

# Solution of Homogeneous Linear Fractional Differential Equations Involving Conformable Fractional Derivative

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## Abstract

In this paper, we have found the solution of linear sequential fractional differential equations (LSFDE) involving conformable fractional derivatives of order  $\alpha$  with constant coefficients. For this purpose, we first discussed the fundamental properties of the conformable derivative and then obtained successive conformable derivatives of the fractional exponential function. After this, we determined the analytic solution of homogeneous LSFDE in terms of a fractional exponential function. We have demonstrated this developed method with a few examples of homogeneous LSFDE. This method gives a conjugation with the method to solve classical linear differential equations with constant coefficients.

**Keywords:** Fractional exponential function, Riemann–Liouville derivative, Caputo fractional derivative, Conformable fractional derivative.

## 1. Introduction

A modern area of study in applied sciences, including biology, mathematics, engineering, and physics, is fractional calculus. Numerous authors have already defined the fractional derivative.

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The Riemann–Liouville (R-L) definition, the Caputo definition of the fractional derivative, and the Jumarie left-handed modification of the R-L fractional derivative are a few major contributors. [1-9]. To date, there is not any standard approach to solving fractional differential equation (FDE) because different forms of fractional derivatives give different types of solutions. There are various integral transforms (Natural transform, Laplace transform, Fourier transform etc.) to solve classical differential equations of integer order [6] [10,11]. Some authors have extensively studied the solutions of FDE via integral transform. Thus, approaches for solving FDE and interpretation of these solutions is an emerging fields of applied mathematics. Many researchers have used the definition of fractional derivatives to solve FDE. Here we are presenting some definitions of fractional derivatives as follows.

**Definition 1.1.** The R-L left fractional derivative is defined as [12, 13]

$${}_a D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha+1)} \left(\frac{d}{dx}\right)^{m+1} \int_a^x (x-u)^{m-\alpha} f(u) du \tag{1}$$

where  $m \leq \alpha < m + 1$ ,  $m$  is positive integer.

particularly when,  $0 \leq \alpha < 1$ , then

$${}_a D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha+1)} \frac{d}{dx} \int_a^x (x-u)^{-\alpha} f(u) du \tag{2}$$

The right R-L fractional derivative is defined as;

$${}_x D_b^\alpha f(x) = \frac{1}{\Gamma(m-\alpha+1)} \left(-\frac{d}{dx}\right)^{m+1} \int_x^b (u-x)^{m-\alpha} f(u) du \tag{3}$$

where  $m \leq \alpha < m + 1$ ,  $m$  is a positive integer.

The classical derivative of a constant is always zero, but R-L definitions (left & right) give a non-zero value for the derivative of a constant.

**Definition 1.2.** To overcome the drawback of the R-L definition (a non-zero value of the derivative of a constant) of fractional derivative in 1967, Prof. Caputo modified it. Caputo’s definition of fractional derivative goes as follows [1];

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-u)^{n-\alpha-1} f^n(u) du \tag{4}$$

where  $n - 1 \leq \alpha < n$

According to this definition first differentiate  $f(x)$ ,  $n$  times and then integrate  $n - \alpha$  times.

FDE of Caputo’s type and classical differential equation have similar initial conditions while, the R-L type differential equation has initial conditions fractional types i.e.,  $\lim_{x \rightarrow a} {}_a D_x^{\alpha-1} f(x) = b$ . Caputo’s definition was also having a shortcoming that the function  $f(x)$  must be differentiable  $n$  times then the derivative of the order  $\alpha$  will exist, where  $n - 1 \leq \alpha < n$ . Thus, this method becomes inapplicable for non-differentiable functions.

**Definition 1.3.** The Jumarie modified existing definition of the fractional derivative of the function  $f(x)$  in the interval  $[a, b]$  as follows [14-18];

$${}_a D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_a^x (x-u)^{-\alpha-1} f(u) du, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-u)^{-\alpha} [f(u) - f(a)] du, & 0 < \alpha < 1 \\ [f^{\alpha-m}(x)]^m, & m \leq \alpha < m + 1 \end{cases} \quad (5)$$

If  $x < a$  then  $f(x) - f(a) = 0$ .

First line in “Eq. (5)” represents fractional integration; the second expression is for R-L derivative of order  $0 < \alpha < 1$  of offset function i.e  $f(x) - f(a)$ . The third expression is used for  $\alpha > 1$ . This definition exhibits all the properties which we need to be fully consistent with the definition via a fractional derivative. It gives the fractional derivative of a constant function is zero which was a notable drawback of the R-L fractional derivative definition.

When we studied the previous definitions of fractional derivatives, we found that there were some disturbances with those definitions. The following shortcomings with those fraction derivatives are as follows:

- (i) R-L definitions (left & right) give a non-zero value of the fractional derivative of a constant.
- (ii) All types of fractional derivatives do not satisfy the product rule of differentiation.

$$D_a^\alpha [u(x).v(x)] = u(x)D_a^\alpha [g(x)] + v(x)D_a^\alpha [f(x)]$$

- (iii) All types of fractional derivatives do not satisfy the quotient rule of differentiation.

$$D_a^\alpha \left[ \frac{u(x)}{v(x)} \right] = \frac{v(x)D_a^\alpha[u(x)] - u(x)D_a^\alpha[v(x)]}{[v(x)]^2}$$

- (iv) All types of fractional derivatives do not satisfy the chain rule of differentiation

$$D_a^\alpha[u(x) \circ v(x)] = D_a^\alpha[u(v(x))]D_a^\alpha[v(x)]$$

- (v) In Caputo definition of fractional derivative assumes that the function  $f(x)$  is differentiable.

## 2. Conformable Fractional Derivative

To overcome the shortcomings of fractional derivatives, Khalil, et al. [7] presented a completely new definition of a fractional derivative named conformable derivative.

**Definition 2.1.** The conformable fractional derivative of a function  $f: [0, \infty] \rightarrow \mathbb{R}$  is defined as [7-9]

$$T_\alpha[f(x)] = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon} \tag{6}$$

For all  $x > 0, \alpha \in (0,1)$ . If  $f$  is  $\alpha$  differentiable in some interval  $(0, a)$ ,  $a > 0$  and  $\lim_{x \rightarrow 0^+} T_\alpha[f(x)]$  exist then  $T_\alpha(o) = \lim_{x \rightarrow 0^+} T_\alpha[f(x)]$ .

The function  $f(x)$  is said to be  $\alpha$ -differentiable when the conformable fractional derivative of  $f(x)$  of order  $\alpha$  exists, this definition of the conformable fractional derivative of a function coincides with the classical definitions of Riemann–Liouville. This definition of the conformable fractional derivative of a function is also coincides with a Caputo definition of the derivative of polynomials (up to a constant multiple). The conformable fractional derivative is more practicable and usual.

If  $\alpha \in (0,1)$  and  $\alpha \in (n, n+1]$  then the definition of conformable derivative would be as the following.

**Definition 2.2.** Let  $f$  be a  $n$  times differentiable function for all  $x > 0$  and  $\alpha \in (n, n + 1)$  then the conformable fractional derivative of function  $f$  is defined as [9]

$$T_{\alpha}[f(x)] = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(x+\varepsilon x^{[\alpha]-\alpha}) - f^{([\alpha]-1)}(x)}{\varepsilon} \text{ where } [\alpha] \text{ is smallest integer function.}$$

Khalil et al. [7] proved that a function  $f: [0, \infty) \rightarrow \mathbb{R}$ , which is  $\alpha$ -differentiable at  $x_0 > 0, \alpha \in (0,1)$ , then  $f$  is continuous at  $x_0$ .

**Corollary 2.1.** Let  $f(x)$  and  $g(x)$  be  $\alpha$ -differentiable functions at a point  $x > 0$  and  $\alpha \in (0,1)$ , then

$$(i) T_{\alpha}[af(x) + bg(x)] = a T_{\alpha}[f(x)] + b T_{\alpha}[g(x)]$$

$$(ii) T_{\alpha}[f(x) g(x)] = f(x) T_{\alpha}[g(x)] + g(x) T_{\alpha}[f(x)]$$

$$(iii) T_{\alpha} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) T_{\alpha}[f(x)] - f(x) T_{\alpha}[g(x)]}{[g(x)]^2}$$

**Theorem 2.1.** The following various formulae are hold for conformable fractional derivatives:

$$(i) T_{\alpha}(c) = 0, \text{ where } c \text{ is a constant.}$$

$$(ii) T_{\alpha}(x^p) = px^{p-\alpha}$$

$$(iii) T_{\alpha}(\sin ax) = ax^{1-\alpha} \cos ax$$

$$(iv) T_{\alpha}(\cos ax) = -ax^{1-\alpha} \sin ax$$

$$(v) T_{\alpha}(e^{ax}) = ax^{1-\alpha} e^{ax}$$

$$(vi) T_{\alpha} \left( \frac{1}{\alpha} x^{\alpha} \right) = 1$$

**Proof:**

$$(i) \text{ Using "Eq. (6)"}$$

$$T_{\alpha}[f(x)] = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}$$

$$T_{\alpha}[c] = \lim_{\varepsilon \rightarrow 0} \frac{c - c}{\varepsilon}$$

$$T_{\alpha}[c] = 0$$

$$\begin{aligned}
 \text{(ii)} \quad T_\alpha[x^p] &= \lim_{\epsilon \rightarrow 0} \frac{(x+\epsilon x^{1-\alpha})^p - x^p}{\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \frac{x^p + \binom{p}{1} x^{p-1} (\epsilon x^{1-\alpha}) + \dots + \binom{p}{p} \epsilon^{p-1} (x^{1-\alpha})^p - x^p}{\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \frac{\epsilon [p x^{p-1} (x^{1-\alpha}) + \dots + \binom{p}{p} \epsilon^{p-1} (x^{1-\alpha})^p]}{\epsilon} \\
 &= p x^{p-1} (x^{1-\alpha}) \\
 T_\alpha[x^p] &= p x^{p-\alpha}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad T_\alpha[\sin ax] &= \lim_{\epsilon \rightarrow 0} \frac{\sin(x+\epsilon x^{1-\alpha}) - \sin ax}{\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \frac{(\sin ax) \cos(a\epsilon x^{1-\alpha}) - \cos ax \sin(a\epsilon x^{1-\alpha}) - \sin ax}{\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \sin ax \left[ \frac{\cos(a\epsilon x^{1-\alpha}) - 1}{\epsilon} \right] + \lim_{\epsilon \rightarrow 0} \left[ \frac{\cos ax \sin(a\epsilon x^{1-\alpha})}{\epsilon} \right] \\
 &= \lim_{\epsilon \rightarrow 0} (\sin ax)(x^{1-\alpha}) \left[ \frac{\cos(a\epsilon x^{1-\alpha}) - 1}{\epsilon x^{1-\alpha}} \right] + \lim_{\epsilon \rightarrow 0} \left[ x^{1-\alpha} \frac{\cos ax \sin(a\epsilon x^{1-\alpha})}{\epsilon x^{1-\alpha}} \right]
 \end{aligned}$$

Let  $\epsilon x^{1-\alpha} = h$

$$T_\alpha[\sin ax] = \lim_{h \rightarrow 0} (\sin ax)(x^{1-\alpha}) \left[ \frac{\cos(ah) - 1}{h} \right] + \lim_{h \rightarrow 0} \left[ x^{1-\alpha} \frac{\cos ax \sin(ah)}{h} \right]$$

Using L. Hospital's rule

$$T_\alpha[\sin ax] = (x^{1-\alpha}) \sin ax \lim_{h \rightarrow 0} (-\sin ah) + x^{1-\alpha} a \cos ax$$

$$T_\alpha[\sin ax] = a x^{1-\alpha} \cos ax$$

$$\text{(iv)} \quad T_\alpha[\cos ax] = \lim_{\epsilon \rightarrow 0} \frac{\cos(x+\epsilon x^{1-\alpha}) - \cos ax}{\epsilon}$$

$$\begin{aligned}
 &= \lim_{\epsilon \rightarrow 0} \frac{(\cos ax) \cos(a\epsilon x^{1-\alpha}) - \sin ax \sin(a\epsilon x^{1-\alpha}) - \cos ax}{\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \cos ax \left[ \frac{\cos(a\epsilon x^{1-\alpha}) - 1}{\epsilon} \right] + \lim_{\epsilon \rightarrow 0} \left[ \frac{\sin ax \sin(a\epsilon x^{1-\alpha})}{\epsilon} \right] \\
 &= \lim_{\epsilon \rightarrow 0} (\cos ax)(x^{1-\alpha}) \left[ \frac{\cos(a\epsilon x^{1-\alpha}) - 1}{\epsilon x^{1-\alpha}} \right] + \lim_{\epsilon \rightarrow 0} \left[ x^{1-\alpha} \frac{\sin ax \sin(a\epsilon x^{1-\alpha})}{\epsilon x^{1-\alpha}} \right]
 \end{aligned}$$

Let  $\epsilon x^{1-\alpha} = h$

$$T_\alpha[\cos ax] = \lim_{h \rightarrow 0} (\cos ax)(x^{1-\alpha}) \left[ \frac{\cos(ah) - 1}{h} \right] + \lim_{h \rightarrow 0} \left[ x^{1-\alpha} \frac{\sin ax \sin(ah)}{h} \right]$$

Using L. Hospital's rule

$$\begin{aligned}
 T_\alpha[\cos ax] &= (x^{1-\alpha}) \cos ax \lim_{h \rightarrow 0} (-\sin ah) - x^{1-\alpha} a \sin ax \lim_{h \rightarrow 0} \cos ah \\
 T_\alpha[\cos ax] &= -a x^{1-\alpha} \sin ax
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad T_\alpha[e^{ax}] &= \lim_{\epsilon \rightarrow 0} \frac{e^{a(x+\epsilon x^{1-\alpha})} - e^{ax}}{\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \frac{e^{ax} e^{a\epsilon x^{1-\alpha}} - e^{ax}}{\epsilon} \\
 &= \lim_{\epsilon \rightarrow 0} \frac{e^{ax}(e^{a\epsilon x^{1-\alpha}} - 1)}{\epsilon} \\
 &= x^{1-\alpha} \lim_{\epsilon \rightarrow 0} \frac{e^{ax}(e^{a\epsilon x^{1-\alpha}} - 1)}{\epsilon x^{1-\alpha}}
 \end{aligned}$$

Let  $\epsilon x^{1-\alpha} = h$

$$T_\alpha[e^{ax}] = x^{1-\alpha} \lim_{\epsilon \rightarrow 0} \frac{e^{ax}(e^{ah} - 1)}{h}$$

Using L. Hospital's rule

$$T_\alpha[e^{ax}] = a x^{1-\alpha} e^{ax}$$

$$\begin{aligned}
 \text{(vi)} \quad T_\alpha \left[ \frac{1}{\alpha} x^\alpha \right] &= \lim_{\varepsilon \rightarrow 0} \frac{\frac{(x+\varepsilon x^{1-\alpha})^\alpha}{\alpha} - \frac{1}{\alpha} x^\alpha}{\varepsilon} \\
 &= \frac{1}{\alpha} \lim_{\varepsilon \rightarrow 0} \frac{(x + \varepsilon x^{1-\alpha})^\alpha - x^\alpha}{\varepsilon} \\
 &= \frac{1}{\alpha} \lim_{\varepsilon \rightarrow 0} \frac{x^\alpha + \binom{\alpha}{1} x^{\alpha-1} \varepsilon x^{1-\alpha} + \dots + \binom{\alpha}{\alpha-1} \varepsilon^{\alpha-1} x^{\alpha-1} + \binom{\alpha}{\alpha} \varepsilon^\alpha (x^{1-\alpha})^\alpha - x^\alpha}{\varepsilon} \\
 &= \frac{1}{\alpha} \lim_{\varepsilon \rightarrow 0} \frac{\binom{\alpha}{1} x^{\alpha-1} \varepsilon x^{1-\alpha} + \dots + \binom{\alpha}{\alpha-1} \varepsilon^{\alpha-1} x^{\alpha-1} + \binom{\alpha}{\alpha} \varepsilon^\alpha (x^{1-\alpha})^\alpha - x^\alpha}{\varepsilon} \\
 &= \frac{1}{\alpha} \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon \left[ \binom{\alpha}{1} + \dots + \binom{\alpha}{\alpha-1} \varepsilon^{\alpha-2} x^{\alpha-1} + \binom{\alpha}{\alpha} \varepsilon^{\alpha-1} (x^{1-\alpha})^\alpha - x^\alpha \right]}{\varepsilon} \\
 &= \frac{1}{\alpha} \alpha \\
 T_\alpha \left( \frac{1}{\alpha} x^\alpha \right) &= 1
 \end{aligned}$$

**Theorem 2.2.** If  $f(x)$  is differentiable function for of  $x > 0$  and  $\alpha \in (0,1)$ , then

$$T_\alpha[f(x)] = x^{1-\alpha} \frac{d}{dx}[f(x)] \tag{7}$$

Proof: By definition of conformable  $\alpha$  differentiation of a function  $f(x)$

$$T_\alpha[f(x)] = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}$$

Let  $\varepsilon x^{1-\alpha} = h$

$$T_\alpha[f(x)] = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{hx^{\alpha-1}}$$

$$T_\alpha[f(x)] = x^{1-\alpha} \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

We know that  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$  is first order derivative of  $f(x)$ . Therefore



$$T_\alpha[f(x)] = x^{1-\alpha} \lim_{h \rightarrow 0} \frac{d}{dx} [f(x)]$$

A function could be  $\alpha$ -differentiable at a point but it is not necessary that it would be differentiable at that point. For example, let  $f(t)=2\sqrt{x}$ , then,  $T_{\frac{1}{2}}f(0) = \lim_{t \rightarrow 0} T_{\frac{1}{2}}f(x) = 1$ , when  $T_{\frac{1}{2}}f(x) = 1$ , for all  $x > 0$ , but  $T_{\frac{1}{2}}f(0) =$  does not exist.

**Theorem 2.3.** Let  $T_\alpha$  be conformable fractional derivative for  $\alpha \in (0,1)$  and  $a \in \mathbb{R}$ , then the following formulae are hold:

- (i)  $T_\alpha \left( e^{\frac{1}{\alpha}x^\alpha} \right) = \left( e^{\frac{1}{\alpha}x^\alpha} \right)$
- (ii)  $T_\alpha \left( \sin \frac{1}{\alpha}x^\alpha \right) = \cos \left( \frac{1}{\alpha}x^\alpha \right)$
- (iii)  $T_\alpha \left( \cos \frac{1}{\alpha}x^\alpha \right) = -\sin \left( \frac{1}{\alpha}x^\alpha \right)$

**Proof:**

- (i) Using theorem 2.3  $T_\alpha \left( e^{\frac{1}{\alpha}x^\alpha} \right) = x^{1-\alpha} \frac{d}{dx} \left( e^{\frac{1}{\alpha}x^\alpha} \right)$   
 $T_\alpha \left( e^{\frac{1}{\alpha}x^\alpha} \right) = x^{1-\alpha} \left( e^{\frac{1}{\alpha}x^\alpha} \right) \frac{1}{\alpha} \alpha x^{\alpha-1}$   
 $T_\alpha \left( e^{\frac{1}{\alpha}x^\alpha} \right) = \left( e^{\frac{1}{\alpha}x^\alpha} \right)$
- (ii)  $T_\alpha \left( \sin \frac{1}{\alpha}x^\alpha \right) = x^{1-\alpha} \frac{d}{dx} \cos \left( \frac{1}{\alpha}x^\alpha \right) \frac{1}{\alpha} \alpha x^{\alpha-1}$   
 $T_\alpha \left( \sin \frac{1}{\alpha}x^\alpha \right) = \cos \left( \frac{1}{\alpha}x^\alpha \right)$
- (iii)  $T_\alpha \left( \cos \frac{1}{\alpha}x^\alpha \right) = -x^{1-\alpha} \frac{d}{dx} \left( \sin \frac{1}{\alpha}x^\alpha \right) \frac{1}{\alpha} \alpha x^{\alpha-1}$   
 $T_\alpha \left( \cos \frac{1}{\alpha}x^\alpha \right) = -\sin \left( \frac{1}{\alpha}x^\alpha \right)$

**Theorem 2.4.** The conformable fractional derivative of order  $n\alpha$  of fractional exponential function  $\exp \left( \frac{x^\alpha}{\psi\alpha} \right)$  is  $\frac{1}{\psi^n} \exp \left( \frac{x^\alpha}{\psi\alpha} \right)$ .

$$T_{n\alpha} \left[ \exp \left( \frac{x^\alpha}{\psi\alpha} \right) \right] = \frac{1}{\psi^n} \exp \left( \frac{x^\alpha}{\psi\alpha} \right)$$

Where  $T_{n\alpha} = T_\alpha T_\alpha T_\alpha \dots \dots \dots$  upto  $n -$  times.

**Proof:**

We know that  $T_\alpha[f(x)] = x^{1-\alpha} \frac{d}{dx} [f(x)]$

Let  $f(x) = \exp\left(\frac{x^\alpha}{\psi\alpha}\right)$

$$T_\alpha \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right] = x^{1-\alpha} \frac{d}{dx} \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right]$$

$$T_\alpha \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right] = x^{1-\alpha} \frac{1}{\psi\alpha} \alpha x^{\alpha-1} \exp\left(\frac{x^\alpha}{\psi\alpha}\right)$$

$$T_\alpha \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right] = \frac{1}{\psi} \exp\left(\frac{x^\alpha}{\psi\alpha}\right)$$

Now  $T_\alpha T_\alpha \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right] = T_\alpha \left[ \frac{1}{\psi} \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right]$

$$T_\alpha T_\alpha \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right] = \frac{1}{\psi} T_\alpha \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right]$$

$$T_\alpha T_\alpha \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right] = \frac{1}{\psi^2} \exp\left(\frac{x^\alpha}{\psi\alpha}\right)$$

Similarly

$$T_{n\alpha} \left[ \exp\left(\frac{x^\alpha}{\psi\alpha}\right) \right] = \frac{1}{\psi^n} \exp\left(\frac{x^\alpha}{\psi\alpha}\right)$$

**3. Conformable Fractional Integral**

Suppose we have a continuous function  $f(x) = x^p$  and let  $\alpha \in (0,1), p \in \mathbb{R}$  such that  $\alpha \neq -p$  then  $\alpha$ -integral of  $f(x)$  is defined as

$$I_\alpha(x^p) = \frac{x^{p+\alpha}}{p + \alpha}$$

If  $\alpha=1$ , then  $I_1(x^p) = \frac{x^{p+1}}{p+1}$ , Which is the classical integral of  $x^p$ .

**Definition 3.1** If  $f(t)$  is a continuous function, then the fractional integral of order  $\alpha$  of  $f(t)$  is defined as [7]

$$I_a^\alpha f(t) = I_1^a [t^{\alpha-1} f(t)] = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where  $a > 0$ ,  $\alpha \in (0,1)$  and the integral  $\int_a^t \frac{f(x)}{x^{1-\alpha}} dx$  is the classical Riemann improper integral.

Khalil et al. [7] proved that a continuous function  $f$  in the domain of  $I_\alpha$ , then

$$T_\alpha I_\alpha^a [f(t)] = f(t), \text{ for all } t \geq a.$$

### 4. Solution of Fractional Homogeneous Linear Differential Equation

If orders of derivatives are in a sequence in a FDE, then it is called a sequential FDE. A differential equation of the form

$$(a_0 T_{n\alpha} + a_1 T_{(n-1)\alpha} + a_2 T_{(n-1)\alpha} + \dots + a_{n-1} T_\alpha + a_n)y = 0 \tag{8}$$

where  $a_0, a_1, a_2, \dots, a_n$ , is known as sequential fractional homogeneous linear differential equation of order  $n\alpha$  with constant coefficients. In this equation  $T_{n\alpha}$  is conformable fractional derivative of order  $n\alpha$  and  $\alpha \in (0,1)$ .

Rewriting "Eq. (8)"

$$f(T_\alpha)y(x) = 0$$

where  $f(T_\alpha)$  is a linear fractional differential operator.

For  $\alpha=1$ , "Eq. (8)" becomes  $n^{\text{th}}$  order classical ordinary differential equation.

Consider  $y(x) = \exp\left(\frac{x^\alpha}{\psi\alpha}\right)$ ,  $\psi \neq 0$

From "Eq. (8)"

$$(a_0 T_{n\alpha} + a_1 T_{(n-1)\alpha} + a_2 T_{(n-1)\alpha} + \dots + a_{n-1} T_\alpha + a_n) \exp\left(\frac{x^\alpha}{\psi\alpha}\right) = 0$$

$$\left( a_0 T_{n\alpha} \left[ \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) \right] + \right. \\ \left. ca_1 T_{(n-1)\alpha} \left[ \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) \right] + \dots + a_{n-1} T_\alpha \left[ \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) \right] + a_n \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) \right) = 0$$

Using theorem 2.5

$$a_0 \frac{1}{\psi^n} \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) + a_1 \frac{1}{\psi^{n-1}} \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) + \dots \dots \dots + a_{n-1} \frac{1}{\psi} \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) + \\ a_n \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) = 0$$

$$\left[ a_0 \frac{1}{\psi^n} + a_1 \frac{1}{\psi^{n-1}} + a_2 \frac{1}{\psi^{n-2}} + \dots \dots \dots + a_{n-1} \frac{1}{\psi} + a_n \right] \exp \left( \frac{x^\alpha}{\psi^\alpha} \right) = 0 \\ f(\sigma) \exp \left( \sigma \frac{x^\alpha}{\alpha} \right) = 0$$

where  $f(\sigma) = a_0 \sigma^n + a_1 \sigma^{n-1} + a_2 \sigma^{n-2} + \dots \dots \dots + a_{n-1} \sigma + a_n$ . If  $\sigma$  is any root of an algebraic equation  $f(\sigma) = 0$ , then  $f(T_\alpha) \exp \left( \sigma \frac{x^\alpha}{\alpha} \right) = 0$ .

This shows that  $y(x) = \exp \left( \sigma \frac{x^\alpha}{\alpha} \right)$  is a solution of homogeneous LSFDE  $f(T_\alpha)y(x) = 0$ .

The algebraic equation  $f(\sigma) = 0$  is known as auxiliary equation of FDE  $f(T_\alpha)y(x) = 0$ .

**Theorem 4.1.** If  $y_1, y_2, y_3, \dots, y_n$  are linearly independent solutions of  $f(T_\alpha)y(x) = 0$ , then the linear combination  $y = C_1y_1 + C_2y_2 + C_3y_3 + \dots \dots \dots + C_ny_n$  is also its solution for the arbitrary constants  $C_i$  where  $i = 1, 2, 3, \dots \dots \dots n$ .

**Proof:** Let  $y_1, y_2, y_3, \dots, y_n$  be the solutions of linear fractional homogeneous differential equation  $f(T_\alpha)y(x) = 0$  with constant coefficients, then

$$f(T_\alpha)y_1 = f(T_\alpha)y_2 = f(T_\alpha)y_3 = \dots \dots \dots = f(T_\alpha)y_n = 0 \tag{9}$$

The linear combination of solutions is  $y = C_1y_1 + C_2y_2 + C_3y_3 + \dots \dots \dots + C_ny_n$

Operating  $f(T_\alpha)$ , we have

$$f(T_\alpha)y = f(T_\alpha)[C_1y_1 + C_2y_2 + C_3y_3 + \dots \dots \dots + C_ny_n]$$

Using linear property

$$f(T_\alpha)y = [C_1f(T_\alpha)y_1 + C_2f(T_\alpha)y_2 + C_3f(T_\alpha)y_3 + \dots \dots + C_n f(T_\alpha)y_n]$$

Using "Eq. (9)"

$$f(T_\alpha)y(x) = 0$$

**Theorem 4.2.** If  $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n$  are n real and distinct roots of the auxiliary equation of homogeneous LSFDE with constant coefficients, then its solution is

$$y(x) = C_1 \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) + C_2 \exp\left(\sigma_2 \frac{x^\alpha}{\alpha}\right) + C_3 \exp\left(\sigma_3 \frac{x^\alpha}{\alpha}\right) + \dots \dots \dots + C_n \exp\left(\sigma_n \frac{x^\alpha}{\alpha}\right)$$

**Proof:** Let  $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n$  be n real and distinct roots of  $f(\sigma) = 0$ , then we can write

$$f(\psi) = (\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3) \dots \dots \dots (\sigma - \sigma_n)$$

$$(T_\alpha - \sigma_1)\exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = T_\alpha \left[\exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] - \sigma_1 \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)$$

$$(T_\alpha - \sigma_1)\exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = 0$$

$$f(T_\alpha)\exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = 0$$

**Theorem 4.3.** If the roots of the auxiliary equation of homogeneous LSFDE with constant coefficients have r repeated roots ( $\sigma_1 = \sigma_2 = \sigma_3 = \dots = \sigma_r$ ) for  $1 \leq r \leq n$ , then its solution is

$$y(x) = C_1 \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) + x^\alpha C_2 \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) + x^{2\alpha} C_3 \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) + \dots \dots \dots + x^{(r-1)\alpha} C_r \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)$$

**Proof:** Let  $f(\sigma) = 0$  has r repeated roots  $\sigma_1 = \sigma_2 = \sigma_3 = \dots = \sigma_r$  for  $1 \leq r \leq n$

Then  $f(\sigma)$  can be written as  $f(\sigma) = g(\sigma)(\sigma - \sigma_1)^r$ , where  $g(m)$  is a polynomial of degree  $n-r$  satisfying  $g(m_1) \neq 0$

$$\begin{aligned}
 & (T_\alpha - \sigma_1)\exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = 0 \\
 & (T_\alpha - \sigma_1)^2 x^\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = (T_\alpha - \sigma_1)(T_\alpha - \sigma_1)x^\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) \\
 & = (T_\alpha - \sigma_1)\left[T_\alpha \left\{x^\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right\} - \sigma_1 x^\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] \\
 & = (T_\alpha - \sigma_1)\left[x^\alpha T_\alpha \left\{\exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right\} + \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) T_\alpha(x^\alpha) \right. \\
 & \quad \left. - \sigma_1 x^\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] \\
 & = (T_\alpha - \sigma_1)\left[x^\alpha \sigma_1 \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) + \alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) - \sigma_1 x^\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] \\
 & = (T_\alpha - \sigma_1)\left[\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = \alpha (T_\alpha - \sigma_1) \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] \\
 & (T_\alpha - \sigma_1)^2 x^\alpha \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = 0
 \end{aligned}$$

Similarly,  $(T_\alpha - \sigma_1)^r x^{(r-1)\alpha} \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) = 0$

Now

$$\begin{aligned}
 f(T_\alpha)y(x) &= C_1 f(T_\alpha) \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) + C_2 f(T_\alpha) \left[x^{2\alpha} \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] + \\
 C_3 f(T_\alpha) \left[x^{2\alpha} \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] + \dots & \dots \dots \dots + C_r f(T_\alpha) \left[x^{(r-1)\alpha} \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right)\right] \\
 f(T_\alpha)y(x) &= 0
 \end{aligned}$$

**Theorem 4.4** If the roots of the auxiliary equation of homogeneous LSFDE are complex  $\gamma \pm i\delta$ , then its solution is

$$y(x) = \exp\left(\gamma \frac{x^\alpha}{\alpha}\right) \left[ A \cos_\alpha \left(\left(\frac{\delta}{\alpha}\right) x^\alpha\right) + i B \sin_\alpha \left(\left(\frac{\delta}{\alpha}\right) x^\alpha\right) \right]$$

Where A and B are constants.

**Proof:** Consider  $\gamma \pm i\delta$  be two complex roots of auxiliary equation of  $f(T_\alpha)y(x) = 0$ .

Let  $\sigma_1 = \gamma + i\delta$  and  $\sigma_2 = \gamma - i\delta$

Using Theorem 2.2  $y(x) = C_1 \exp\left(\sigma_1 \frac{x^\alpha}{\alpha}\right) + C_2 \exp\left(\sigma_2 \frac{x^\alpha}{\alpha}\right)$

$$\begin{aligned}
y(x) &= C_1 \exp\left((\gamma + i\delta) \frac{x^\alpha}{\alpha}\right) + C_2 \exp\left((\gamma - i\delta) \frac{x^\alpha}{\alpha}\right) \\
&= C_1 \exp\left(\gamma \frac{x^\alpha}{\alpha}\right) \exp\left(i\delta \frac{x^\alpha}{\alpha}\right) + C_2 \exp\left(\gamma \frac{x^\alpha}{\alpha}\right) \exp\left(-i\delta \frac{x^\alpha}{\alpha}\right) \\
y(x) &= \exp\left(\gamma \frac{x^\alpha}{\alpha}\right) \left[ A \cos\left(\left(\frac{\delta}{\alpha}\right) x^\alpha\right) + iB \sin\left(\left(\frac{\delta}{\alpha}\right) x^\alpha\right) \right]
\end{aligned}$$

Now we will find the solutions of homogeneous LSFDE and will also provide the graphs of solutions of LSFDE and compare them with the solutions of the differential equation of integer order.

**Example 4.1.** [12] Consider  $(T_1 + 4T_{1/2} + 3)y(x) = 0$  be a homogeneous LSFDE.

### Solution

If  $\alpha = \frac{1}{2}$ , then

$$(T_{2\alpha} + 4T_\alpha + 3)y(x) = 0$$

The auxiliary equation is

$$\sigma^2 + 4\sigma + 3 = 0$$

$$\sigma = -1 \text{ and } -3$$

Hence, the solution is  $y(x) = C_1 \exp(-2x^{1/2}) + C_2 \exp(-6x^{1/2})$

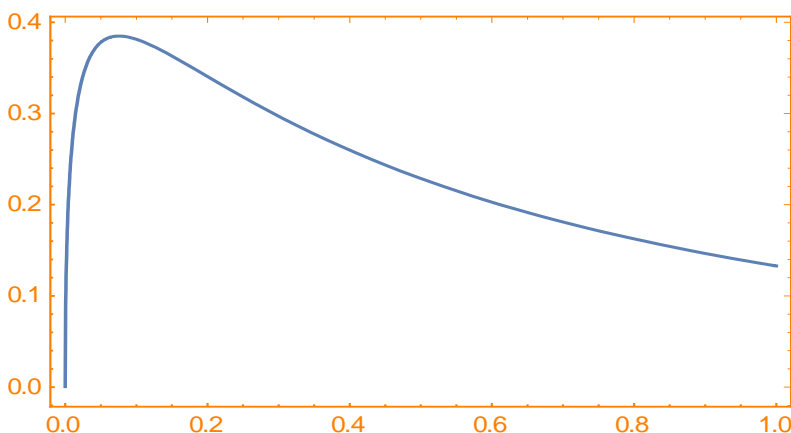


Figure 1: Solution of Example 4.1 for  $C_1 = 1$  and  $C_2 = -1$  and  $\alpha = \frac{1}{2}$

**Example 4.2.** Consider a homogeneous LSFDE  $D^{0.5}D^{0.5}y(x) - y(x) = 0$ ,  $D$  is a conformable fractional differential operator.

**Solution** This equation can be written as

$$D^{2\alpha}y(x) - y(x) = 0, \text{ where } \alpha = 0.5$$

The auxiliary equation of this FDE is

$$\sigma^2 - 1 = 0$$

$$\sigma = 1, -1$$

Hence, the solution is  $y(x) = C_1 e^{\left(\frac{1}{0.5}\right)x^{0.5}} + C_2 e^{\left(\frac{-1}{0.5}\right)x^{0.5}}$

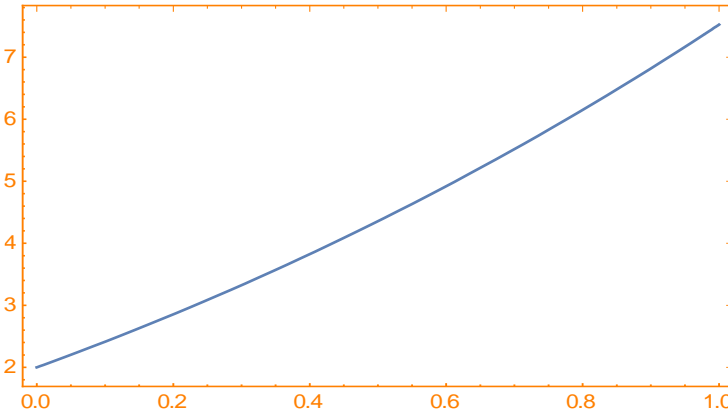


Figure 2: Solution of Example 4.2 when  $C_1 = C_2 = 1$  and  $\alpha = 0.5$

**Example 4.3** Let  $D^{0.5}D^{0.5}D^{0.5}((y(x))) = 0$  be a FDE.

**Solution** This equation can be written as  $D^{3\alpha}y(x) = 0$ , where  $\alpha = 0.5$

The auxiliary equation is  $\sigma^3 = 0$

Hence, the solution is  $y(x) = C_1 + \sqrt{x} C_2 + x C_3$



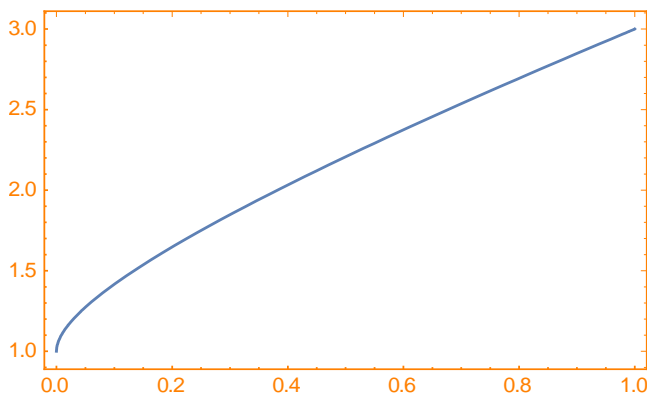


Figure 3: Solution of Example 4.3 when  $C_1 = C_2 = 1$  and  $\alpha = 0.5$

**Example 4.4** Consider a homogeneous LSFDE  $D^{0.25}D^{0.25}y(x) + y(x) = 0$ , where  $D$  is a conformable fractional differential operator.

**Solution** This equation can be written as  $D^{2\alpha}y(x) = 0$ , where  $\alpha = 0.25$

$$\sigma^2 + 1 = 0$$

$$\sigma = \pm i$$

Hence, the solution is  $y(x) = A\cos(4x^{0.25}) + iB\sin(4x^{0.25})$ , if  $A = B = 1$ , then the graph of solution of this LFSDE is given by

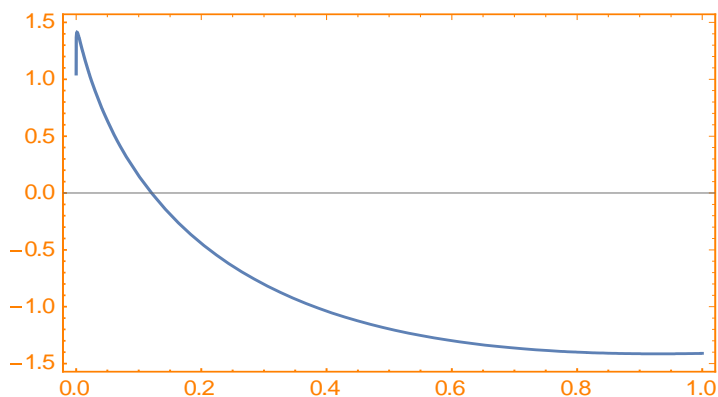


Figure 4: Solution of Example 4.4 when  $A = B = 1$  and  $\alpha = 0.25$

**Example 4.5.** Consider a homogeneous LSFDE

$$\frac{d^{3\alpha}y}{dx^{3\alpha}} + 6 \frac{d^{2\alpha}y}{dx^{2\alpha}} + 11 \frac{d^\alpha y}{dx^\alpha} + 6y = 0 \text{ where } \alpha = 0.25, 0.5, 1.$$

**Solution** The auxiliary equation of this FDE is

$$\sigma^3 + 6\sigma^2 + 11\sigma + 6 = 0 \text{ and } \sigma = -1, -2, -3$$

The solution is

(i)  $y(x) = C_1 \exp(-4 \sqrt[4]{x}) + C_2 \exp(-8 \sqrt[4]{x}) + C_3 \exp(-12 \sqrt[4]{x})$   
for  $\alpha = 0.25$

(ii)  $y(x) = C_1 \exp(-2 \sqrt{x}) + C_2 \exp(-4 \sqrt{x}) + C_3 \exp(-6 \sqrt{x})$  for  
 $\alpha = 0.5$

$y(x) = C_1 \exp(-x) + C_2 \exp(-2x) + C_3 \exp(-3x)$  for  $\alpha = 1$

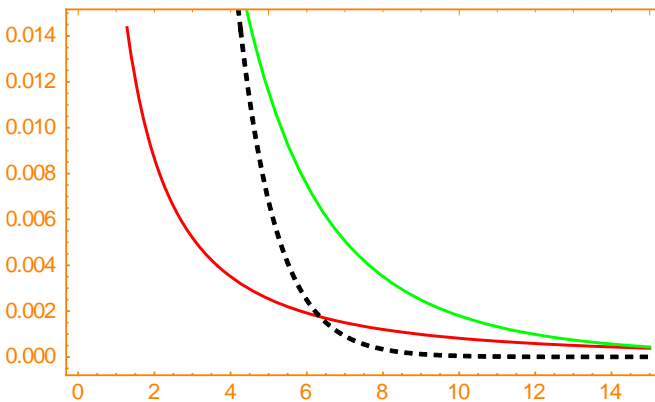


Figure 5: Red colour curve and green colour curve are solutions of Example 4.5 for  $\alpha = 0.25$  and  $\alpha = 0.5$  while dashed black colour is the solution for classical liner differential equation (for  $\alpha = 1$ ) corresponding to Example 4.5.

## 5. Conclusion

In this research paper, we developed an analytical method to solve homogeneous LSFDE with constant coefficients. This method depends on finding auxiliary equations of FDE and gives an association with the solutions of classical differential equations. This method is easier and more accurate.

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