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The study of $\boldsymbol{S}$-curvature on a homogeneous Finsler space with Randers-Matsumoto metric

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Abstract

In this article, we have focused on the study of S-curvature of Randers-Matsumoto metric on a homogeneous Finsler space. We have deduced the condition for an isometry of Finsler homogeneous space with Randers-Matsumoto metric to be an isometry of Riemannian homogeneous space and proved that the group of isometries of Finsler space are closed subgroups of that of Riemannian space. We have examined the existence of an invariant vector field. Further, we have derived the formula for S-curvature on the reductive homogeneous space, discussed the condition for isotropic S-curvature, and derived the E-curvature of the Randers-Matsumoto metric for the homogenous space by using S-curvature formula.

Keywords: Randers-Matsumoto metric, Homogeneous Finsler space, invariant vector field, S-curvature, E-curvature.
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1. Introduction

To solve variational problems in the spaces, P. Finsler proposed Finsler geometry in 1918. Matsumoto first presented the idea of Finsler spaces with (α, β)-metric [12]. There are several applications of it in mathematics, physics, mechanics, seismology, biology, and ecology [1, 18, 23]. Matsumoto proposed (α, β)-metric of the form $L = \frac{\alpha^2}{\alpha - \beta}$, $\alpha = \sqrt{a_{ij}(\mu)\eta^i\eta^j}$, and $\beta = b_i(\mu)\eta^i$, is said to be slope of a mountain metric as well as Matsumoto metric [14]. With the use of this metric, Finsler geometry has been enhanced, and researchers have a useful working tool [18, 24]. Randers change of Finsler metricis defined by $L(\mu,\eta) \to F(\mu,\eta) = L(\mu,\eta) + b_i(\mu)\eta^i$. Matsumoto advanced the notion of a Randers change, and Hashiguchi and Ichijyo [10] named it and Shibata studied it from a detailed perspective [22]. An (α, β)-metric $L(\mu,\eta) = \frac{\alpha^2}{\alpha-\beta} + \beta$ is said tobe Randers-Matsumoto metric. The characteristics of a Finsler space with the Randers-Matsumoto metric were recently discussed by Nagaraja and Pradeep Kumar [15]. Matsumoto [13] presented the theory of Finslerian hypersurface. The geometrical characteristics of hypersurfaces in a few unique Finsler spaces have been studied by Gupta and Pandey [8, 9].

In Finsler geometry, S-curvature is one of the important non-Riemannian curvatures. The structure of S-curvature was first initiated by Shen [21]. S-curvature measures the rate of change of volume form of Finsler space and is subtly related to flag curvature of Finsler metrics. It has lots of applications in biology, physics, mechanics, and the medical field. Recently so, many authors have worked on this S-curvature [2, 7, 16, 19, 25, 26, 27]. In 2013, Shaoqiang Deng and Zhiguang Hu [5], who have discussed the curvatures of homogeneous Randers space, positive flag curvature. In 2017, Laurian-Ioan Piscoran and Vishnu Narayan Mishra [17] investigated the S-curvature, Cartan and mean Cartan torsion, Landsberg curvature, and studied the bounded Cartan torsion metrics classes; also obtained the if and only if condition for Finsler metrics to be Riemannian or locally Minkowskian.

In this present article, we have considered the Randers-Matsumoto metric $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ to study the S-curvature on homogeneous Finsler metric. This presents a new approach to analyse the curvature properties of homogenous Finsler space using the concept of change of metrics. The discussion of this article is designed as below: In 1st portion, here we introduced S-curvature in a brief way and defined some basic definitions related to the article. In section 2, the existence of an invariant vector field is proved by some theorems and lemmas. Finally, in part 3 and 4, in homogeneous Finsler spaces, the S-curvature and E-curvature formula for can be found using the Randers-Matsumoto metric.

Definition 1.1. [17] Let M be an n-dimensional smooth manifold, $F: TM \to [0, +\infty)$ be a non-negative function on the tangent bundle. F is known as Finsler metric on M if it satisfies the following conditions:

(1) $F(\mu, \lambda\eta) = \lambda F(\mu, \eta),\ \forall\, \lambda > 0$ (Positively homogeneous),

(2) $F(\mu, \eta)$ is smooth on $TM_0 = TM \backslash \{0\}$,

(3) For any non-zero vector η, the Hessian matrix $g_{ij}(\mu, \eta) = \frac{1}{2}\frac{\partial^2 F^2}{\partial\eta^i \partial\eta^j}$, is positive definite.

A differentiable manifold M equipped with a Finsler metric F is called a Finsler manifold or Finsler space denoted by (M, F).

We look back on the lemma, is proved by Shen [20], which is:

Lemma 1.1. For a Riemannian metric α and 1-form β with $\|\beta\|_\alpha < b_0$, let $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, ϕ is smooth on an interval

$(-b_0, b_0)$. Then F is a Finsler metric if and only if ϕ satisfies: $(b^2 - s^2)\phi''(s) - s\phi'(s) + \phi(s) > 0, \phi(s) > 0, \forall |s| \leq b < b_0$.

Definition 1.2. A smooth manifold G together with an abstract group structure is defined to be a Lie group if the mapping $(\mu, \eta) \to \mu\eta^{-1}$ from $G \times G \to G$ is C^∞. For a Lie group G on M, a C^∞ manifold, if there exists a C^∞ mapping $v: G \times M \to M$ which satisfies,

(1) $v(e, \mu) = \mu$, where e, the identity of G, $\forall\, \mu \in G$,

(2) $v\big(g_1, v(g_2, \mu)\big) = v(g_1 g_2, \mu), \forall\, \mu \in M$ and $\forall g_1, g_2 \in G$,

then G is said to act smoothly on M and G as the group of Lie transformations of M.

Theorem 1.1. [4] Let G be a Lie group, H be a closed subgroup of G, G/H be the space of left cosets gH with natural topology. Then G/H has a unique analytic structure with the property that G is a Lie transformation group of G/H.

Definition 1.3. [11] A connected Finsler space (M, F) and the group of isometries of (M, F) is $I(M, F)$, and is a Lie, transformation group. If $I(M, F)$ acts transitively on M, then (M, F) is known as homogeneous Finsler space.

A Finsler space (M, F), $G = I(M, F)$ be the group of isometries of M. Let $a \in M$. Then the isotropy subgroup $H = I_a(M, F)$ of G is a subgroup of G, it is closed as well as compact. We write M as the quotient space G/H.

Definition 1.4. For the Lie group G with g as its Lie algebra and its identity element be e, the map $exp: g \to G$ is characterized by

$exp(tY) = v(t), \forall\, t \in R$, where $v: R \to G$ is defined as unique one-parameter subgroup of G and $\dot{v}(0) = Y_e$.

In a homogeneous reductive manifold, at the origin $eH = H$, we can recognise the tangent space $T_H(G/H)$ of G/H with m through the map,

$$Y \to \frac{d}{dt} exp(tY)H|_{t=0}, Y \in m,$$

since G/H is recognised as M and for any Lie group that contains Lie algebra G described as $T_e G$.

2. An invariant vector field on homogeneous Finsler space

It is shown here that there is an invariant vector field associated with β for the homogeneous Finsler space with (α, β)-metric $F = \frac{\alpha^2}{\alpha - \beta} + \beta$. For that, we are in need of the following lemmas:

Lemma 2.1. A Riemannian space (M, α) and $\beta = b_i \eta^i$ be a 1-form, with $\|\beta\| = \sqrt{b_i b^i} < 1$. On M, corresponding to β, $\exists$ a smooth vector field ξ with $\alpha(\xi|_\mu) < 1, \forall\, \mu \in M$ such that the metric F of a Finsler space (M, F) can be described through α along with ξ as

$$F(\mu, \eta) = \frac{\alpha^2(\mu,\eta)}{\alpha(\mu,\eta) - \langle \xi|_\mu, \eta \rangle} + \langle \xi|_\mu, \eta \rangle,\ \mu \in M,\ \eta \in T_\mu M, \qquad (2.1)$$

where the inner product $\langle , \rangle$ is extracted from the Riemannian metric α.

Proof. We know that an inner product constrains a Riemannian metric to $T_\mu M$ for some $\mu \in M$. Therefore, the bilinear form $\langle u, v \rangle = a_{ij} u^i v^j, \forall\, u, v \in T_\mu M$ is an inner product on $T_\mu M$ for $\mu \in M$, and this induces the inner product on $T_\mu^* M$, the co-tangent space of M at μ which gives us $\langle d\mu_i, d\mu_j \rangle = a^{ij}$. With the existence of linear isomorphism between $T_\mu^* M$ and $T_\mu M$, the inner product is defined. This follows that a smooth vector field ξ corresponds to the 1-form β on M and is given by $\xi|_\mu = b^i \frac{\partial}{\partial \mu^i}$, where $b^i = a^{ij} b_j$. Now, for $\eta \in T_\mu M$, we have

$$\langle \xi|_\mu, \eta \rangle = \langle b^i \frac{\partial}{\partial \mu^i}, \eta^j \frac{\partial}{\partial \mu^j} \rangle = b^i \eta^j a_{ij} = b_j \eta^j = \beta(\eta).$$

Also, we have $\alpha^2(\mu, \eta) = a_{ij} \eta^i \eta^j$,

$$\Rightarrow \alpha^2\left(\xi|_\mu\right) = a_{ij} b^i b^j = ||\beta||^2 < 1,$$

equivalent to $\alpha\left(\xi|_\mu\right) < 1$. ■

Lemma 2.2. A Finsler space (M, F) with (α, β)-metric $F = \frac{\alpha^2}{\alpha - \beta} + \beta$. Let φ be an isometry of (M, F). Then φ is an isometry of (M, α) if and only if $\langle \xi|_\mu, \eta \rangle = \langle \xi|_{\varphi(\mu)}, d\varphi_\mu(\eta) \rangle$.

Proof. Let $\mu \in M$ and $\varphi : (M, F) \to (M, F)$ be an isometry. Consequently,

$$F(\mu, \eta) = F\left(\varphi(\mu), d\varphi_\mu(\eta)\right), \forall\, \eta \in T_\mu M. \tag{2.2}$$

By the Lemma 2.1, we get

$$\frac{\alpha^2(\mu, \eta)}{\alpha(\mu, \eta) - \langle \xi|_\mu, \eta \rangle} + \langle \xi|_\mu, \eta \rangle$$
$$= \frac{\alpha^2\left(\varphi(\mu), d\varphi_\mu(\eta)\right)}{\alpha\left(\varphi(\mu), d\varphi_\mu(\eta)\right) - \langle \xi|_{\varphi(\mu)}, d\varphi_\mu(\eta) \rangle} + \langle \xi|_{\varphi(\mu)}, d\varphi_\mu(\eta) \rangle . \tag{2.3}$$

Replacing η by $-\eta$ in equation (2.3), we get

$$\frac{\alpha^2(\mu,\eta)}{\alpha(\mu,\eta)+\langle\xi|_{\mu},\eta\rangle}-\langle\xi|_{\mu},\eta\rangle$$
$$=\frac{\alpha^2\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)}{\alpha\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)+\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle}-\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle. \qquad (2.4)$$

Adding the equations (2.3) and (2.4),

$$\frac{\alpha^3(\mu,\eta)}{\alpha^2(\mu,\eta)-\langle\xi|_{\mu},\eta\rangle^2}=\frac{\alpha^3\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)}{\alpha^2\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)-\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle^2},$$

$$\frac{\alpha^3(\mu,\eta)-\alpha^3\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)}{\alpha^3\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)}$$
$$=\frac{\alpha^2(\mu,\eta)-\langle\xi|_{\mu},\eta\rangle^2-\alpha^2\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)+\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle^2}{\alpha^2\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)-\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle^2}.$$

If $\alpha(\mu,\eta)=\alpha\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)$, then we have

$$\langle\xi|_{\mu},\eta\rangle^2=\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle^2,$$

implies $$\langle\xi|_{\mu},\eta\rangle=\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle, \qquad (2.5)$$

and thus $d\varphi\left(\xi|_{\mu}\right)=\xi|_{\varphi(\mu)}$.

Now, subtracting (2.4) from (2.3),

$$\frac{\alpha^2(\mu,\eta)\langle\xi|_{\mu},\eta\rangle}{\alpha^2(\mu,\eta)-\langle\xi|_{\mu},\eta\rangle^2}+\langle\xi|_{\mu},\eta\rangle$$
$$=\frac{\alpha^2\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle}{\left(\alpha^2\left(\varphi(\mu),d\varphi_{\mu}(\eta)\right)-\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle^2\right)}+\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle.$$

Again, if $\langle\xi|_{\mu},\eta\rangle=\langle\xi|_{\varphi(\mu)},d\varphi_{\mu}(\eta)\rangle$, then

$$\frac{\alpha^2(\mu,\eta)}{\alpha^2(\mu,\eta)-\langle\xi|_\mu,\eta\rangle^2}=\frac{\alpha^2\left(\varphi(\mu),d\varphi_\mu(\eta)\right)}{\alpha^2\left(\varphi(\mu),d\varphi_\mu(\eta)\right)-\langle\xi|_\mu,\eta\rangle^2},$$

Implies

$$\alpha^2(\mu,\eta)\alpha^2\left(\varphi(\mu),d\varphi_\mu(\eta)\right)-\alpha^2(\mu,\eta)\langle\xi|_\mu,\eta\rangle^2$$
$$=\alpha^2(\mu,\eta)\alpha^2\left(\varphi(\mu),d\varphi_\mu(\eta)\right)-\alpha^2\left(\varphi(\mu),d\varphi_\mu(\eta)\right)\langle\xi|_\mu,\eta\rangle^2,$$

thus $\alpha(\mu,\eta)=\alpha(\varphi(\mu),d\varphi_\mu(\eta))$. (2.6)

Hence, we proved. ■

Lemma 2.3. Let (M,F) be a Finsler space with $F=\frac{\alpha^2}{\alpha-\beta}+\beta$. Let $I(M,F)$ be the group of isometries of (M,F) and $I(M,\alpha)$ be that of Riemannian space (M,α). Then $I(M,F)$ is a closed subgroup of $I(M,\alpha)$ if and only if $\langle\xi|_\mu,\eta\rangle=\langle\xi|_{\varphi(\mu)},d\varphi_\mu(\eta)\rangle$.

$Proof$. Let $I(M,F)$ be a closed subgroup of $I(M,\alpha)$. Thus, if φ is an isometry of (M,F), φ is an isometry of (M,α). Then from the lemma (2.2), we have $\langle\xi|_\mu,\eta\rangle=\langle\xi|_{\varphi(\mu)},d\varphi_\mu(\eta)\rangle$.

On the other hand, suppose φ is an isometry of (M,F) and satisfies $\langle\xi|_\mu,\eta\rangle=\langle\xi|_{\varphi(\mu)},d\varphi_\mu(\eta)\rangle$, then from lemma (2.2) φ is an isometry of (M,α). Thus $I(M,F)$ is a closed subgroup of $I(M,\alpha)$.

Hence the proof. ■

From the above Lemma, we determine that for the metric $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ if (M,F) is a homogeneous Finsler space, then (M,α) is homogeneous.

Therefore, a Finsler homogeneous space with the Randers-Matsumoto metric can be stated as a connected Lie group's coset space with metric F. Considering the metric $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ as G-invariant on M.

Theorem 2.1. Let $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ be a G-invariant Randers-Matsumoto metric on G/H, ξ be a corresponding vector field to β. Then α is a G-invariant if and only if ξ is also G-invariant.

Proof. As F is a G-invariant on G/H,

$F(\eta) = F(Ad(h)\eta), \forall\, h \in H, \eta \in m.$

By the Lemma 2.1, we get

$$\frac{\alpha^2(\eta)}{\alpha(\eta)-\langle\xi,\eta\rangle} + \langle\xi,\eta\rangle = \frac{\alpha^2(Ad(h)\eta)}{\alpha(Ad(h)\eta)-\langle\xi,Ad(h)\eta\rangle} + \langle\xi, Ad(h)\eta\rangle. \qquad (2.7)$$

Replacing y by $-\eta$ in equation (2.7), we get

$$\frac{\alpha^2(\eta)}{\alpha(\eta)+\langle\xi,\eta\rangle} - \langle\xi,\eta\rangle = \frac{\alpha^2(Ad(h)\eta)}{\alpha(Ad(h)\eta)+\langle\xi,Ad(h)\eta\rangle} - \langle\xi, Ad(h)\eta\rangle. \qquad (2.8)$$

Adding the equations (2.7) and (2.8), implies that

$$\frac{\alpha^3(\eta)}{\alpha^2(\eta) - \langle\xi,\eta\rangle^2} = \frac{\alpha^3\big(Ad(h)(\eta)\big)}{\alpha^2\big(Ad(h)(\eta)\big) - \langle\xi, Ad(h)(\eta)\rangle^2},$$

$$\frac{\alpha^3(\eta)-\alpha^3\big(\xi,Ad(h)(\eta)\big)}{\alpha^3\big(Ad(h)(\eta)\big)} = \frac{\alpha^2(\mu,\eta)-\langle\xi,\eta\rangle^2-\alpha^2\big(\xi,Ad(h)(\eta)\big)+\langle\xi,Ad(h)(\eta)\rangle^2}{\alpha^2\big(Ad(h)(\eta)\big) - \langle\xi,Ad(h)(\eta)\rangle^2}.$$

If $\alpha(\eta) = \alpha\big(Ad(h)(\eta)\big)$, then

$$\langle\xi,\eta\rangle^2 = \langle\xi, Ad(h)(\eta)\rangle^2,$$

implies $\langle\xi,\eta\rangle = \langle\xi, Ad(h)(\eta)\rangle$ (2.9)

Now, subtracting equation (2.8) from equation (2.7), we get

$$\frac{\alpha^2(\eta)\langle\xi,\eta\rangle}{\alpha^2(\eta)-\langle\xi,\eta\rangle^2}+\langle\xi,\eta\rangle=\frac{\alpha^2\big(Ad(h)(\eta)\big)\langle\xi,Ad(h)(\eta)\rangle}{\alpha^2\big(Ad(h)(\eta)\big)-\langle\xi,Ad(h)(\eta)\rangle^2}+\langle\xi,Ad(h)(\eta)\rangle.$$

Again, if $\langle\xi,\eta\rangle=\langle\xi,Ad(h)(\eta)\rangle$, then

$$\frac{\alpha^2(\eta)}{\alpha^2(\eta)-\langle\xi,\eta\rangle^2}=\frac{\alpha^2\big(\xi,Ad(h)(\eta)\big)}{\alpha^2\big(\xi,Ad(h)(\eta)\big)-\langle\xi,\eta\rangle^2},$$

implies

$$\alpha^2(\eta)\alpha^2\big(\xi,Ad(h)(\eta)\big)-\alpha^2(\eta)\langle\xi,\eta\rangle^2$$
$$=\alpha^2(\eta)\alpha^2\big(\xi,Ad(h)(\eta)\big)-\alpha^2\big(\xi,Ad(h)(\eta)\big)\langle\xi,\eta\rangle^2,$$

thus $\alpha(\eta)=\alpha\big(\xi,Ad(h)(\eta)\big)$. (2.10)

Therefore, α is a G-invariant Riemannian metric if and only if $Ad(h)\xi=\xi$. ■

3. S-curvature of Randers-Matsumoto metric

Now, we derive the equation of S-curvature of $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ for the Finsler space . Assume that F is a Minkowski norm on V and V, an n-dimensional real space with α_i as basis. Let $Vol(B)$ be the volume of $B\subset R^n$, and B^n be the open unit ball. The distortion $\tau=\tau(\eta)$ is defined as

$$\tau(\eta)=\ln\left(\frac{\sqrt{det\big(g_{ij}(\eta)\big)}}{\sigma_F}\right),\eta\in V-\{0\},$$

where, $\sigma_F=\frac{Vol(B^n)}{Vol\{(\eta^i)\in R^n : F(\eta^i\alpha_i)<1\}}$ is the distortion of *(V, F)*.

For a Finsler space (M,F), distortion of Minkowski norm F_μ on $T_\mu M$ is$\tau=\tau(\mu,\eta)$, $\mu\in M$. Let γ be a geodesic with $\gamma(0)=\mu$, $\dot\gamma(0)=\eta$, where $\eta \in T_\mu M$, then along the geodesic γ, the rate of change of distortion defines the S-curvature, denoted as $S(\mu,\eta)$, i.e.,

$$S(\mu,\eta) = \frac{d}{dt}\{\tau(\gamma(t),\dot{\gamma}(t))\}|_{t=0}.$$

Here, $S(\mu,\eta)$ is illustrated as 1(p)-homogeneous function, i.e., for $\lambda > 0$, we have $S(\mu,\lambda\eta) = \lambda S(\mu,\eta)$.

A Finsler space's S-curvature is interconnected to a volume form. The Busemann-Hausdorff $(dV_{BH} = \sigma_{BH}(\mu)d\mu)$ and the Holmes-Thompson $(dV_{HT} = \sigma_{HT}(\mu)d\mu)$ volume forms are significant volume forms in Finsler geometry:

$\sigma_{BH}(\mu) = \frac{Vol(B^n)}{Vol(A)}$, and $\sigma_{HT}(\mu) = \frac{1}{Vol(B^n)}\int_A det(g_{ij})d\eta$ respectively,

where, $A = \left\{(\eta^i) \in R^n : F\left(\mu,\eta^i\frac{\partial}{\partial\mu^i}\right) < 1\right\}$. If we consider Riemannian metric instead of Finsler metric F, then dV_{HT} and dV_{BH} are reduced to single Riemannian volume form $dV_{HT} = dV_{BH} = \sqrt{det\left(g_{ij}(\mu)\right)}d\mu$. Subsequently, the function $T(s) = \phi(\phi - s\phi')^{n-2}\{(b^2 - s^2)\phi'' - s\phi' + \phi\}$, $dV = dV_{BH}$ (or dV_{HT}) is given by $dV = f(b)dV_\alpha$,

where

$$f(b) = \begin{cases} \dfrac{\int_0^\pi \sin^{n-2} t\, dt}{\int_0^\pi \dfrac{\sin^{n-2} t}{\phi(b\cos t)^n} dt}, & if\ dV = dV_{BH} \\[2ex] \dfrac{\int_0^\pi (\sin^{n-2} t) T(b\cos t) dt}{\int_0^\pi \sin^{n-2} t\, dt}, & if\ dV = dV_{HT} \end{cases}$$

and $dV_\alpha = \sqrt{det(a_{ij})}d\mu$ is the Riemannian volume form of α.

In a local coordinate system, the S-curvature formula was given by Cheng and Shen [3] and is written as:

$$S = \left(2v - \frac{f'(b)}{bf(b)}\right)(r_0 + s_0) - \frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0), \qquad (3.1)$$

where

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Delta = 1 + sQ + (b^2 - s^2)Q', \quad v = \frac{Q'}{2\Delta},$$

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), r_j = b^i r_{ij}, r_0 = r_i\eta^i, \quad r_{00} = r_{ij}\eta^i\eta^j,$$

$$\Phi = (sQ' - Q)(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q'',$$

$$s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad s_j = b^i s_{ij}, \quad s_0 = s_i\eta^i.$$

It is commonly known that $r_0 + s_0 = 0$ if b, the Riemannian length, is constant.
Hence,

$$S = -\frac{\Phi}{2\alpha\Delta^2}(r_{00} - 2\alpha Q s_0).$$

Theorem 3.1. [6] Let $F = \alpha\phi(s)$ be a G-invariant (α, β)-metric on the reductive homogeneous manifold G/H with $g = h + m$. Then the S-curvature formula for F is of the form

$$S(H,\eta) = \frac{\Phi}{2\alpha\Delta^2}(\langle[v,\eta]_{\mathfrak{m}}, \eta\rangle + \alpha Q\langle[v,\eta]_{\mathfrak{m}}, v\rangle), \qquad (3.2)$$

where $v \in m$ corresponds to the 1-form β and on G/H, m is verified with the tangent space $T_H(G/H)$ at the origin H.

Now, with metric $F = \frac{\alpha^2}{\alpha-\beta} + \beta$, we invent a formula for S-curvature of homogeneous Finsler spaces:

Theorem 3.2. Consider G/H as a reductive homogeneous Finsler space with Lie algebra decomposition $g = h + m$, and $F = \frac{\alpha^2}{\alpha-\beta} + \beta$ be a G-invariant Randers-Matsumoto metric on G/H, then S-curvature is written as,

$$S(H,\eta) = \left[\frac{\begin{array}{r}6s^4-(9n+15)s^3+(6b^2n+27n+21)s^2-(16b^2n+10b^2+14n+14)s\\ +4b^2n+10b^2+2n+2\end{array}}{2(s^2-s-1)(2b^2-3s+1)^2}\right]$$

$$\times\left(\frac{s^2-2s+2}{1-2s}\langle[v,\eta]_{\mathfrak{m}},v\rangle+\frac{1}{\alpha}\langle[v,\eta]_{\mathfrak{m}},\eta\rangle\right), \qquad (3.3)$$

where $v\in\mathfrak{m}$ corresponds to the 1-form β and on G/H, $\mathfrak{m}$ is verified with the tangent space $T_H(G/H)$ at the origin H.

Proof. For a metric $F=\alpha\phi(s)$, where $\phi(s)=\frac{1}{1-s}+s$, the components of equation (3.1) have the following values:

$$Q=\frac{-s^2+2s-2}{2s-1}, \quad Q'=\frac{-2s^2+2s+2}{(2s-1)^2}, \quad Q''=-\frac{10}{(2s-1)^3},$$

$$\Delta=-\frac{(s^2-s-1)(2b^2-3s+1)}{(2s-1)^2},$$

$$\Phi=\frac{(s^2-s-1)\left(\begin{array}{r}6s^4-3(3n+5)s^3+3(2nb^2+9n+7)s^2-\big((16n+10)b^2\\ +14(n+1)\big)s+(4n+10)b^2+2(n+1)\end{array}\right)}{(2s-1)^4}.$$

When these values are substituted in equation (3.2), we obtain a required S-curvature formula as shown in (3.3). ■

Corollary 3.1. Let G/H be reductive homogeneous Finsler space with Lie algebra decomposition $g=h+m$ and $F=\frac{\alpha^2}{\alpha-\beta}+\beta$ be a G-invariant Randers-Matsumoto metric on G/H. Then $(G/H,F)$ has and isotropic S-curvature if and only if it has a vanishing S-curvature.

Proof.

Necessary condition:

Consider G/H has an isotropic S-curvature, then $S(\mu,\eta)=(n+1)c(\mu)F(\eta), \mu\in G/H, \eta\in T_{\mu}(G/H)$.

Taking $\mu = H$ and $\eta = v$ and using equation (3.3), we get $c(H) = 0$. Consequently, $S(H, \eta) = 0$, $\forall\, \eta \in T_H(G/H)$. Since F is a homogeneous metric, we have $S = 0$ everywhere and sufficient condition is obvious. ■

4. Mean Berwald Curvature

The mean Berwald curvature [20], a quantity associated with S-curvature, is given by

$$E_{ij} = \frac{1}{2}\frac{\partial^2}{\partial\eta^i\partial\eta^j}S(\mu,\eta) = \frac{1}{2}\frac{\partial^2}{\partial\eta^i\partial\eta^j}\left(\frac{\partial G^m}{\partial\eta^m}\right)(\mu,\eta),$$

where G^m are spray coefficients. On $TM\backslash\{0\}$, $E := E_{ij}d\mu^i \otimes d\eta^j$ is tensor, which is E tensor. A group of symmetric forms of E tensor considered as $E_\eta : T_\mu M \times T_\mu M \to R$ defined as $E_\eta(u,v) = E_{ij}(\mu,\eta)u^i v^j$, where $u = u^i \frac{\partial}{\partial\mu^i}|_\mu, v = v^i\frac{\partial}{\partial\mu^i}|_\mu \in T_\mu M$. Then the collection $\{E_\eta : \eta \in TM\backslash\{0\}\}$ is said to be E-curvature or mean Berwald curvature.

With $F = \frac{\alpha^2}{\alpha-\beta} + \beta$, we determine the mean Berwald curvature of a homogeneous Finsler space in this section. For this, the following is required: At the origin, $a_{ij} = \delta^i_j$,

therefore, $\eta_i = a_{ij}\eta^j = \delta^i_j\eta^j = \eta^i$, $\alpha_{\eta^i} = \frac{\eta_i}{\alpha}$, $\beta_{\eta^i} = b_i$,

$$s_{\eta^i\eta^j} = \frac{\partial}{\partial\eta^j}\left(\frac{b_i\alpha - s\eta}{\alpha^2}\right)$$

$$= \frac{\alpha^2\left[b_i\frac{\eta_j}{\alpha} - \left(\frac{b_j\alpha - s\eta_j}{\alpha^2}\right)\eta_i - s\delta^i_j\right] - (b_i\alpha - s\eta_i)2\eta_j}{\alpha^4},$$

$$= \frac{-(b_i\eta_j + b_j\eta_i)\alpha + 3s\,\eta_i\eta_j - \alpha^2 s\delta^i_j}{\alpha^4}.$$

In $S(H,\eta)$, we are assuming

$$A = \left[\frac{6s^4-(9n+15)s^3+(6b^2n+27n+21)s^2-(16b^2n+10b^2+14n+14)s\ +4b^2n+10b^2+2n+2}{2(s^2-s-1)(2b^2-3s+1)^2}\right],$$

then

$$\frac{\partial A}{\partial \eta^j}=\frac{k_1s^5+k_2s^4+k_3s^3+k_4s^2+k_5s+k_6}{2(s^2-s-1)^2(2b^2-3s+1)^3}s_{\eta^j},$$

and

$$\frac{\partial^2 A}{\partial \eta^i\eta^j}=\frac{l_1s^7+l_2s^6+l_3s^5+l_4s^4+l_5s^3+l_6s^2+l_7s+l_8}{(s^2-s-1)^3(2b^2-3s+1)^4}s_{\eta^i}s_{\eta^j}+\frac{k_1s^5+k_2s^4+k_3s^3+k_4s^2+k_5s+k_6}{2(s^2-s-1)^2(2b^2-3s+1)^3}s_{\eta^i\eta^j},$$

where,

$k_1=24b^2-27n-15$, $k_2=(18n-66)b^2+153n+129$,

$k_3=-\big((126n+78)b^2+216n+228\big)$,

$k_4=20(n+1)b^4+(182n+266)b^2+122n+146$,

$k_5=-(40(n\ +\ 1)b^4+(124n+172)b^2+34n+22)$,

$k_6=40(n+1)b^4+(28n-8)b^2+4n+4$,

$l_1=72b^2-81n-45$, $l_2=24b^4+(54n-300)b^2+675n+573$,

$l_3=-(72b^4+(702n+288)b^2-1566n-1584)$,

$l_4=240(n+1)b^4+(1860n+2220)b^2+1815n+1995$,

$l_5=-(40(n+1)b^6+(780n+660)b^4+(2730n+3420)b^2+1175n+1145)$

$l_6=120(n+1)b^6+1440(n+1)b^4+(1890n+1674)b^2+438n+330$

$l_7 = -(240(n+1)b^6 + (990n+972)b^4 + (432n+90)b^2 + 75n + 129)$

and $l_8 = 120(n+1)b^6 + (60n+36)b^4 + (6n+144)b^2 + 3n - 3.$

Theorem 4.1. Let G/H be a reductive homogeneous Finsler space with Lie algebra decomposition $g = h + m$, and a G-invariant metric $F = \frac{\alpha^2}{(\alpha-\beta)} + \beta$ on G/H. Then for the homogeneous space with F, the mean Berwald curvature is given by

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial\eta^i\partial\eta^j} = \frac{1}{2}\left(\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j} + \frac{\partial^2 v_1}{\partial\eta^i\partial\eta^j}\right),$$

where $\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j}$ and $\frac{\partial^2 v_1}{\partial\eta^i\partial\eta^j}$ are as in equation (4.1) and (4.2) respectively.

Proof. From equation $S(H,\eta)$, S-curvature at the origin can be expressed as follows:

$$S(H,\eta) = \phi_1 + v_1,$$

where

$$\phi_1 = \frac{A}{\alpha}\langle[v,y]_{\mathrm{m}},\eta\rangle \quad \text{and} \quad v_1 = \frac{(s-1)^2+1}{1-2s}A\langle[v,\eta]_{\mathrm{m}},v\rangle.$$

Therefore, mean Berwald curvature is

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S}{\partial\eta^i\partial\eta^j} = \frac{1}{2}\left(\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j} + \frac{\partial^2 v_1}{\partial\eta^i\partial\eta^j}\right),$$

where $\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j}$ and $\frac{\partial^2 v_1}{\partial\eta^i\partial\eta^j}$ are calculated as follows:

$$\frac{\partial\phi_1}{\partial\eta^j} = \frac{\partial}{\partial\eta^j}\left(\frac{A}{\alpha}\langle[v,\eta]_m,\eta\rangle\right)$$
$$= \left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^j} - \frac{A}{\alpha^2}\frac{\eta_j}{\alpha}\right)\langle[v,\eta]_m,\eta\rangle + \frac{A}{\alpha}\left(\langle[v,v_j]_m,\eta\rangle + \langle[v,\eta]_m,v_j\rangle\right),$$

$$\frac{\partial^2\phi_1}{\partial\eta^i\partial\eta^j} = \frac{\partial}{\partial\eta^i}\left(\left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^j} - \frac{A\eta_j}{\alpha^3}\right)\langle[v,\eta]_{\mathfrak{m}},\eta\rangle + \frac{A}{\alpha}\left(\left\langle[v,v_j]_{\mathfrak{m}},\eta\right\rangle + \langle[v,\eta]_{\mathfrak{m}},v_j\rangle\right)\right)$$

$$= \left(\frac{1}{\alpha}\frac{\partial^2 A}{\partial\eta^i\partial\eta^j} - \frac{\eta_i}{\alpha^3}\frac{\partial A}{\partial\eta^j} - \frac{\eta_j}{\alpha^3}\frac{\partial A}{\partial\eta^i} - \frac{A}{\alpha^3}\delta_i^j + \frac{3A}{\alpha^5}\eta_i\eta_j\right)\langle[v,\eta]_{\mathfrak{m}},\eta\rangle$$
$$+ \left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^j} - \frac{A\eta}{\alpha^3}\right)\left(\langle[v,v_i]_{\mathfrak{m}},\eta\rangle\langle[v,\eta]_{\mathfrak{m}},v_i\rangle\right)$$
$$+ \left(\frac{1}{\alpha}\frac{\partial A}{\partial\eta^i} - \frac{A}{\alpha^3}\eta_i\right)\left(\langle[v,v_j]_{\mathfrak{m}},\eta\rangle\langle[v,\eta]_{\mathfrak{m}},v_j\rangle\right)$$
$$+ \frac{A}{\alpha}\left(\langle[v,v_j]_{\mathfrak{m}},v_i\rangle + \langle[v,v_i]_{\mathfrak{m}},v_j\rangle\right), \tag{4.1}$$

and

$$\frac{\partial v_1}{\partial\eta^j} = \frac{\partial}{\partial\eta^j}\left(\frac{(s-1)^2+1}{1-2s}A\langle[v,\eta]_m,v\rangle\right)$$
$$= \left(\frac{(s-1)^2+1}{1-2s}\frac{\partial A}{\partial\eta^j} - \frac{2s^2-2s-2}{(2s-1)^2}As_{\eta^j}\right)\langle[v,\eta]_m,v\rangle$$
$$+ \frac{(s-1)^2+1}{1-2s}A\langle[v,v_j]_m,v\rangle,$$

$$\frac{\partial^2 v_1}{\partial\eta^i\partial\eta^j} = \frac{\partial}{\partial\eta^i}\left[\left(\frac{(s-1)^2+1}{1-2s}\frac{\partial A}{\partial\eta^j} - \frac{2s^2-2s-2}{(2s-1)^2}As_{\eta^j}\right)\langle[v,\eta]_{\mathfrak{m}},v\rangle + \frac{(s-1)^2+1}{1-2s}A\langle[v,v_j]_{\mathfrak{m}},v\rangle\right]$$

$$= \left(\frac{(s-1)^2 + 1}{1-2s} \frac{\partial^2 A}{\partial \eta^i \partial \eta^j} - \frac{2s^2 - 2s - 2}{(2s-1)^2} s_{\eta^i} \frac{\partial A}{\partial \eta^j} - \frac{2s^2 - 2s - 2}{(2s-1)^2} s_{\eta^j} \frac{\partial A}{\partial \eta^i} - \frac{10A}{(2s-1)^3} s_{\eta^i} s_{\eta^j} - \frac{2s^2 - 2s - 2}{(2s-1)^2} A s_{\eta^i} s_{\eta^j} \right) \langle [v, \eta]_{\mathfrak{m}}, v \rangle$$

$$+ \left(\frac{(s-1)^2 + 1}{1-2s} \frac{\partial A}{\partial \eta^j} - \frac{2s^2 - 2s - 2}{(2s-1)^2} A s_{\eta^j} \right) \langle [v, v_i]_{\mathfrak{m}}, v \rangle$$

$$+ \left\{ \frac{(s-1)^2 + 1}{1-2s} \frac{\partial A}{\partial \eta^i} - \frac{2s^2 - 2s - 2}{(2s-1)^2} A s_{\eta^i} \right\} \langle [v, v_j]_{\mathfrak{m}}, v \rangle, \quad (4.2)$$

substituting (4.1) and (4.2) in E_{ij}, we get the formula $E_{ij}(H, \eta)$. ■

5. Conclusion:

In this paper, we have obtained the exact formula of S-curvature for the (α, β)-metric of a Finsler homogeneous space with vanishing S-curvature, we proved the existence of invariant vector field and derived the mean Berwald curvature of Randers-Matsumoto metric. The study of S-curvature helps to study the topological properties of Einstein (α, β)-metric with vanishing S-curvature, homogeneous Ricci flat (α, β) space, and homogeneous (α, β)-metrics with flag curvature.

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