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Trapezoidal Spherical Fuzzy Numbers and its Application to Fuzzy Risk Analysis

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Abstract

Spherical fuzzy sets are a broader type of fuzzy sets that have the ability to handle various scenarios using their membership, non-membership, and neutral membership grades. These sets require that the total of the squares of these grades be no greater than one. This condition extends the possible values for the three grades and enables decision makers to have a wider range of options when assessing a situation. In solving real-life problems, it is necessary to describe a real number as a spherical fuzzy set to incorporate the fuzziness; thus, the need to use trapezoidal spherical fuzzy numbers (TSFN). This paper discusses the membership functions of the TSFN, their arithmetic operations, and their properties. Also, a ranking function is proposed to order the TSFNs. All these are used to solve a fuzzy risk analysis problem whose parameters are presented as TSFNs.

Keywords— Trapezoidal Fuzzy Number, Spherical Fuzzy Set, Fuzzy Risk Analysis, Ranking Function

Introduction

Nobody can accurately predict the risk involved in a production unless they are familiar with every component of the risk system they are researching. In real-world situations, it is impossible to

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prevent the risk assessment gaps that lead to fuzziness. As a result, we must cope with a risk system's fuzziness.

Fuzzy set theory was first introduced by (Zadeh, 1965) to mathematically represent data that is uncertain or vague. In this theory, a fuzzy subset A of a universal set X is represented by a set of ordered pairs, where each element x in X is associated with a degree of membership $\mu_A(x)$ that ranges from 0 to 1. If the degree of membership is zero for all elements in X, then A is an empty set. If the set of elements in X with a non-zero degree of membership is finite, then A is a discrete fuzzy set. A fuzzy set becomes a crisp set when $\mu_A(x)$ takes only the values 0 or 1. However, in fuzzy sets, the non-membership value of an element cannot be derived from its membership value. To address this, Atanassov developed intuitionistic fuzzy sets (Atanassov, 1986), which include both membership and non-membership degrees for each element. Later, interval-valued intuitionistic fuzzy sets were developed (T.Atanassov & G.Gargov, 1989) as an extension of intuitionistic fuzzy sets. However, some situations may arise where the sum of the two membership grades exceeds 1, which cannot be handled by these fuzzy sets. To overcome this issue, Pythagorean fuzzy sets were developed by (R.Yager, 2013) , with the condition that the sum of squares of the two membership grades is up to 1. Picture fuzzy sets were then developed by (Cuong & Kreinovich, 2013), using three grades of membership, non-membership, and neutral membership, whose sum does not exceed 1. However, there are still situations where the sum of the three considered membership grades exceeds 1, which cannot be handled by these fuzzy sets. To address this, (Gündoğdu & Kahraman, 2019a) introduced the idea of spherical fuzzy sets, where the sum of squares of the three membership grades does not exceed 1. Spherical fuzzy sets are more realistic and versatile than other fuzzy sets, and have been applied in various fields, such as multi-criteria decision making (Ashraf et al., 2019; Balin, 2020; Zeng et al., 2019), medical diagnosis (Mahmood et al., 2019), pattern recognition (Mahmood et al., 2021; Ullah et al., 2018; Wu et al., 2019, 2020), clustering (Özlü & Karaaslan, 2021; Ullah et al., 2020), and selection problems (Gündoğdu & Kahraman, 2019b; Kahraman et al., 2019; Kutlu Gündoğdu & Kahraman, 2019; Liu et al., 2019; Mathew et al., 2020).

This article aims to employ the notion of spherical fuzzy sets to establish a type of fuzzy number called Trapezoidal Spherical Fuzzy Number (TSFN). Section 2 recalls the preliminary definitions and results related to spherical fuzzy sets. Section 3 discusses membership functions, arithmetic operations, and properties. Also, Section 3 presents a ranking function to order the TSFNs. Section 4 discusses the fuzzy risk analysis problem, an algorithm to solve it and an illustration. Finally, the conclusion is presented in section 5.

Preliminaries

To ensure thoroughness, we will review the necessary definitions and results.

Definition 1. (Gündoğdu & Kahraman, 2019a) Spherical fuzzy sets $\tilde{A}_S$ of the universe of discourse U is given by

$$\tilde{A}_S = \{\left(x, \langle \mu_{\tilde{A}_S}(x), \nu_{\tilde{A}_S}(x), \eta_{\tilde{A}_S}(x) \rangle\right)\}$$

where

$$\mu_{\tilde{A}_S}: U \to [0,1], \nu_{\tilde{A}_S}: U \to [0,1], \eta_{\tilde{A}_S}: U \to [0,1]$$

and

$$0 \le \mu_{\tilde{A}_S}(x)^2 + \nu_{\tilde{A}_S}(x)^2 + \eta_{\tilde{A}_S}(x)^2 \le 1, \forall x \in U$$

for each x, the numbers $\mu_{\tilde{A}_S}(x), \nu_{\tilde{A}_S}(x)$ and $\eta_{\tilde{A}_S}(x)$ are the degree of membership, non-membership and neutral(hesitancy)-membership of x to $\tilde{A}_S$, respectively.

Remark 1. The quantity

$$\rho_{\tilde{A}_S}(x) = \sqrt{1 - \left(\mu_{\tilde{A}_S}(x)^2 + \nu_{\tilde{A}_S}(x)^2 + \eta_{\tilde{A}_S}(x)^2\right)}$$

is considered as the degree of refusal-membership.

Definition 2. (Gündoğdu & Kahraman, 2019a) Suppose we have two spherical fuzzy sets, $\tilde{A}_S$ and $\tilde{B}_S$, and a positive real number λ. The arithmetic operations are specified in the following manner:

Theorem 1. *(Kahraman & Gündoğdu, 2021)The subsequent properties are valid for any two spherical fuzzy sets.* $\tilde{A}_S$ *and* $\tilde{B}_S$ *and* $\lambda, \lambda_1, \lambda_2$ *being positive real numbers:*

$$\tilde{A}_S \oplus \tilde{B}_S = \left\{\left(x, \langle \sqrt{\mu^2_{\tilde{A}_S}(x) + \mu^2_{\tilde{B}_S}(x) - \mu^2_{\tilde{A}_S}(x)\mu^2_{\tilde{B}_S}(x)}, \nu_{\tilde{A}_S}(x)\nu_{\tilde{B}_S}(x), \sqrt{\left[1-\mu^2_{\tilde{B}_S}(x)\right]\eta^2_{\tilde{A}_S}(x) + \left[1-\mu^2_{\tilde{A}_S}(x)\right]\eta^2_{\tilde{B}_S}(x) - \eta^2_{\tilde{A}_S}(x)\eta^2_{\tilde{B}_S}(x)}\rangle\right)\right\}$$

$$\tilde{A}_S \otimes \tilde{B}_S = \left\{\left(x, \langle \mu_{\tilde{A}_S}(x)\mu_{\tilde{B}_S}(x), \sqrt{\nu^2_{\tilde{A}_S}(x) + \nu^2_{\tilde{B}_S}(x) - \nu^2_{\tilde{A}_S}(x)\nu^2_{\tilde{B}_S}(x)}, \sqrt{\left[1-\nu^2_{\tilde{B}_S}(x)\right]\eta^2_{\tilde{A}_S}(x) + \left[1-\nu^2_{\tilde{A}_S}(x)\right]\eta^2_{\tilde{B}_S}(x) - \eta^2_{\tilde{A}_S}(x)\eta^2_{\tilde{B}_S}(x)}\rangle\right)\right\}$$

$$\lambda\tilde{A}_S = \left\{\left(x, \langle \sqrt{1-\left(1-\mu^2_{\tilde{A}_S}(x)\right)^\lambda}, \nu^\lambda_{\tilde{A}_S}(x), \sqrt{\left[1-\mu^2_{\tilde{A}_S}(x)\right]^\lambda - \left[1-\mu^2_{\tilde{A}_S}(x)-\eta^2_{\tilde{A}_S}(x)\right]^\lambda}\rangle\right)\right\}$$

$$\tilde{A}_S^\lambda = \left\{\left(x, \langle \mu^\lambda_{\tilde{A}_S}(x), \sqrt{1-\left(1-\nu^2_{\tilde{A}_S}(x)\right)^\lambda}, \sqrt{\left[1-\nu^2_{\tilde{A}_S}(x)\right]^\lambda - \left[1-\nu^2_{\tilde{A}_S}(x)-\eta^2_{\tilde{A}_S}(x)\right]^\lambda}\rangle\right)\right\}$$

(1) $\tilde{A}_S \oplus \tilde{B}_S = \tilde{B}_S \oplus \tilde{A}_S$

(2) $\tilde{A}_S \otimes \tilde{B}_S = \tilde{B}_S \otimes \tilde{A}_S$

(3) $\lambda(\tilde{A}_S \oplus \tilde{B}_S) = \lambda\tilde{A}_S \oplus \lambda\tilde{B}_S$

(4) $\lambda_1\tilde{A}_S \oplus \lambda_2\tilde{A}_S = (\lambda_1+\lambda_2)\tilde{A}_S$

(5) $(\tilde{A}_S \otimes \tilde{B}_S)^\lambda = \tilde{A}_S^\lambda \otimes \tilde{B}_S^\lambda$

(6) $\tilde{A}_S^{\lambda_1} \otimes \tilde{A}_S^{\lambda_2} = \tilde{A}_S^{\lambda_1+\lambda_2}$

Trapezoidal Spherical Fuzzy Number

In this section trapezoidal spherical fuzzy numbers are defined based on the definitions of spherical fuzzy set (Gündoğdu & Kahraman, 2019a), spherical fuzzy number(Deli & Çağman, 2021) and intuitionistic fuzzy number(P. Burillo, 1994).

Definition 3. A trapezoidal spherical fuzzy number is denoted by $\tilde{a} = (a_1, a_2, a_3, a_4; \langle\alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}}\rangle)$ such that $a_i \in \mathbb{R}, i = 1, \cdots, 4$, $a_1 \leq a_2 \leq a_3 \leq a_4$, $\alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \in [0,1]$ and $0 \leq \alpha^2_{\tilde{a}} + \beta^2_{\tilde{a}} + \gamma^2_{\tilde{a}} \leq 1$ whose membership, non-membership and neutral functions are respectively given by

$$\mu_{\tilde{a}}(x)=\begin{cases}\dfrac{(x-a_1)\alpha_{\tilde{a}}}{a_2-a_1} & a_1\le x\le a_2\\ \alpha_{\tilde{a}} & a_2\le x\le a_3\\ \dfrac{(a_4-x)\alpha_{\tilde{a}}}{a_4-a_3} & a_3\le x\le a_4\\ 0 & \text{otherwise}\end{cases}$$

$$\nu_{\tilde{a}}(x)=\begin{cases}\dfrac{(a_2-x)+(x-a_1)\beta_{\tilde{a}}}{a_2-a_1} & a_1\le x\le a_2\\ \beta_{\tilde{a}} & a_2\le x\le a_3\\ \dfrac{(x-a_3)+(a_4-x)\beta_{\tilde{a}}}{a_4-a_3} & a_3\le x\le a_4\\ 1 & \text{otherwise}\end{cases}$$

$$\eta_{\tilde{a}}(x)=\begin{cases}\dfrac{(x-a_1)\gamma_{\tilde{a}}}{a_2-a_1} & a_1\le x\le a_2\\ \gamma_{\tilde{a}} & a_2\le x\le a_3\\ \dfrac{(a_4-x)\gamma_{\tilde{a}}}{a_4-a_3} & a_3\le x\le a_4\\ 0 & \text{otherwise}\end{cases}$$

Arithmetic Operations on TSFNs

Definition 4. Let $\tilde{a}=(a_1,a_2,a_3,a_4;\langle\alpha_{\tilde{a}},\beta_{\tilde{a}},\gamma_{\tilde{a}}\rangle)$ and $\tilde{b}=(b_1,b_2,b_3,b_4;\langle\alpha_{\tilde{b}},\beta_{\tilde{b}},\gamma_{\tilde{b}}\rangle)$ be two trapezoidal spherical fuzzy numbers and λ be a positve real number, the arithmetic operations are defined as follows:

$$\tilde{a}\oplus\tilde{b}=(a_1+b_1,a_2+b_2,c_1+c_2,d_1+d_2;\left\langle\sqrt{\alpha_{\tilde{a}}^2+\alpha_{\tilde{b}}^2-\alpha_{\tilde{a}}^2\alpha_{\tilde{b}}^2},\beta_{\tilde{a}}\beta_{\tilde{b}},\sqrt{[1-\alpha_{\tilde{b}}^2]\gamma_{\tilde{a}}^2+[1-\alpha_{\tilde{a}}^2]\gamma_{\tilde{b}}^2-\gamma_{\tilde{a}}^2\gamma_{\tilde{b}}^2}\right\rangle)$$

$$\lambda\tilde{a}=\begin{cases}\left(\lambda a_1,\lambda a_2,\lambda a_3,\lambda a_4;\langle\sqrt{1-(1-\alpha_{\tilde{a}}^2)^{\lambda}},\beta_{\tilde{a}}^{\lambda},\sqrt{[1-\alpha_{\tilde{a}}^2]^{\lambda}-[1-\alpha_{\tilde{a}}^2-\gamma_{\tilde{a}}^2]^{\lambda}}\rangle\right) & \text{if } \lambda\ge 0\\ \left(\lambda a_4,\lambda a_3,\lambda a_2,\lambda a_1;\langle\sqrt{1-(1-\alpha_{\tilde{a}}^2)^{-\lambda}},\beta_{\tilde{a}}^{-\lambda},\sqrt{[1-\alpha_{\tilde{a}}^2]^{-\lambda}-[1-\alpha_{\tilde{a}}^2-\gamma_{\tilde{a}}^2]^{-\lambda}}\rangle\right) & \text{if } \lambda<0\end{cases}$$

$$\tilde{a}\otimes\tilde{b}=\left(\begin{matrix}\min(c_{1,4}),\min(c_{2,3}),\max(c_{2,3}),\max(c_{1,4});\langle\alpha_{\tilde{a}}\alpha_{\tilde{b}},\sqrt{\beta_{\tilde{a}}^2+\beta_{\tilde{b}}^2-\beta_{\tilde{a}}^2\beta_{\tilde{b}}^2},\\ \sqrt{[1-\beta_{\tilde{b}}^2]\gamma_{\tilde{a}}^2+[1-\beta_{\tilde{a}}^2]\gamma_{\tilde{b}}^2-\gamma_{\tilde{a}}^2\gamma_{\tilde{b}}^2}\rangle\end{matrix}\right)$$

where $c_{i,j}=\{a_ib_i,a_ib_j,a_jb_i,a_jb_j\}$

$$\tilde{a}^{\lambda} = \begin{cases} \left(a_1^{\lambda}, a_2^{\lambda}, a_3^{\lambda}, a_4^{\lambda}; \langle \alpha_{\tilde{a}}^{\lambda}, \sqrt{1-\left(1-\beta_{\tilde{a}}^2\right)^{\lambda}}, \sqrt{[1-\beta_{\tilde{a}}^2]^{\lambda} - [1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2]^{\lambda}} \rangle\right) & \text{if } \lambda \geq 0 \\ \left(a_4^{\lambda}, a_3^{\lambda}, a_2^{\lambda}, a_1^{\lambda}; \langle \alpha_{\tilde{a}}^{-\lambda}, \sqrt{1-\left(1-\beta_{\tilde{a}}^2\right)^{-\lambda}}, \sqrt{[1-\beta_{\tilde{a}}^2]^{-\lambda} - [1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2]^{-\lambda}} \rangle\right) & \text{if } \lambda < 0 \end{cases}$$

Proposition 1. *Let $\tilde{a} = (a_1, a_2, a_3, a_4; \langle \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle)$ and $\tilde{b} = (b_1, b_2, b_3, b_4; \langle \alpha_{\tilde{b}}, \beta_{\tilde{b}}, \gamma_{\tilde{b}} \rangle)$ be any two trapezoidal spherical fuzzy numbers and λ be a positve real number, then $\tilde{a} \oplus \tilde{b}, \tilde{a} \otimes \tilde{b}, \lambda\tilde{a}, \tilde{a}^{\lambda}$ are all trapezoidal spherical fuzzy numbers.*

Proof. By the definition of the arithmetic operations, the order in the four components of the trapezoidal spherical fuzzy number are preserved. The spherical condition on the membership, non-membership and neutral function on these operations are to be proved. Since $\tilde{a}$ and $\tilde{b}$ are trapezoidal spherical fuzzy numbers, we have the following two spherical conditions:

$$0 \leq \alpha_{\tilde{a}}^2 + \beta_{\tilde{a}}^2 + \gamma_{\tilde{a}}^2 \leq 1; \quad 0 \leq \alpha_{\tilde{b}}^2 + \beta_{\tilde{b}}^2 + \gamma_{\tilde{b}}^2 \leq 1$$

By definition *4*, the membership, non-membership and neutral function on the addition of two trapezoidal spherical fuzzy numbers are

$$\begin{aligned} \alpha_{\tilde{a}\oplus\tilde{b}} &= \sqrt{\alpha_{\tilde{a}}^2 + \alpha_{\tilde{b}}^2 - \alpha_{\tilde{a}}^2\alpha_{\tilde{b}}^2} \\ \beta_{\tilde{a}\oplus\tilde{b}} &= \beta_{\tilde{a}}\beta_{\tilde{b}} \\ \gamma_{\tilde{a}\oplus\tilde{b}} &= \sqrt{[1-\alpha_{\tilde{b}}^2]\gamma_{\tilde{a}}^2 + [1-\alpha_{\tilde{a}}^2]\gamma_{\tilde{b}}^2 - \gamma_{\tilde{a}}^2\gamma_{\tilde{b}}^2} \end{aligned}$$

Consider

$$\begin{aligned} 0 \leq \alpha_{\tilde{a}\oplus\tilde{b}}^2 + \beta_{\tilde{a}\oplus\tilde{b}}^2 + \gamma_{\tilde{a}\oplus\tilde{b}}^2 &= \alpha_{\tilde{a}}^2 + \alpha_{\tilde{b}}^2 - \alpha_{\tilde{a}}^2\alpha_{\tilde{b}}^2 + \beta_{\tilde{a}}^2\beta_{\tilde{b}}^2 + [1-\alpha_{\tilde{b}}^2]\gamma_{\tilde{a}}^2 + [1-\alpha_{\tilde{a}}^2]\gamma_{\tilde{b}}^2 - \gamma_{\tilde{a}}^2\gamma_{\tilde{b}}^2 \\ &\leq \alpha_{\tilde{a}}^2 + \alpha_{\tilde{b}}^2 - \alpha_{\tilde{a}}^2\alpha_{\tilde{b}}^2 + (1-\alpha_{\tilde{a}}^2-\gamma_{\tilde{a}}^2)(1-\alpha_{\tilde{b}}^2-\gamma_{\tilde{b}}^2) \\ &\quad + [1-\alpha_{\tilde{b}}^2]\gamma_{\tilde{a}}^2 + [1-\alpha_{\tilde{a}}^2]\gamma_{\tilde{b}}^2 - \gamma_{\tilde{a}}^2\gamma_{\tilde{b}}^2 \\ &= \alpha_{\tilde{a}}^2 + \alpha_{\tilde{b}}^2 - \alpha_{\tilde{a}}^2\alpha_{\tilde{b}}^2 + 1 - \alpha_{\tilde{b}}^2 - \gamma_{\tilde{b}}^2 + \alpha_{\tilde{a}}^2\alpha_{\tilde{b}}^2 + \alpha_{\tilde{a}}^2\gamma_{\tilde{b}}^2 - \gamma_{\tilde{a}}^2 + \gamma_{\tilde{a}}^2\alpha_{\tilde{b}}^2 + \gamma_{\tilde{a}}^2\gamma_{\tilde{b}}^2 \\ &\quad + \gamma_{\tilde{a}}^2 - \alpha_{\tilde{b}}^2\gamma_{\tilde{a}}^2 + \gamma_{\tilde{b}}^2 - \alpha_{\tilde{a}}^2\gamma_{\tilde{b}}^2 - \gamma_{\tilde{a}}^2\gamma_{\tilde{b}}^2 \\ &= 1 \end{aligned}$$

This proves that the addition of two TSFNs is again a TSFN. Now, consider the membership, non-membership and neutral function of a scalar multiplication of a TSFN $\tilde{a}$

$$\begin{aligned} \alpha_{\lambda\tilde{a}} &= \sqrt{1-\left(1-\alpha_{\tilde{a}}^2\right)^{\lambda}} \\ \beta_{\lambda\tilde{a}} &= \beta_{\tilde{a}}^{\lambda} \\ \gamma_{\lambda\tilde{a}} &= \sqrt{[1-\alpha_{\tilde{a}}^2]^{\lambda}-[1-\alpha_{\tilde{a}}^2-\gamma_{\tilde{a}}^2]^{\lambda}} \end{aligned}$$

Then,

$$\begin{aligned} 0 \leq \alpha_{\lambda\tilde{a}}^2+\beta_{\lambda\tilde{a}}^2+\gamma_{\lambda\tilde{a}}^2 &= 1-\left(1-\alpha_{\tilde{a}}^2\right)^{\lambda}+\left(\beta_{\tilde{a}}^{\lambda}\right)^2+\left[1-\alpha_{\tilde{a}}^2\right]^{\lambda}-\left[1-\alpha_{\tilde{a}}^2-\gamma_{\tilde{a}}^2\right]^{\lambda} \\ &\leq 1-\left(1-\alpha_{\tilde{a}}^2\right)^{\lambda}+\left(1-\alpha_{\tilde{a}}^2-\gamma_{\tilde{a}}^2\right)^{\lambda}+\left[1-\alpha_{\tilde{a}}^2\right]^{\lambda}-\left[1-\alpha_{\tilde{a}}^2-\gamma_{\tilde{a}}^2\right]^{\lambda} \\ &= 1 \end{aligned}$$

Thus the scalar multiplication of a TSFN with a non-negative scalar is again a TSFN. The same is true for negative scalar also. Similarly it can be shown that the product of two TSFNs and the power of a TSFN are also TSFNs. Hence the proof. □

Remark 1. *Theorem* 1 *holds good for TSFNs.*

Proposition 2. *Let $\tilde{a}^{(1)}, \tilde{a}^{(2)}, \dots, \tilde{a}^{(n)}$ be n TSFNs given by $\tilde{a}^{(i)} = \left(a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, a_4^{(i)}; \langle \alpha_{\tilde{a}^{(i)}}, \beta_{\tilde{a}^{(i)}}, \gamma_{\tilde{a}^{(i)}} \rangle\right)$, then the sum of these n TSFNs is*

$$sum\left(\tilde{a}^{(1)}, \tilde{a}^{(2)}, \dots, \tilde{a}^{(n)}\right) = \left(\sum_{i=1}^{n} a_1^{(i)}, \sum_{i=1}^{n} a_2^{(i)}, \sum_{i=1}^{n} a_3^{(i)}, \sum_{i=1}^{n} a_4^{(i)}; \langle \sqrt{1-\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)}, \prod_{i=1}^{n} \beta_{\tilde{a}^{(i)}}, \sqrt{\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)-\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2-\gamma_{\tilde{a}^{(i)}}^2\right)} \rangle\right)$$

Proof. Let us prove this by induction.
Trivial Case (n=2):

$$
\begin{aligned}
sum(\tilde{a}^{(1)},\tilde{a}^{(2)}) &= \Big(a_1^{(1)}+a_1^{(2)},a_2^{(1)}+a_2^{(2)},a_3^{(1)}+a_3^{(2)},a_4^{(1)}+a_4^{(2)};\\
&\quad \langle\sqrt{1-\left(1-\alpha^2_{\tilde{a}^{(1)}}\right)\left(1-\alpha^2_{\tilde{a}^{(2)}}\right)}),\beta_{\tilde{a}^{(1)}}\beta_{\tilde{a}^{(2)}},\\
&\quad \sqrt{\left(1-\alpha^2_{\tilde{a}^{(1)}}\right)\left(1-\alpha^2_{\tilde{a}^{(2)}}\right)-\left(1-\alpha^2_{\tilde{a}^{(1)}}-\gamma^2_{\tilde{a}^{(1)}}\right)\left(1-\alpha^2_{\tilde{a}^{(2)}}-\gamma^2_{\tilde{a}^{(2)}}\right)}\rangle\Big)\\
&= \Big(a_1^{(1)}+a_1^{(2)},a_2^{(1)}+a_2^{(2)},a_3^{(1)}+a_3^{(2)},a_4^{(1)}+a_4^{(2)};\\
&\quad \langle\sqrt{1-\left(1-\alpha^2_{\tilde{a}^{(1)}}-\alpha^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(1)}}\alpha^2_{\tilde{a}^{(2)}}\right)},\beta_{\tilde{a}^{(1)}}\beta_{\tilde{a}^{(2)}},\\
&\quad \sqrt{\begin{matrix}\left(1-\alpha^2_{\tilde{a}^{(1)}}-\alpha^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(1)}}\alpha^2_{\tilde{a}^{(2)}}\right)\\ -\left(1-\alpha^2_{\tilde{a}^{(1)}}-\alpha^2_{\tilde{a}^{(2)}}-\gamma^2_{\tilde{a}^{(1)}}-\gamma^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(1)}}\gamma^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(2)}}\gamma^2_{\tilde{a}^{(1)}}+\gamma^2_{\tilde{a}^{(1)}}\gamma^2_{\tilde{a}^{(1)}}\right)\end{matrix}}\rangle\Big)\\
&= \Big(a_1^{(1)}+a_1^{(2)},a_2^{(1)}+a_2^{(2)},a_3^{(1)}+a_3^{(2)},a_4^{(1)}+a_4^{(2)};\\
&\quad \langle\sqrt{\alpha^2_{\tilde{a}^{(1)}}+\alpha^2_{\tilde{a}^{(2)}}-\alpha^2_{\tilde{a}^{(1)}}\alpha^2_{\tilde{a}^{(2)}}}),\beta_{\tilde{a}^{(1)}}\beta_{\tilde{a}^{(2)}},\\
&\quad \sqrt{\begin{matrix}\left(1-\alpha^2_{\tilde{a}^{(1)}}-\alpha^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(1)}}\alpha^2_{\tilde{a}^{(2)}}\right)\\ -\left(1-\alpha^2_{\tilde{a}^{(1)}}-\alpha^2_{\tilde{a}^{(2)}}-\gamma^2_{\tilde{a}^{(1)}}-\gamma^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(1)}}\alpha^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(1)}}\gamma^2_{\tilde{a}^{(2)}}+\alpha^2_{\tilde{a}^{(2)}}\gamma^2_{\tilde{a}^{(1)}}+\gamma^2_{\tilde{a}^{(1)}}\gamma^2_{\tilde{a}^{(1)}}\right)\end{matrix}}\rangle\Big)\\
&= \Big(a_1^{(1)}+a_1^{(2)},a_2^{(1)}+a_2^{(2)},a_3^{(1)}+a_3^{(2)},a_4^{(1)}+a_4^{(2)};\\
&\quad \langle\sqrt{\alpha^2_{\tilde{a}^{(1)}}+\alpha^2_{\tilde{a}^{(2)}}-\alpha^2_{\tilde{a}^{(1)}}\alpha^2_{\tilde{a}^{(2)}}}),\beta_{\tilde{a}^{(1)}}\beta_{\tilde{a}^{(2)}},\\
&\quad \sqrt{\gamma^2_{\tilde{a}^{(1)}}+\gamma^2_{\tilde{a}^{(2)}}-\alpha^2_{\tilde{a}^{(1)}}\gamma^2_{\tilde{a}^{(2)}}-\alpha^2_{\tilde{a}^{(2)}}\gamma^2_{\tilde{a}^{(1)}}-\gamma^2_{\tilde{a}^{(1)}}\gamma^2_{\tilde{a}^{(1)}}})\rangle\Big)\\
&= \tilde{a}^{(1)}\oplus\tilde{a}^{(2)},
\end{aligned}
$$

Let us assume the statement is true for the cases less than n and prove for n. Let

$$\tilde{b}=(b_1,b_2,b_3,b_4;\langle\alpha_{\tilde{b}},\beta_{\tilde{b}},\gamma_{\tilde{b}}\rangle)=sum\left(\tilde{a}^{(1)},\tilde{a}^{(2)},\ldots,\tilde{a}^{(n-1)}\right)$$

then

$$
\begin{aligned}
b_1 &= \sum_{i=1}^{n-1}a_1^{(i)},\quad b_2=\sum_{i=1}^{n-1}a_2^{(i)}\\
b_3 &= \sum_{i=1}^{n-1}a_3^{(i)},\quad b_4=\sum_{i=1}^{n-1}a_4^{(i)}\\
\alpha_{\tilde{b}} &= \sqrt{1-\prod_{i=1}^{n-1}\left(1-\alpha^2_{\tilde{a}^{(i)}}\right)},\quad \beta_{\tilde{b}}=\prod_{i=1}^{n-1}\beta_{\tilde{a}^{(i)}}\\
\gamma_{\tilde{b}} &= \sqrt{\prod_{i=1}^{n-1}\left(1-\alpha^2_{\tilde{a}^{(i)}}\right)-\prod_{i=1}^{n-1}\left(1-\alpha^2_{\tilde{a}^{(i)}}-\gamma^2_{\tilde{a}^{(i)}}\right)}
\end{aligned}
$$

Now consider

$$\begin{aligned}\tilde{b}\oplus\tilde{a}^{(n)} &= sum\left(\tilde{b},\tilde{a}^{(n)}\right), \text{ since the statement is true for the trivial case}\\ &= sum\left((b_1,b_2,b_3,b_4;\langle\alpha_{\tilde{b}},\beta_{\tilde{b}},\gamma_{\tilde{b}}\rangle),\left(a_1^{(n)},a_2^{(n)},a_3^{(n)},a_4^{(n)};\langle\alpha_{\tilde{a}^{(n)}},\beta_{\tilde{a}^{(n)}},\gamma_{\tilde{a}^{(n)}}\rangle\right)\right)\\ &= \Bigg(b_1+a_1^{(n)},b_2+a_2^{(n)},b_3+a_3^{(n)},b_4+a_4^{(n)};\\ &\qquad \langle\sqrt{1-\left(1-\alpha_{\tilde{b}}^2\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)},\beta_{\tilde{b}}\beta_{\tilde{a}^{(n)}},\\ &\qquad \sqrt{\left(1-\alpha_{\tilde{b}}^2\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)-\left(1-\alpha_{\tilde{b}}^2-\gamma_{\tilde{b}}^2\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2-\gamma_{\tilde{a}^{(n)}}^2\right)}\rangle\Bigg)\\ &= \Bigg(\sum_{i=1}^{n}a_1^{(i)},\sum_{i=1}^{n}a_2^{(i)},\sum_{i=1}^{n}a_3^{(i)},\sum_{i=1}^{n}a_4^{(i)};\\ &\qquad \langle\sqrt{1-\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)},\prod_{i=1}^{n}\beta_{\tilde{a}^{(i)}},\sqrt{\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)-\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2-\gamma_{\tilde{a}^{(i)}}^2\right)}\rangle\Bigg)\\ &= sum\left(\tilde{a}^{(1)},\tilde{a}^{(2)},\ldots,\tilde{a}^{(n)}\right)\end{aligned}$$

For,

$$\begin{aligned}1-\left(1-\alpha_{\tilde{b}}^2\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2\right) &= 1-(1-\left(1-\prod_{i=1}^{n-1}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)\\ &= 1-\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)+\left(1-\prod_{i=1}^{n-1}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)\\ &= 1-\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)+\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)-\prod_{i=1}^{n-1}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)\\ &= 1-\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)\end{aligned}$$

and

$$\begin{aligned}&\left(1-\alpha_{\tilde{b}}^2\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)-\left(1-\alpha_{\tilde{b}}^2-\gamma_{\tilde{b}}^2\right)\left(1-\alpha_{\tilde{a}^{(n)}}^2-\gamma_{\tilde{a}^{(n)}}^2\right)\\ &= \left[1-\left(1-\prod_{i=1}^{n-1}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)\right)\right]\left(1-\alpha_{\tilde{a}^{(n)}}^2\right)\\ &\quad -\left[1-\left(1-\prod_{i=1}^{n-1}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)\right)-\left(\prod_{i=1}^{n-1}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)-\prod_{i=1}^{n-1}\left(1-\alpha_{\tilde{a}^{(i)}}^2-\gamma_{\tilde{a}^{(i)}}^2\right)\right)\right]\left(1-\alpha_{\tilde{a}^{(n)}}^2-\gamma_{\tilde{a}^{(n)}}^2\right)\\ &= \prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2\right)-\prod_{i=1}^{n}\left(1-\alpha_{\tilde{a}^{(i)}}^2-\gamma_{\tilde{a}^{(i)}}^2\right)\end{aligned}$$

□

In a similar pattern one can prove the following statement about the product of n TSFNs.

Proposition 3. Let $\tilde{a}^{(1)}, \tilde{a}^{(2)}, \ldots, \tilde{a}^{(n)}$ be n TSFNs given by $\tilde{a}^{(i)} = \left(a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, a_4^{(i)}; \langle \alpha_{\tilde{a}^{(i)}}, \beta_{\tilde{a}^{(i)}}, \gamma_{\tilde{a}^{(i)}} \rangle\right)$, then the product of these n TSFNs is

$$product\left(\tilde{a}^{(1)}, \tilde{a}^{(2)}, \ldots, \tilde{a}^{(n)}\right) = \left(\prod_{i=1}^{n} a_1^{(i)}, \prod_{i=1}^{n} a_2^{(i)}, \prod_{i=1}^{n} a_3^{(i)}, \prod_{i=1}^{n} a_4^{(i)}; \langle \prod_{i=1}^{n} \alpha_{\tilde{a}^{(i)}}, \sqrt{1 - \prod_{i=1}^{n}\left(1 - \beta_{\tilde{a}^{(i)}}^2\right)}, \sqrt{\prod_{i=1}^{n}\left(1 - \beta_{\tilde{a}^{(i)}}^2\right) - \prod_{i=1}^{n}\left(1 - \beta_{\tilde{a}^{(i)}}^2 - \gamma_{\tilde{a}^{(i)}}^2\right)} \rangle \right)$$

provided each $a_1^{(i)} \geq 0$.

Ranking of TSFNs

The α −cut set of the membership function $\mu_{\tilde{a}}(x)$ of the TSFN $\tilde{a} = (a_1, a_2, a_3, a_4; \langle \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle)$ can be obtained from the membership function as $[L_{\tilde{a}}(h), R_{\tilde{a}}(h)], h \in [0, \alpha_{\tilde{a}}]$ where $L_{\tilde{a}}(h) = a_1 + (a_2 - a_1)\frac{h}{\alpha_{\tilde{a}}}, R_{\tilde{a}}(h) = a_4 - (a_4 - a_3)\frac{h}{\alpha_{\tilde{a}}}$. Also observe that the membership degree of TSFN lies in $\left[\alpha_{\tilde{a}}, \sqrt{1 - \beta_{\tilde{a}}^2 - \gamma_{\tilde{a}}^2}\right]$

Definition 5. The ranking funtion $\mathcal{R}$ of a TSFN $\tilde{a} = (a_1, a_2, a_3, a_4; \langle \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle)$ is defined as

$$\mathcal{R}_\omega(\tilde{a}) = \frac{1}{2}\left[\int_0^{\alpha_{\tilde{a}}} [(1-\omega)L_{\tilde{a}}(h) + \omega R_{\tilde{a}}(h)]\, dh + \int_0^{\sqrt{1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2}} [(1-\omega)L_{\tilde{a}}(h) + \omega R_{\tilde{a}}(h)]\, dh\right]$$

where $0 \leq \omega \leq 1$. The value of ω represents the risk preference.

Remark 2: For TSFN $\tilde{a} = (a_1, a_2, a_3, a_4; \langle \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle)$ with $\omega = \frac{1}{2}$, the ranking value is

$$\begin{aligned}
\mathcal{R}(\tilde{a}) &= \frac{1}{4}\left[\int_0^{\alpha_{\tilde{a}}}[L_{\tilde{a}}(h)+R_{\tilde{a}}(h)]\,dh+\int_0^{\sqrt{1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2}}[L_{\tilde{a}}(h)+R_{\tilde{a}}(h)]\,dh\right]\\
&= \frac{1}{4}\left[\int_0^{\alpha_{\tilde{a}}}\left[a_1+(a_2-a_1)\frac{h}{\alpha_{\tilde{a}}}+a_4-(a_4-a_3)\frac{h}{\alpha_{\tilde{a}}}\right]dh\right.\\
&\quad \left.+\int_0^{\sqrt{1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2}}\left[a_1+(a_2-a_1)\frac{h}{\alpha_{\tilde{a}}}+a_4-(a_4-a_3)\frac{h}{\alpha_{\tilde{a}}}\right]dh\right]\\
&= \frac{1}{4\alpha_{\tilde{a}}}\left[\int_0^{\alpha_{\tilde{a}}}[(a_1+a_4)\alpha_{\tilde{a}}-(a_1+a_4-a_2-a_3)h]\,dh\right.\\
&\quad \left.+\int_0^{\sqrt{1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2}}[(a_1+a_4)\alpha_{\tilde{a}}-(a_1+a_4-a_2-a_3)h]\,dh\right]\\
&= \frac{1}{4\alpha_{\tilde{a}}}\left[\left[(a_1+a_4)\alpha_{\tilde{a}}h-(a_1+a_4-a_2-a_3)\frac{h^2}{2}\right]_0^{\alpha_{\tilde{a}}}\right.\\
&\quad \left.+\left[(a_1+a_4)\alpha_{\tilde{a}}h-(a_1+a_4-a_2-a_3)\frac{h^2}{2}\right]_0^{\sqrt{1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2}}\right]\\
&= \frac{1}{4\alpha_{\tilde{a}}}\left[\alpha_{\tilde{a}}(a_1+a_4)\left(\alpha_{\tilde{a}}+\sqrt{1-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2}\right)-\frac{1}{2}(a_1+a_4-a_2-a_3)(1+\alpha_{\tilde{a}}^2-\beta_{\tilde{a}}^2-\gamma_{\tilde{a}}^2)\right]
\end{aligned}$$

Remark 3. The above defined ranking function is used to sort the TSFNs as $\tilde{a} \prec \tilde{b}$ if and only if

- $\mathcal{R}(\tilde{a}) < \mathcal{R}(\tilde{b})$ *or*
- $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$ *and* $\delta_{\tilde{a}} > \delta_{\tilde{b}}$ *, where the quantity* $\delta_{\tilde{a}} = \sqrt{1-(\alpha_{\tilde{a}}^2+\beta_{\tilde{a}}^2+\gamma_{\tilde{a}}^2)}$ *is the refusal degree of the TSFN* $\tilde{a}$*.*

Fuzzy Risk Analysis Problems using TSFNs

In this section, we apply the TSFNs and their arithmetic and ranking operations to deal with fuzzy risk analysis problems. Assume that there are n manufacturers $C_1, C_2, \cdots C_n$ who manufactures the same component. Suppose that manufacturer C_i produces component A_i, which comprises of m sub-components $A_{i1}, A_{i2}, \ldots, A_{im}$. To assess the likelihood of A_i failing, we employ two evaluation measures, R_{ik} and W_{ik}, which respectively represent the probability of failure and the severity of loss associated with each sub-component A_{ik} of each manufacturer C_i, where $1 \le i \le n$ and $1 \le k \le m$. It should be noted that the probability of failure and the severity of loss are determined using TSFNs. The subsequent procedure outlines the risk analysis algorithm:

- **Step 1:**

Using the TSFNs R_{ik} and W_{ik} for each sub-component A_{ik} belonging to component A_i manufactured by C_i, where $1 \leq k \leq m$ and $1 \leq i \leq n$, determine the probability of failure R_i of component A_i made by C_i by the following formula.

$$\tilde{R}_i = \frac{\sum_{k=1}^{m} \tilde{R}_{ik} \otimes \widetilde{W}_{ik}}{\sum_{k=1}^{m} \widetilde{W}_{ik}}$$

- **Step 2:**

Calculate the ranking function for each calculated probability of failure $\tilde{R}_i$. The higher the value of the ranking function, higher the probability of failure of component A_i made by the manufacturer C_i.

Illustration

Consider a problem of the production of a component by three manufacturers C_1, C_2, C_3 and name it A_1, A_2, A_3. Let A_{i1}, A_{i2}, A_{i3} and A_{i4} be the sub-components required for the component $A_i, 1 \leq i \leq 3$. Let the following tables describe the two evaluating terms: $\tilde{R}_{ik}$ and W_{ik} of the sub-components A_{ik}.

Manufacturer	**Sub-component**	**Probability of Failure (R_{ik})**	**Severity of Loss *(Wik)***
C_1	A_{11}	[0.17, 0.22, 0.36, 0.42; (0.8, 0.4, 0.3)]	[0.04, 0.1, 0.18, 0.23; (0.8, 0.3, 0.3)]
	A_{12}	[0.32, 0.41, 0.58, 0.65; (0.6, 0.5, 0.3)]	[0.58, 0.63, 0.80, 0.86; (0.65, 0.4, 0.2)]
	A_{13}	[0.58, 0.63, 0.80, 0.86; (0.8, 0.7, 0.3)]	[0, 0, 0, 0; (0.5, 0.6, 0.1)]
	A_{14}	[0.30, 0.40, 0.50, 0.60; (0.9, 0.1, 0.2)]	[0.2, 0.3, 0.35, 0.42; (0.6, 0.4, 0.2)]
C_2	A_{21}	[0.93, 0.98, 1.0, 1.0; (0.85, 0.3, 0.2)]	[0.04, 0.1, 0.18, 0.23; (0.8, 0.2, 0.1)]

Manufacturer	**Sub-component**	**Probability of Failure** ***(R_{ik})***	**Severity of Loss** ***(Wik)***
	A_{22}	[0.58, 0.63, 0.80, 0.86; (0.6, 0.3, 0.2)]	[0.58, 0.63, 0.80, 0.86; (0.65, 0.4, 0.2)]
	A_{23}	[0.32, 0.41, 0.58, 0.65; (0.7, 0.4, 0.2)]	[0, 0.2, 0.28, 0.4; (0.8, 0.3, 0.1)]
	A_{24}	[0.74, 0.78, 0.80, 0.85; (0.6, 0.3, 0.4)]	[0.2, 0.25, 0.31, 0.38; (0.7, 0.3, 0.2)]
C_3	A_{31}	[0.17, 0.22, 0.36, 0.42; (0.9, 0.1, 0.1)]	[0.04, 0.1, 0.18, 0.23; (0.8, 0.3, 0.3)]
	A_{32}	[0.72, 0.78, 0.92, 0.97; (0.5, 0.4, 0.3)]	[0.5, 0.58, 0.63, 0.7; (0.7, 0.3, 0.2)]
	A_{33}	[0.58, 0.63, 0.83, 0.87; (1.0, 0, 0)]	[0.21, 0.25, 0.32, 0.38; (0.8, 0.1, 0.2)]
	A_{34}	[0.61, 0.66, 0.72, 0.80; (0.5, 0.4, 0.2)]	[0.4, 0.45, 0.52, 0.6; (0.7, 0.3, 0.1)]

Table 1: Evaluating Terms of the sub-components by the 3 manufacturers

Sub - comp	$\tilde{R}_{ik} \otimes \tilde{W}_{ik}$	$\sum_{k=1}^{m} \tilde{R}_{ik} \otimes \tilde{W}_{ik}$	$\sum_{k=1}^{m} \tilde{W}_{ik}$
A_{11}	(0.0068, 0.022, 0.0648, 0.0966; ⟨0.64, 0.485, 0.387⟩)	(0.2524, 0.4003, 0.7038, 0.9076; ⟨0.838, 0.099, 0.368⟩)	(0.82, 1.03, 1.33, 1.51; ⟨0.949, 0.029, 0.188⟩)
A_{12}	(0.1856, 0.2583, 0.464, 0.559; ⟨0.39, 0.608, 0.319⟩)		
A_{13}	(0.0, 0.0, 0.0, 0.0; ⟨0.4, 0.821, 0.249⟩)		

Sub - comp	$\tilde{R}_{ik} \otimes \widetilde{W}_{ik}$	$\sum_{k=1}^{m} \tilde{R}_{ik} \otimes \widetilde{W}_{ik}$	$\sum_{k=1}^{m} \widetilde{W}_{ik}$
A_{14}	(0.06, 0.12, 0.175, 0.252; ⟨0.54, 0.41, 0.268⟩)		
A_{21}	(0.0372, 0.098, 0.18, 0.23; ⟨0.68, 0.356, 0.217⟩)	(0.5216, 0.7719, 1.2304, 1.5526; ⟨0.861, 0.035, 0.314⟩)	(0.82, 1.18, 1.57, 1.87; ⟨0.981, 0.007, 0.085⟩)
A_{22}	(0.3364, 0.3969, 0.64, 0.7396; ⟨0.39, 0.485, 0.262⟩)		
A_{23}	(0.0, 0.082, 0.1624, 0.26; ⟨0.56, 0.485, 0.211⟩)		
A_{24}	(0.148, 0.195, 0.248, 0.323; ⟨0.42, 0.415, 0.419⟩)		
A_{31}	(0.0068, 0.022, 0.0648, 0.0966; ⟨0.72, 0.315, 0.312⟩)	(0.7326, 0.9289, 1.2844, 1.5862 ;⟨0.931, 0.008, 0.235⟩)	(1.15, 1.38, 1.65, 1.91; ⟨0.983, 0.003, 0.116⟩)
A_{32}	(0.36, 0.4524, 0.5796, 0.679; ⟨0.35, 0.485, 0.335⟩)		
A_{33}	(0.1218, 0.1575, 0.2656, 0.3306; ⟨0.8, 0.1, 0.2⟩)		
A_{34}	(0.244, 0.297, 0.3744, 0.48; ⟨0.35, 0.485, 0.211⟩)		

Table 2: Calculation of the product of Probability of Failure and Severity of Loss

The probability of failure R_i of the component A_i made by the manufacturer C_i and their respective ranking function are as follows:

$$R_1 = (0.1672, 0.301, 0.6833, 1.1068; \langle 0.795, 0.103, 0.407 \rangle), \quad \mathcal{R}(R_1) = 0.476$$
$$R_2 = (0.279, 0.4916, 1.0428, 1.8934; \langle 0.845, 0.036, 0.324 \rangle), \quad \mathcal{R}(R_2) = 0.821$$
$$R_3 = (0.3836, 0.563, 0.9307, 1.3794; \langle 0.915, 0.009, 0.261 \rangle), \quad \mathcal{R}(R_3) = 0.764$$

Therefore, the order in which manufacturers are ranked based on their risk is $C_2 > C_3 > C_1$..

Conclusion

Trapezoidal fuzzy number is the most common form of fuzzy number in the literature. Spherical fuzzy sets are effective as they incorporate the membership, non-membership and neutral membership grades of a set. It is important to combine the two concepts to apply to the real life problems. Also, the arithmetic operations and ranking function are the two main features required to apply the fuzziness to a problem. This paper discusses the trapezoidal spherical fuzzy number, ranking function, arithmetic operations and related properties so that the TSFNs are applicable to decision problems. All these are applied to a fuzzy risk analysis problem.

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