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Stability Analysis of Visco-Elastically Damped Structure through Bagley Torvik Equation

Priyadharsini S*

ABSTRACT

The stability of fractional-order visco-elastically damped linear system Bagley Torvik equation is analyzed in this paper. The fundamental novelty of this paper is the application of Caputo derivative. Prevailing sufficient spectral conditions are considered to guarantee the stability of linear models. Laplace transform, and Mittag-Leffler functions are utilized to develop the results. Furthermore, asymptotical stability of linear fractional-order models are also achieved through spectral values of the characteristic polynomials. Numerical examples are given to display the effectiveness of suggested method.

Keywords: Stability, Newtonian fluid, Spectral values, Fractional system, Bagley Torvik equation, Viscosity, Linear systems
Mathematics Subject Classification (2010): 26A33, 76A05, 33Q35

INTRODUCTION

Fractional Calculus (FC) performing an exceptionally significant role in mathematical modelling for a range of physical systems in various discipline particularly in thermal conduction, diffusion wave, population dynamics, predator-prey model[1], plankton–fish model [2], control theory, controller in air and ocean circulation [3], models that includes memory, delayed model [4-5] and etc. In recent times, FC used as a precise representation of the real-life phenomena. Viscosity in a liquid (Newtonian element) is inevitable in the practical

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* Department of Mathematics, Sri Krishna Arts & Science College, Kuniamuthur, Coimbatore – 641 008; Email: priyamaths24@gmail.com

mechanical structures. It frequently affects stability, shifting of the control, performances and dynamic behaviours. Bagley Torvik equation [6] takes place in visco-elastically damped structures modeling, in which a plate (rigid) is fully submerged in a Newtonian fluid. Researchers made an attempt to describe Viscoelastic solids and its mechanical properties. Due to the lack of clear connection between the theoretical models, explaining the function of such elements and the relevant physical laws are still complex. To depict mechanical properties of these materials, the researcher has been forced to adopt empirical approaches.

A complex constant as material modulus, numerical methods, and the standard linear viscoelastic model are the classical models. The negative aspect of this model is that the huge derivative terms performing on stress and strain. Several viscoelastic materials have to be analyzed, which signify the frequency and damping property. The stress relaxation phenomenon seems to be proportional to time raised to fractional powers, according to early measurements of the mechanical characteristics of viscoelastic materials. Additionally, it can be deduced from the experimental findings that several metals and glasses had genuine fractional derivative connections. This article proposes a broader model of fractional derivatives. It is crucial to remember that fractional derivative interactions come from the ground up. The viscoelastic media's mechanical behavior can be established with the help of fractional model and its mathematical form. This model not only convinces second law of thermodynamics but also estimates elliptic stress-strain loops for the proposed viscoelastic materials.

Autonomous fractional dynamical systems and its stability have been analyzed by various methods. One of the influential methods is stability theorem [7-8], in which the position of spectral values in the complex plane is utilized to guarantee the stability of the given system. Other well-known methods are Fractional Lyapunov direct method, Linear Matrix Inequality (LMI), Mittag-Leffler Stability, operational matrix method [10], finite time stability [11] etc. Although it has been investigated in earlier research, the stability problem for fractional nonlinear systems is still unresolved. Computational methods may be employed to solve nonlinear fractional systems. Many numerical techniques are suggested for approximating solutions of the nonlinear system including wavelet Operational matrix method [12], Homotopy Perturbation Method [13-14], Variation Iteration Methods [15], Adomine decomposition method [16], Collocation method [17], Chebyshev Method [18], Euler algorithm [19], Argument condition [20]. Momani and Odobat's Numerical approach specified in [21-22], to solving the linear fractional system has been utilized in this work.

To model the viscoelastic material and calculate the structure's response to general loading conditions, only a small number of empirical parameters are required. Certain issues must be resolved before a visco-elastically damped structure's response can be determined successfully. For viscoelastic materials, it is necessary to construct stress-strain relationships based on the literature review. Fractional calculus is used to find closed-form solutions to the equations of motion for visco-elastically damped structures using these relationships. The organization of this paper is as follows. Section 2 consists of preliminary concepts and basic definitions. In section 3, the primary idea of investigating stability is summarized. Section 4 illustrates the considered theory through examples, in which graphical solutions are obtained through MATLAB. Finally, section 6 gives the conclusion.

PRELIMINARIES

This section contains basic definitions and standard results.

Definition 2.1: [23] (Riemann - Liouville Fractional Integral).

$$I^{\alpha}y(x) = \frac{1}{\Gamma\{(\alpha)\}}\int_0^x (x-s)^{\{\alpha-1\}}y(s)\,ds \qquad (1)$$

where $y \in L^1(R^+),\ \alpha > 0$.

Definition 2.: [23] ((Riemann - Liouville Fractional Derivative).

$$D^{\alpha}y(x) = D^{n}I^{\{n-\alpha\}}y(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_0^x (x-s)^{\{n-\alpha-1\}}y(s)ds \qquad (2)$$

where $\alpha > 0, n-1 < \alpha < n, n \in N, y^{\{n-1\}}(x) \in AB(R)^+$.

Definition 2.3: [23] (Caputo Fractional Derivative).

$${}^{C}_{F}D^{\alpha}y(x) = I^{\{n-\alpha\}}y^{n}(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-s)^{\{n-\alpha-1\}}y^{n}(s)ds \qquad (3)$$

where $\alpha > 0, n-1 < \alpha < n, n \in N, y^{n-1}(x) \in AB(R)^{+}$.

In particular

$$ {}_{F}^{C}D^{\alpha}y(x) = I^{1-\alpha}y'(x) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{x}(x-s)^{-\alpha}y'(s)ds \tag{4}$$

Definition 2.4: [23] Mittag-Leffler function. The Mittag-Leffler function can be defined in terms of two parameters

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}, \qquad (\alpha,\beta > 0, z \in \mathbb{C}). \tag{5}$$

Definition 2.5: [23] Laplace transform.

$$ {}_{F}^{C}D^{\alpha}y\,(x) = s^{\alpha}(L(y))(s) - y_1 s^{\{\alpha-1\}}, \text{when } 0 < \alpha < 1 \tag{6}$$

$$ {}_{F}^{C}D^{\alpha}y\,(x) = s^{\alpha}(L(y)(s) - y_1 s^{\{\alpha-1\}} + y_{2s}^{\{\alpha-1\}}, \text{when } 1 < \alpha < 2 \tag{7}$$

here $y_1 = y(0), y_2 = y'(0)$.

$$\mathcal{L}\{x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \left(\mathcal{R}(s) > |\lambda|^{\frac{1}{\alpha}}\right), \tag{8}$$

where $x \geq 0, \lambda \in R$.

Definition 2.6: Linear Fractional Differential System

Consider

$$ {}_{F}^{C}D^{\alpha}\,y(x) = P\,y(x), \tag{9}$$

with initial value $y(0) = y_0 = (y_{10}, y_{20}, \dots, y_{n0})^{T}$ where the above-mentioned system (9) is said to be

- stable if for any x_0, , there exists $\epsilon > 0$ such that $||y(x)|| \leq \epsilon$ for $x \geq 0$,
- asymptotically stable if $\lim_{x\to\infty} ||y(x)|| = 0$.

Theorem 2.7: Take the following system into consideration

$${}_{F}^{C}D^{\alpha}\,y(x) = P\,y(x),\ \alpha \in (0,1) \text{ and initial condition } y_0 = y(0) \tag{10}$$

i) asymptotically stable if and only if $\left|arg\left(spec(P)\right)\right| > \frac{\alpha\pi}{2}$

ii) stable if and only if either it is asymptotically stable, or those critical spectral values satisfy $\left|arg\left(spec(P)\right)\right| = \frac{\alpha\pi}{2}$ with geometric multiplicity one.

Note: Here *spec(P)* denotes the spectral values of the matrix. If the critical spectral values satisfying $\left|arg\left(spec(P)\right)\right| = \frac{\alpha\pi}{2}$ and all non-zero spectral values of satisfying have the same algebraic and geometric multiplicities, and the zero spectral value of *P* has the same algebraic and geometric multiplicities, then the system's zero solution is stable. As long as *P* matrix has a zero spectral value, this system's zero solution is never asymptotically stable.

3. STABILITY ANALYSIS OF LINEAR DIFFERENTIAL SYSTEM

Take the nonlinear system into consideration

$$ {}_{F}^{C}D^{\alpha}\, y(x) = P\, y(x) + f(x) \tag{11}$$

where $\alpha \in (0,1), f(x) \in C(R \times R^n, R^n), f(x) = 0$

With IC $y(0) = y_0 = (y_{10}, y_{20}, \dots, y_{n0})^T,\ P \in R^{n \times n}$.

Lemma 3.1: If all the spectral values of P meet the requirement

$$|\arg(spec(P))| > \frac{\alpha\pi}{2} \tag{12}$$

then there exists a constant value *K>0* such that,

$$\int_0^x ||\xi^{\alpha-1}\ \mathbb{E}_{\alpha,\alpha}(P\xi^{\alpha})||d\xi \le K. \tag{13}$$

Theorem 3.2: Suppose $f(x) \le M$ and all the spectral values of *P* meet the requirement (12). Then, the solution of system [11] is asymptotically stable.

Proof. The solution of (11) can be represented as follows

$$y(x) = E_{\alpha}(P\, x^{\alpha})y_0 + \int_0^x (x-s)^{\alpha-1}\ \mathbb{E}_{\alpha,\alpha}(P(x-s)^{\alpha})\ f\big(s, y(s)\big)ds \tag{14}$$

From which it follows that,

$$\left\|y(x)\right\| \leq\leq \left\|E_{\alpha}(P\,x^{\alpha})y_0\right\| + \int_0^x \|(x-s)^{\alpha-1}\ \mathbb{E}_{\alpha,\alpha}(P(x-s)^{\alpha})\|\,\|\,f(s,y(s))\|ds$$

$$\leq \left\|E_{\alpha}(P\,x^{\alpha})y_0\right\| + \int_0^x \|\xi^{\alpha-1}\ \mathbb{E}_{\alpha,\alpha}(P\xi^{\alpha})\|\,\|\,f(x-\ \xi, y(t-\xi))\|d\xi$$

$$\leq \left\|E_{\alpha}(P\,x^{\alpha})y_0\right\| + M\int_0^x \|\xi^{\alpha-1}\ \mathbb{E}_{\alpha,\alpha}(P\xi^{\alpha})\|d\xi$$

From Gronwall Inequality, we have

$$\left\|y(x)\right\| \leq \ \left\|E_{\alpha}(P\,x^{\alpha})y_0\right\| \exp\left\{ M\int_0^x \|\xi^{\alpha-1}\ \mathbb{E}_{\alpha,\alpha}(P\xi^{\alpha})\|\,d\xi\right\} \qquad (15)$$

By using Lemma 3.1, we have

$\exp\{ M\ \int_0^x \|\xi^{\alpha-1}\ \mathbb{E}_{\alpha,\alpha}(P\xi^{\alpha})\|\,d\xi\}$is bounded. (16)

Further, $\left\|E_{\alpha}(P\,x^{\alpha})y_0\right\| \to\ 0$ as $x \to \infty$. Hence, we have $\lim_{x\to\infty} y(x) \to 0$. Therefore, there exists an asymptotically stable solution.

MAIN WORK - BAGLEY TORVIK EQUATION

Internal friction or air induces the damping effect, which makes the vibration of system to stop after time. This dampening effect depends on the air resistance to the movement of mass, and the corresponding resistance is proportional to the velocity of the moving mass. Thereby, fractional form of the damping force occurs in rigid pate immersed in a Newtonian fluid model. This type of system is discussed in [24-25].

Considering the forces, we have

$$my''(x) = mf(x) - ky(x) - 2S\sigma(x,0)$$

Using the relationships

$$\sigma(x,z) = \sqrt{(\mu\rho)}D_x^{1/2}v(s,z), v_p(x,0) = y'(x)$$

We get the below simplified form.

$$y''(x) +\ A\,D_x^{\left\{\frac{3}{2}\right\}y(x)} + B\,y(x) = f(x)(x>0),$$

$$A = \left(\frac{2S}{m}\right)\sqrt{\backslash mu\ \rho}, B = \frac{k}{m}.$$

A real, conventionally formulated physical system is described using the fractional derivative. For further details see [24].

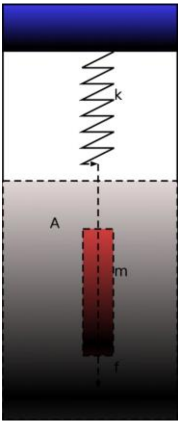

Figure 1: Bagley-Torvik Equation

Analytic Solution

This equation can be represented as follows.

$$ {}^{C}D^{2}y(x) + A^{C}D^{\frac{3}{2}}y(x) + By(x) = f(x). \tag{17}$$

The system of fractional differential equation is given by

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}y_1(x) &= y_2(x), \\ {}^{C}D^{\frac{1}{2}}y_2(x) &= y_3(x), \\ {}^{C}D^{\frac{1}{2}}y_3(x) &= y_4(x), \\ {}^{C}D^{\frac{1}{2}}y_4(x) &= -By_1(x) - Ax_4(x) + f(x), \end{aligned} \tag{18}$$

where $y_1(x) = y(x)$, with IC $y_1(0) = y_2(0) = y_3(0) = y_4(0) = 0$. The standard form is given by

$$ {}^{C}D^{\alpha}y(x) = Py(x) + F(x) \tag{19}$$

and $F(x) = f(x) = (0,0,0,15)^T/A \;\; 0 < x \leq 1$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -B & 0 & 0 & -A \end{bmatrix} \tag{20}$$

Laplace transform Technique

$$^{C}D^{\alpha}y(x) + Py(x) = f(x) \qquad (21)$$

with $y(0) = 0, 0 < \alpha < 1$ with the help of Laplace transform we reach,

$$s^{\alpha}Y(s) - y(0) + PY(s) = \frac{1}{s} \qquad (22)$$

$$Y(s) = \frac{1}{s(s^{\alpha}+P)} \qquad (23)$$

By taking inverse Laplace transform

$$y(x) = L^{-1}\frac{1}{s(s^{\alpha}+P)} = x^{\alpha}E_{\alpha,\alpha-1}(-Px^{\alpha}) \qquad (24)$$

which is the required analytical solution.

Stability Analysis

The stability analysis depends on the characteristic functions and spectral values of the matrix. Characteristic equation is given by

$$\lambda^4 + A\lambda^3 + B = 0 \qquad (25)$$

The spectral values of P are

$$\lambda_1 = -0.25A + 0.5\sqrt{(I_1)} + 0.5\sqrt{(I_2)} \qquad (26)$$

$$\lambda_2 = -0.25A + 0.5\sqrt{(I_1)} - 0.5\sqrt{(I_2)} \qquad (27)$$

$$\lambda_3 = -0.25A - 0.5\sqrt{(I_1)} + 0.5\sqrt{(I_2)} \qquad (28)$$

$$\lambda_4 = -0.25A - 0.5\sqrt{(I_1)} - 0.5\sqrt{(I_2)} \qquad (29)$$

Where, $I_1 = 0.25A^2 + \frac{3.4943B}{I_3} + 0.3816I_3, I_2 = 0.75A^2 - I_1 - \frac{0.25A^2}{\sqrt{I_1}}, I_3 = \left(9A^2B + 1.732\sqrt{(I_4)}\right)^{1/3}, I_4 = 27A^4B^2 - 256B^3$. Next, different cases are analyzed, in which the cases satisfies the $\left|\arg(spec(P))\right| > \frac{\pi}{4}$ conditions are concentrated for the stability analysis.

Case 1: $I_1 < 0, I_2 < 0$

In this case the spectral values, the real part will be -0.25 A. In particular, if *A>0*, then all the spectral values have negative real part. Hence, there exists an asymptotically stable solution. If *A<0,* further investigation is required to guarantee the stability.

Case 2: $I_1 > 0, I_2 < 0$

In this case the spectral values are $-0.25(A \pm 2\sqrt{(I_1)})$. In particular if $A \pm 2\sqrt{(I_1)} > 0$, then all the spectral values have negative real part, and hence there exists an asymptotically stable solution. If $A \pm 2\sqrt{(I_1)} < 0$ further investigation is required to guarantee the stability.

Case 3: $I_1 < 0, I_2 > 0$

In this case the spectral values, whose real part will be $-0.25(A \pm 2\sqrt{(I_2)})$. In particular if $A \pm 2\sqrt{(I_2)} > 0$, then all the spectral values have negative real part, and hence there exists an asymptotically stable solution. If $A \pm 2\sqrt{(I_2)} < 0$, further investigation is required to guarantee the stability.

Case 4: $I_1 > 0, I_2 > 0$

In this case the real part of the spectral values will be $-0.25(A \pm 2\sqrt{(I_1)})$. In particular, if $(A \pm 2\sqrt{(I_1)} \pm 2\sqrt{(I_2)}) > 0$, then all the spectral values have negative real part, hence there exists an asymptotically stable solution.

If any one of the root satisfies condition $(A \pm 2\sqrt{(I_1)} \pm 2\sqrt{(I_2)} < 0$, then the spectral values would not have negative real root. In this case the argument condition is not satisfied. Thereby, the given system is not stable.

Case 5: $B = 0$

In this case the spectral values are given by 0 with multiplicity three and -A . In this case the given system is stable for the positive value of A and zeros must have same geometric and algebraic multiplicity. Otherwise, it is not stable.

Case 6: $A = 0$

In this case the spectral values are given by $(-B)^{1/4}$. For $B > 0$, spectral values are given by $0.7071 \pm 0.7071i$, which satisfies the condition $|arg(\lambda)| = \pi/4 = \alpha\pi/2$. In this case the given system is stable but not asymptotically stable. For $B < 0$, spectral values are real and this fails to satisfy the condition$|arg(\lambda)| > \alpha\pi/2$. . In this case the given system is not stable.

5. EXAMPLES

Example 5.1 (Bagley-Torvik equation)

Take the following fractional system into consideration

$$ {}^{C}D^{2}y(x) + A\,{}^{C}D^{3/2}y(x) + By(x) = f(x), \qquad (30)$$

where A, $B \in \mathbb{R}$ and $\alpha = \frac{3}{2}$with IC $y(0) = y'(0) = 0$,

$$f(x) = \begin{cases} 15, & 0 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The stability of Bagley-Torvik equation depends on the parameters and it can be seen from the following. If $A = B = 1$, the standard form is given by ${}^{C}D^{1/2}y(x) = Py(x) + F(x)$ and , where $F(x) = (0,0,0,15)^T \ \ 0 < x \leq 1$,

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \end{bmatrix} \qquad (31)$$

The spectral values of P are $-1.0189 \pm 0.6026i, 0.5189 \pm 0.6666i$, which satisfies $\left|\arg(spec(P))\right| \ > \frac{\pi}{4}$.. It meets all the requirements of Theorem 3.2. Hence, there exists an asymptotically stable solution of (30) see [Fig 2].

If $A = -1, B = 1$, the standard form is ${}^{C}D^{1/2}y(x) = Py(x) + F(x)$ and where $F(x) = (0,0,0,15)^T \ \ 0 < x \leq 1$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad (32)$$

The spectral values of P are $1.0189 \pm 0.6026i, -0.5189 \pm 0.6666i$, which fails to satisfy the spectral condition $\left|\arg(spec(P))\right| > \frac{\pi}{4}$. It does not meet the allthe requirement of the Theorem 3.2. Hence, there exists a non-stable solution of (30) see [Fig 3].

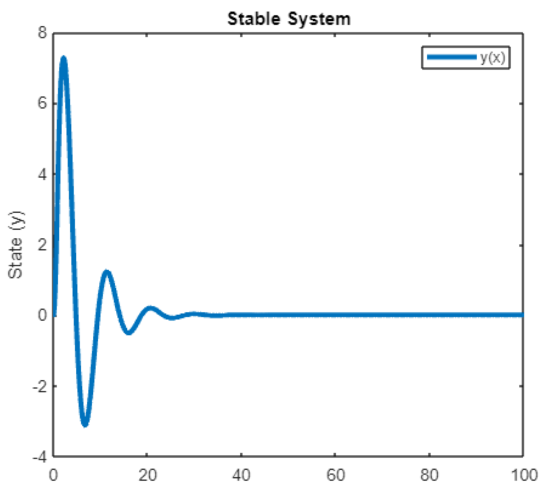

Figure 2: StaFble

Figure 2: Stable systems

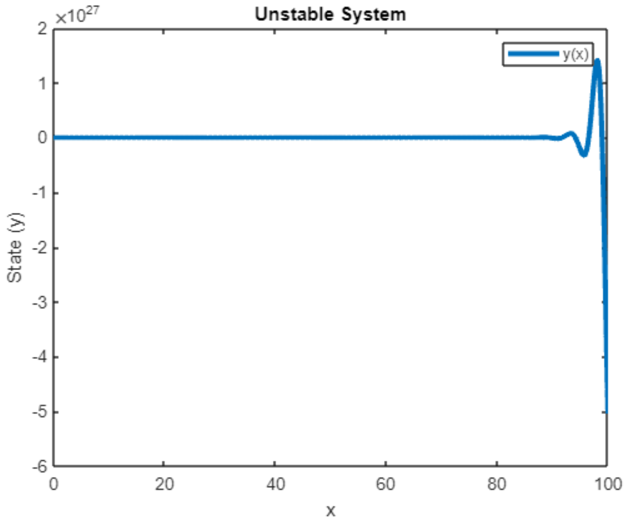

Figure 3: Unstable system

Example 5.2. (Without Dampening effect)

Now take the following system into consideration

$$^{C}D^{2}y(x) + By(x) = f(x), \tag{33}$$

where $B \in \mathbb{R}$ with IC $y(0) = y'(0) = 0$

$$f(x) = \begin{cases} 15, & 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, Bagley-Torvik becomes an ordinary differential equation due to the absence of dampening effect and its stability completely depends on the values of B.

It can be seen from the following.

If B=4, it can be represented by $^{C}D^{1/2}y(x) = Py(x) + F(x)$and $F(x) = (0,0,0,15)^{T}$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

The spectral values of P are $1 \pm i, -1 \pm i$, which meet the requirement of spectral condition $\left|\arg\left(spec(P)\right)\right| = \frac{\pi}{4}$. Hence, there exists an asymptotically stable solution of (33) see [Fig 4].

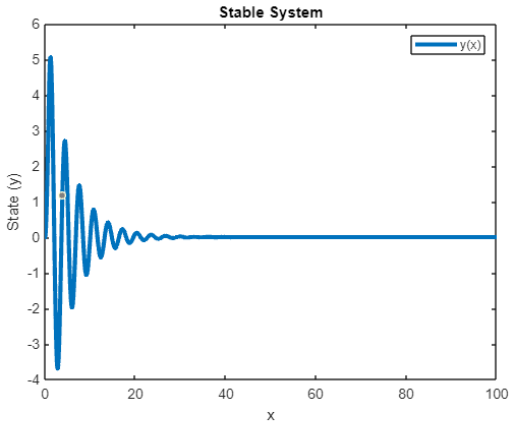

Figure 4: Stable system

If B=-4, it can be represented by ${}^{C}D^{1/2}y(x) = Py(x) + F(x)$ and $F(x) = (0,0,0,15)^T \ \ 0 < x \leq 1$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \qquad (35)$$

The spectral values of P are $\pm 1.4141, 1.4142, \pm 1.4142i$, which fails to satisfy hence the given system (33) does not meet all the requirement of Theorem 3.2. Hence, there exists an asymptotically stable solution (33) see [Fig 5].

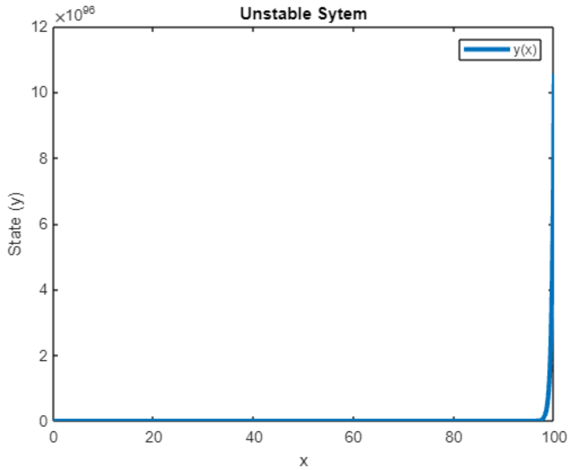

Figure 5: Unstable system

Example 5.3 (Bagley-Torvik Equation)

Now, take the following system into consideration

$$ {}^{C}D^{2}y(x) + A\,{}^{C}D^{\alpha}y(x) = f(x) \qquad (36), $$

where $A \in \mathbb{R}$, and $\alpha = 3/2$, and with IC $y(0) = y'(0) = 0$,

$$ f(x) = \begin{cases} 15, & 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases} $$

The stability of Bagley-Torvik equation depends on the parameters of A. It can be seen from the following.

If A=9 it can be represented by ${}^{C}D^{1/2}y(x) = Py(x) + F(x)$, and $F(x) = (0,0,0,15)^{T}$ $0 < x \le 1$ and, where

$$ P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -9 \end{bmatrix} \qquad (37) $$

The spectral values of P are 0 with multiplicity three and -9 with one. It can be noticed that every spectral value satisfies $|\arg(spec(P))| > \frac{\pi}{4}$. Hence, the given system (36) meets all the requirement of Theorem 3.2. Hence, there exists a stable solution of (36) See [Fig 6].

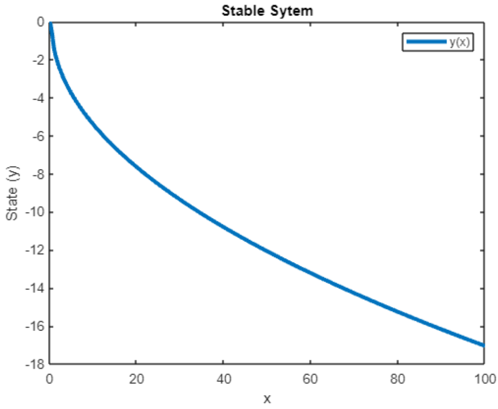

Figure 6: Stable system

If A=-9, it can be represented by ${}^{C}D^{1/2}y(x) = Py(x) + F(x)$ and $\mathrm{F}(x) = (0,0,0,15)^T \quad 0 < x \leq 1$, where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}. \tag{38}$$

The spectral values of P are 0 with multiplicity three and 9 with one. It fails to meet the spectral condition$\left|\arg\left(spec(P)\right)\right| > \frac{\pi}{4}$. Hence, there exists an unstable solution of (36). See [Fig 7].

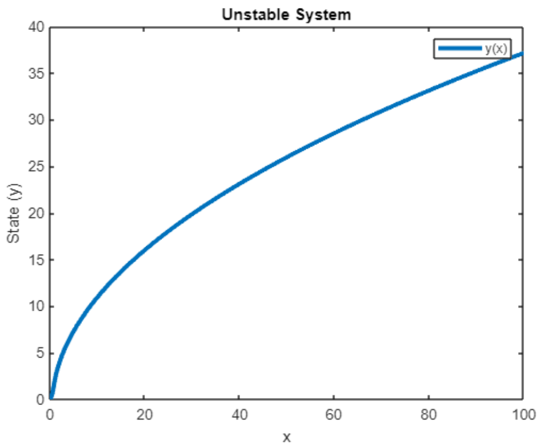

Figure 7: Unstable system

6. CONCLUSION

In this work, a few fascinating linear fractional differential systems emerging in visco-flexibly damped structure have been settled utilizing the Bagley-Torvik condition. The approximate stability argument condition is investigated using spectral values for linear fractional system. The required results are deduced using prevailing properties of the Mittag-Leffler functions. Examples are analytically solved using traditional Laplace Transform technique. Numerical methods and Matlab were utilized to plot the corresponding solutions. Examples provided demonstrate the efficacy of the concepts presented.

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