on *V(G)* is discrete.

4. Interior and Closure of Vertex Induced Subgraphs of a Graph

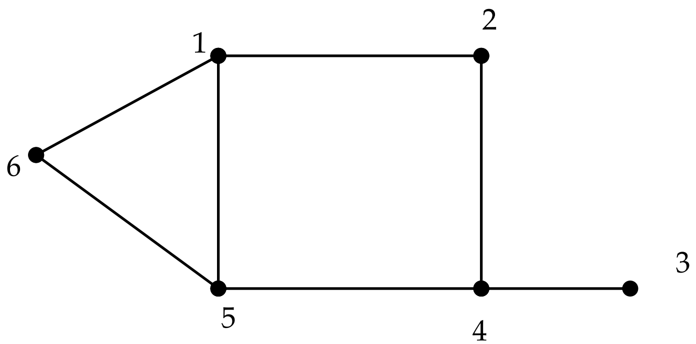
In topology, the interior and closure of a set are dual notions, and the exterior of a set is the complement of the closure. The interior, boundary, and exterior of a subset together partition the whole space into three blocks. Shokry Nada et al.[14] defined closure and interior of vertex set of subgraph $H$of a graph $G$by $cl(V(H)) = V(H) \cup \{v \in V(G) - V(H): vh \in E(G)$ for all $h \in V(H)\}$ and $int(V(H)) = V(G) - cl(V(G) - V(H))$

In this section, interior, closure of vertex induced subgraphs of graphs in terms of adjacency and incidence relations are defined, and basic properties of interior and closure are studied.

Definition 4.1 Let $G = (V(G), X(G))$ be a graph without isolated vertices and $(V(G), T_A)$ $((V(G), T_I)$ respy.) be a graph adjacency topological space (graph incidence topological space). Let Wbe a vertex induced subgraph of G. The closure of $V(W)$ is defined by $cl(V(W)) = V(W) \cup \{v \in V(G): N_v \cap V(W) \neq \varphi\}$

$(cl(V(W)) = V(W) \cup \{v \in V(G): v \in (I(e)), I(e) \cap V(W) \neq \varphi\})$ and interior of $V(W)$ is defined by $int(V(W)) = \{v \in V(G): N_v \subseteq V(W)\}$ $(int(V(W)) = \{v \in I(e): I(e) \subseteq V(W)\})$.

Example 4.2 Consider the following graph



$S_N = \{\{2,5,6\},\{1,4\},\{4\},\{3,2,5\},\{4,1,6\},\{1,5\}\}$.

cl({1,4,3}) = {1.2,3,4,5,6},cl({3,6}) = {1,3,4,5,6}, int({1,4,3}) = {2,3}, int({1,6}) = ϕ

$S_I = \{\{1,2\},\{2,4\},\{3,4\},\{4,5\},\{5,6\},\{6,1\},\{1,5\}\}$.

$cl(\{1,4,3\}) = \{1,2,3,4,5,6\}, int(\{1,4,3\}) = \{3,4\}$.

Theorem 4.3 Let $(V(G), T_A)$ be a graph adjacency topological space. Let W_1 and W_2 be vertex induced subgraphs of G. Then (i) $V(W_1) \subseteq cl(V(W_1))$

(ii) If $V(W_1) \subseteq V(W_2)$, then $cl(V(W_1)) \subseteq cl(V(W_2))$.

Proof: (i) The proof follows trivially from the definition of $cl(V(W_1))$.

(ii) Let v ∈ cl(V(W_1)). Then v ∈ V(W_1) or N_v ∩ V(W_1) ≠ ∅. Since V(W_1) ⊆ V(W_2), it follows that v ∈ V(W_2) or N_v ∩V(W_2) ≠ ∅.

So v ∈ Cl(V(W_2)). Thus cl(V(W_1)) ⊆ cl(V(W_2)).

Theorem 4.4

Let $(V(G), T_A)$ be a graph adjacency topological space. Let W_1 and W_2 be vertex induced subgraphs of G. Then

1. $cl(V(W_1) \cup V(W_2)) = cl(V(W_1)) \cup cl(V(W_2))$.
2. $cl(V(W_1) \cap V(W_2)) \subseteq cl(V(W_1)) \cap cl(V(W_2))$.
3. $V(W_1) \subseteq V(W_2) \Rightarrow int(V(W_1)) \subseteq int(V(W_2))$.
4. $int(V(W_1) \cap V(W_2)) = int(V(W_1)) \cap int(V(W_2))$.
5. $int(V(W_1)) \cup int(V(W_2)) \subseteq int(V(W_1) \cup V(W_2))$.

Proof: 1. Let v ∈ cl(V(W_1)∪V(W_2)). Then v ∈ V(W_1)∪V(W_2) or N_v ∩ (V(W_1) ∪V(W_2)) ≠ ∅ which implies v ∈V(W_1) or v ∈ V(W_2) or (N_v ∩ V(W_1)) ∪ (N_v ∩ V(W_2)) ≠ ∅. Hence v ∈ V(W_1) or $N_v \cap V(W_1) \neq$ ∅ or v ∈ V(W_2) or $N_v \cap V(W_2) \neq$ ∅. So v ∈ cl(V(W_1)) or v ∈ cl(V(W_2)). Thus v ∈cl(V(W_1)) ∪ cl(V(W_2)) and $cl(V(W_1) \cup V(W_2)) \subseteq cl(V(W_1)) \cup cl(V(W_2))$. By reversing the above steps, cl(V(W_1))∪ cl(V(W_2)) ⊆ cl(V(W_1) ∪ cl(V(W_2)) can be proved. Thus $cl(V(W_1) \cup V(W_2)) = cl(V(W_1)) \cup cl(V(W_2))$.

2. Let v $\in$ cl(V(W1) $\cap$ V(W2)). Then v $\in$ V(W1) $\cap$V(W2) or Nv $\cap$ (V(W1) $\cap$ V(W2)) $\neq \emptyset$ which implies v $\in$ V(W1) and v $\in$ V(W2)

or

(Nv $\cap$ $V(W1)) \cap (N_v \cap V(W2)) \neq \emptyset$. Hence v $\in$ V(W1) or Nv $\cap$ $V(W_1) \neq \emptyset$

and v $\in$ V(W2) or Nv $\cap$ $V(W_2) \neq \emptyset$. So v $\in$ cl(V(W1)) and v $\in$ cl(V(W2)).

Thus v $\in$ cl(V(W1)) $\cap$ cl(V(W2)) and $cl(V(W_1) \cap V(W_2)) \subseteq cl(V(W_1)) \cap cl(V(W_2))$.

3. Let v $\in$ int(V(W_1)). Then $N_v \subseteq V(W_1) \subseteq V(W_2)$. So v $\in$ int(V(W_2)) and int(V(W_1)) $\subseteq$ int(V(W_2)).c

4. Let v $\in$ int(V(W_1) $\cap$ V(W_2)). Then $N_v \subseteq V(W_1) \cap V(W_2)$. So $N_v \subseteq$ V(W1) and Nv $\subseteq$ V(W2). Hence v $\in$ int(V(W1) and v $\in$ intV(W2)). Thus $int(V(W_1) \cap V(W_2)) \subseteq int(V(W_1)) \cap int(V(W_2))$. By reversing the above steps, int(V(W_1))$\cap$int(V(W_2)) $\subseteq$ int(V(W_1)$\cap$V(W_2)).

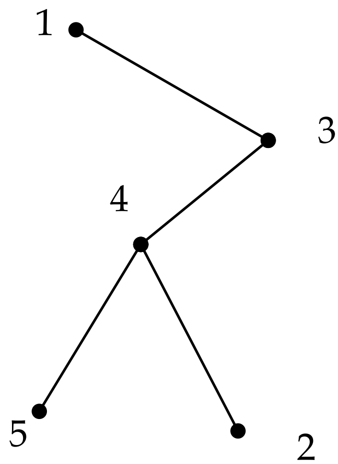
Hence $int(V(W_1) \cap V(W_2)) = int(V(W_1)) \cap int(V(W_2))$.

5. Let v $\in$ int(V(W_1)) $\cup$ int(V(W_2)). Then v $\in$ int(V(W_1)) or v $\in$ int(V(W_2)). So $N_v \subseteq V(W_1)$ or $N_v \subseteq V(W_2)$. Hence $N_v \subseteq V(W_1) \cup V(W_2)$ and v $\in$ int(V(W_1)$\cup$V(W_2)). Thus $int(V(W_1)) \cup int(V(W_2)) \subseteq int(V(W_1) \cup V(W_2))$.

We note that, in general, $int(V(W_1) \cup V(W_2)) \not\subset int(V(W_1)) \cup int(V(W_2))$ and

$cl(V(W_1)) \cap cl(V(W_2)) \not\subset cl(V(W_1) \cap V(W_2))$.

Example 4.5 Consider the following graph



$S_N = \{\{3\},\{4\},\{1,4\},\{2,3,5\},\{4\}\}$.

$int(\{1,2\}) = \varphi$, $int(\{2,3,4\}) = \{1,2,5\}$, $int(\{1,2,3,4\}) = \{1,2,3,5\}$.

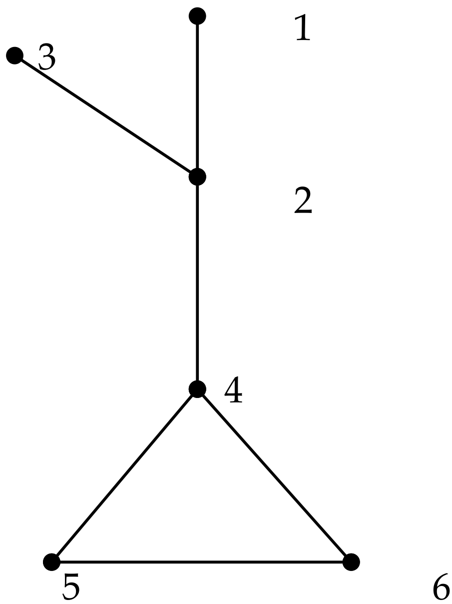
$int(\{1,2\}) \cup int(\{2,3,4\}) = \{1,2,5\}$, $int(\{1,2\} \cup \{2,3,4\})$Ú$int(\{1,2\}) \cup int(\{2,3,4\})$.

$cl(\{2,3\}) = \{1,2,3,4\}$, $cl(\{1,3,4\}) = \{1,2,3,4,5\}$.$cl(\{3\}) = \{1,3,4\}$.

$cl(\{2,3\}) \cap cl(\{1,3,4\}) = \{1,2,3,4\}$.$cl(\{2,3\}) \cap cl(\{1,3,4\})$Ú$cl(\{2,3\} \cap \{1,3,4\})$.

Definition 4.6 In a graph adjacency topological space $(V(G), T_A)$, a vertex induced subgraph H of $G$is said to be a dense subgraph of G if $cl(V(H)) = V(G)$.

Example 4.7 Consider the following graph



$$S_N = \{\{2\}, \{1,3,4\}, \{2\}, \{2,5,6\}, \{4,6\}, \{4,5\}\}$$

$cl(\{2,4\}) = \{1,2,3,4,5,6\}.\{2,4\}$ is dense.

$cl(\{3,4\}) = \{2,3,4,5,6\}.\{3,4\}$ is not dense.

We observe that in a regular graph, every vetex induced subgraph is dense and in a tree, centre with two adjacent vertices is dense.

Example 4.8 In example 2.3, let $V(W) = \{2\}$. Then int(V(W)) = {1}, V(G) – int(V(W)) = {2,3,4}, cl(V(G) – V(W)) = cl({1,3,4}) = {1,2,3,4}. Hence $cl(V(G) - V(W)) \neq V(G) - int(V(W))$. Also, int(V(G) – V(W)) = int({1,3,4} = {2}, cl(V(W)) = cl({2}) = {1,2,3,4}, V(G) – cl(V(W)) = ϕ. Hence $int(V(G) - V(W)) \neq V(G) - cl(V(W))$.

If we define $N_v = \{v\} \cup \{u \in V(G): uv \in X(G)\}$, then it can be proved that $cl(V(G) - V(W)) = V(G) - int(V(W))$ and $int(V(G) - V(W)) = V(G) - cl(V(W))$.

Theorem 4.9 Let $(V(G), T_A)$ be a graph adjacency topological space such that

$N_v = \{v\} \cup \{u \in V(G): uv \in X(G)\}$. Let W be a vertex induced subgraph of G. Then

$cl(V(G) - V(W)) = V(G) - int(V(W))$

$int(V(G) - V(W)) = V(G) - cl(V(W))$

Proof:

1. If $v \in cl(V(G)\text{-}V(W))$, then $v \in V(G)$ and $v \notin V(W)$ or $N_v \not\subseteq V(W)$. Hence $v \in V(G)$ and $v \notin int(V(W))$ or $v \notin int(V(W))$ and so $v \in V(G) - int(V(W))$. Therefore $cl(V(G)\text{-}V(W)) \subseteq V(G) - int(V(W))$.

   If $v \in V(G) - int(V(W))$ then $v \in V(G)$ and $N_v \not\subseteq V(W)$ Hence $v \in V(G) - V(W)$ and $N_v \cap (V(G) - V(W)) \neq \varphi$ and so $v \in cl(V(G) - V(W))$.

   Therefore $V(G) - int(V(W)) \subseteq cl(V(G) - V(W))$ and $cl(V(G) - V(W)) = V(G) - int(V(W))$.

If v ∈ int(V(G)-V(W)), then $N_v \subseteq V(G) - V(W)$. Hence $N_v \cap V(W) = \varphi$ and so v ∉ cl(V(W)). Therefore $v \in V(G) - cl(V(W))$. Reversing the steps proves $int(V(G) - V(W)) = V(G) - cl(V(W))$

Some basic properties of the interior and closure of the vertex induced subgraphs of a graph in a graph incidence topological space can also be proved as earlier.

Proposition 4.10 Let $(V(G), T_I)$ be a graph incidence topological space.Let W_1 and W_2 be vertex induced subgraphs of G. Then

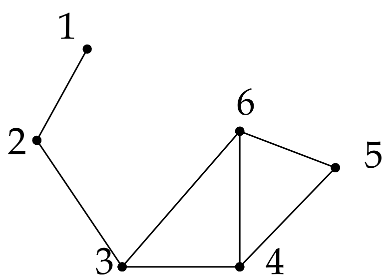
1. $V(W_1) \subseteq cl(V(W_1))$
2. $V(W_1) \subseteq V(W_2) \Rightarrow cl(V(W_1)) \subseteq cl(V(W_2))$
3. $cl(V(W_1) \cup V(W_2)) = cl(V(W_1)) \cup cl(V(W_2))$
4. $cl(V(W_1) \cap V(W_2)) \subseteq cl(V(W_1)) \cap cl(V(W_2))$
5. $V(W_1) \subseteq V(W_2) \Rightarrow int(V(W_1)) \subseteq int(V(W_2))$
6. $int(V(W_1) \cap V(W_2)) \neq int(V(W_1)) \cap int(V(W_2))$
7. $int(V(W_1)) \cup int(V(W_2)) \subseteq int(V(W_1) \cup V(W_2))$.

5. Exterior and Boundary of Vertex Induced Subgraphs of a Graph

Taha H. Jasim et al. [15] defined the exterior and boundary of vertex set of subgraph H of a graph G as $ext(V(H)) = \{v \in V(G): vR \cap V(H) = \varphi\}$ there vR is a post class of a relation R on $V(G)$) and $bd(V(H)) = cl(V(H)) - int(V(H))$. In this section, exterior and boundary of vertex induced subgraphs of graphs in terms of adjacency are introduced and the basic properties of exterior and boundary are studied.

Definition 5.1 Let $G = (V(G), X(G))$ be a graph without isolated vertices and $(V(G), T_A)$ be a graph adjacency topological space. Let W be a vertex induced subgraph of G. Then exterior of $V(W)$ is defined by $ext(V(W)) = \{v \in V(G): N_v \cap V(W) = \varphi\}$ and boundary of $V(W)$ is defined by $bd(V(W)) = cl(V(W)) - int(V(W))$.

Example 5.2 Consider the following graph



S_N = {{2}}, {1,3}, {2,4,6}, {3,6,5}, {4,6}, {3,4,5}}.

cl({1,4,6}) = {1,2,3,4,5,6},int({1,4,6}) = {5}, ext({1,4,6}) = {1}, bd({1,4,6}) = {1,2,3,4,6}

The following theorems describe the properties of different operators on vertex set of graphs.

Theorem 5.3 Let $(V(G), T_A)$ be a graph adjacency topological space. Let $H$and Wbe vertex induced subgraphs of G. Then

i. ext(V(H)) = int(V(G) – V(H))

ii. If V(H) ⊆ V(W), then ext(V(W)) ⊆ ext (V(H))

iii. ext(V(H)∪ V(W)) = ext(V(H)) ∩ ext(V(W))

iv. bd(V(G)) = ∅

v. cl(V(H)) = int(V(H)) ∪ bd(V(H))

vi. bd(V(H)) ∩ int(V(H)) = ∅

vii. bd(V(H)) ∩ ext(V(H)) = ∅

viii. int(V(H)) ∩ ext(V(H)) = ∅

ix. int(V(H)) ∪ ext(V(H)) ∪ bd(V(H)) =V(G)

Proof:

If v ∈ ext(V(H)), then $N_v \cap V(H) = \varphi$. Hence $N_v \subseteq (V(G) - V(H))$ and so v ∈ int(V(G) – V(H)). Reversing the steps proves $ext(V(H)) = int(V(G) - V(H))$

If v ∈ ext(V(W)), then $N_v \cap V(W) = \varphi$ and $N_v \cap V(H) = \varphi$. So v ∈ ext(V(H)) and ext(V(W)) ⊆ *ext(V(H)).*

i) *ext(V(H) ∪ V(W)) = int(V(G) – (V(H) ∪V(W))) = int((V(G) – V(H)) ∩ (V(G) – V(W))) = int(V(G) – V(H)) ∩ int(V(G) – V(W)) = ext(V(H)) ∩ ext(V(W)).*

ii) Since *cl(V(G)) = V(G)* and *int(V(G)) = V(G)* , it follows that *bd(V(G) = ∅.*

iii) *int(V(H)) ∪ bd(V(H)) = int(V(H) ∪ (cl(V(H)) – int(V(H))) = (cl(V(H)) ∩ (V(G) – int(V(H)))) ∪ int(V(H)) = (cl(V(H)) ∪ int(V(H))) ∩ ((V(G) – int(V(H))) ∪ int(V(H))) = cl(V(H)) ∩ V(G) = cl(V(H)).*

iv) If v ∈ *bd(V(H)),* then v ∉ *int(V(H)).* So *bd(V(H)) ∩ int(V(H)) = ∅.*

v) If v ∈ ext(V(H)), then $N_v \cap V(W) = \varphi$ and v ∉cl(V(H)). So v ∉ bd(V(H)) and bd(V(H)) ∩ ext(V(H)) = ∅.

vi) If v ∈ *int(V(H)),* then N_v ⊆ *V(H).* Hence N_v ∩ V(H) ≠ ∅ and $v \notin ext(V(H))$. So $int(V(H)) \cap ext(V(H)) = \varphi$.

vii) By definition, *cl(V(H)) ∪ int(V(H)) ∪ bd(V(H)) = V(G).*

Theorem 5.4 Let $(V(G), T_A)$ be a graph adjacency topological space such that $N_v = \{v\} \cup \{u \in V(G): uv \in X(G)\}$. Let $H$and W be a vertex induced subgraph of G. Then

i. *ext(VH)) ∩ V(H) = ∅*

ii. $ext(V(H)) = V(G) - cl(V(H))$

iii. $bd(V(H)) = cl(V(H)) \cap cl(V(G) - V(H))$

iv. $bd(V(H)) \subseteq bd(V(G) - V(H))$

v. $bd(V(H) \cup V(W)) \subseteq bd(V(H)) \cup bd(V(W))$

Proof:

1. If $v \in ext(VH))$, then $N_v \cap V(H) = \emptyset$. Since $v \in N_v$, $v \notin V(H)$. So $ext(V(H)) \cap V(H) = \varphi$.
2. $ext(V(H)) = int(V(G) - V(H)) = V(G) - cl(V(H))$..
3. If $v \in bd(V(H))$, then $v \in cl(V(H))$ and $v \notin int(V(H))$. Hence $v \in cl(V(H))$ and $v \in V(G) - int(V(H))$. So $v \in cl(V(H))$ and $v \in cl(V(G)\text{-}V(H))$ which gives $v \in cl(V(H)) \cap cl(V(G)\text{-}V(H))$. Reserving the steps proves $bd(V(H)) = cl(V(H)) \cap cl(V(G) - V(H))$.
4. If $v \in bd(V(H))$, then $v \in cl(V(H))$ and $v \notin int(V(H))$. Hence $v \in cl(V(H))$ and $v \in V(G) - int(V(H))$ and so $v \in cl(V(H))$ and $v \in cl(V(G)\text{-}V(H))$. So $v \in (cl(V(G) - V(H))) \cap (cl(V(H)))$ and $v \in (cl(V(G) - V(H))) - (V(G) - cl(V(H)))$.. Therefore $v \in (cl(V(G) - V(H))) - int(V(G) - V(H))$ and so $v \in bd(V(G) - V(H))$.
5. If $v \in bd(V(H) \cup V(W))$, then $v \in cl(V(H) \cup V(W))$ and $v \notin int(V(H) \cup V(W))$. Hence $v \in cl(V(H) \cup cl(V(W))$ and $v \in cl(V(G) - ((V(H) \cup V(W)))$ and $v \in cl(V(H)) \cup cl(V(W))$ and $v \in cl((V(G) - V(H)) \cap (V(G) - V(W)))$. So $\{v \in cl(V(H))$ and $v \in cl(V(G) - V(H))\}$ or $\{v \in cl(V(W))$ and $v \in cl(V(G) - V(H))\}$ and $\{v \in cl(V(H))$ and $v \in V(G) - int(V(H))\}$ or $\{v \in cl(V(W))$ and $v \in V(G) - int(V(H))\}$. Therefore $\{v \in cl(V(H))$ and $v \notin int(V(H))\}$ or $\{v \in cl(V(W))$ and $v \notin int(V(H))\}$ and $v \in bd(V(H)) \cup bd(V(W))$.

Theorem 5.5 Let $(V(G), T_A)$ be a graph adjacency topological space. Let H and W be vertex induced subgraphs of G. If cl(V(H) ∩ cl(V(W)) = ∅, then int(V(H)) ∪ int(V(W)) = int(V(H) ∪ V(W)).

Proof: By Theorem 4.4, $int(V(H)) \cup int(V(W)) \subseteq int(V(H) \cup V(W))$.

To prove the reverse inclusion, let $v \notin int(V(H)) \cup int(V(W))$ and $v \in int(V(H) \cup V(W))$. Hence Nv ⊆ V(H) or Nv ⊆ V(W) or Nv ⊆ V(H) ∩ V(W). If $N_v \subseteq V(H)$, then $v \in int(V(H))$. So $v \in int(V(H)) \cup$

$int(V(W))$, which is a contradiction. Similarly, if $N_v \subseteq V(W)$, then $v \in int(V(W))$. So $v \in int(V(H)) \cup int(V(W))$, which is a contradiction. If $N_v \subseteq V(H) \cap V(W)$, then $N_v \cap V(W) \neq \varphi$ and $N_v \cap V(H) \neq \varphi$. So $v \in cl(V(H)) \cap cl(V(W))$ which is a contradiction. Hence $v \notin int(V(H) \cup V(W))$ and $int(V(H) \cup V(W)) \subseteq int(V(H)) \cup int(V(W))$.

Applications

Complex network theory plays a vital role in bio-chemical and bio-medical fields. Such networks, electrical circuits, and information systems can be modeled using the graph theory notion by representing vertices and edges as the nature of the trend of study. The most important feature of the hydrogen bond is that it possesses direction and hence hydrogen bond networks along with co-operativity and antico-operativity can be modeled as digraphs. Hydrogen bond networks can be represented by digraphs where vertices correspond to the donor and acceptor group, and edges correspond to hydrogen bonds from proton-donor to proton-acceptor. Protein functioning can be shown graphically. Interactions between entities such as proteins, chemicals, or macromolecules can be represented using graphs and it can also be used to describe biological pathways. The most important issue in our biological system is the process of blood circulation and the functioning of kidneys. Medical tests play an important role in the life of rights to make sure that the retreat of diseases, perhaps the most prominent of those analyzes macroeconomic analysis functions. Through the medical application, the system can be modeled graphically. By considering the parts of the heart/kidney as vertices and the flow of blood/liquid between the parts as edges, the system can be modeled as graphs. The Interior and closure of induced subgraphs under the topology generated from the resulting graph of the system will be useful in detecting and predicting the diseases of the heart/kidney.

Conclusion

A synthesis between graph theory and topology has been made. Subbasis for different topologies on vertex set of simple undirected graphs are introduced, and the nature of topology generated by vertex sets of some standard graphs are stated. Some basic properties of closure, interior, exterior, and boundary of vertex induced subgraphs of a graph with respect to graph adjacency topology are studied. The results discussed in this paper will be helpful in further study of some other topological structures and its properties. Also, the results and properties discussed in this paper can be studied further with respect to graph non-adjacency topology,graph incidence topology, and graph non-incidence topology. There are many ways of generating topologies on an edge set of graphs. But there may arise a situation where the edge set of the graph can be empty; so that a topology on the edge set can not be generated. Also, while considering the way of generating topologies on the edge set of graphs, the incidence relation between the vertices and edges of a graph can also be taken into account. This way of generating topologies can be studied further.

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