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Distance pattern distinguishing colouring of graphs

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Abstract

Let f_M be an assignment of subsets of X to the vertices of G such that $f_M(u) = \{d(u,v): v \in M\}$ where, $d(u,v)$ is the usual distance between u and v. We call f_M an M −distance pattern colouring of G if no two adjacent vertices have same f_M. Define $f_M^{\oplus}$ of an edge $e \in E(G)$ as $f_M^{\oplus}(e) = f_M(u) \oplus f_M(v); e = uv$. A distance pattern distinguishing colouring of a graph G is an M distance pattern colouring of G such that both $f_M(G)$ and $f_M^{\oplus}(G)$ are injective. This paper is a study on distance pattern colouring and distance pattern distinguishing colouring of graphs.

Keywords: Distance pattern colouring, colouring

1. Introduction

For all terminology that are not defined in this paper, we refer the reader to F. Harary[8]. All graphs considered in this paper are finite, connected and simple.

Let M be a non-empty subset of vertices of a graph G and $u \in V(G)$. Then, the M −distance pattern of u is the set $f_M(u) = \{d(u,v): v \in M\}$.nIf no two vertices in V (G) have the same M−distance pattern,

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then the set M is said to be a distance pattern distinguishing set of G, and G is called a distance pattern distinguishing graph. The least cardinality of the distance pattern distinguishing set in G is called distance pattern distinguishing number of G, denoted by $\varrho(G)$.

The concept of distance pattern distinguishing sets of graphs was introduced by Dr. B D Acharya and a detailed study on the topic can be found in [3], [5], [6], [9]. It has lot of applications in the fields like site control, robot navigation, radio navigation, molecular graph theory, etc. Also, colourings of graphs that are required to satisfy certain conditions have often been motivated by their effectiveness in various applied fields and their intrinsic mathematical interest. An enormous amount of literature has built upon different types of graph colourings. Motivated by the vast applications of the concept of distance pattern distinguishing sets of graphs, this paper is an attempt at extending the concept of distance pattern of graphs to colouring.

Definition 1.1. Given a connected $(p,q)-$graph $G = (V,E)$ of diameter $d, \emptyset \neq M \subseteq V(G)$ and a nonempty set $X = \{0,1,\dots,d\}$ of colors of cardinality $d + 1$, let f_M be an assignment of subsets of X to the vertices of G such that $f_M(u) = \{d(u,v): v \in M\}$ where, $d(u,v)$ is the usual distance between u and v . We call G an M $-$distance pattern colourable graph if no two adjacent vertices have same f_M. The minimum number of vertices in M that gives a distance pattern colouring to a graph is called the distance pattern colouring number of that graph and is denoted by $\chi_d(G)$.

Theorem 1.2. [3] A cycle C_n is a distance pattern distinguishing graph if and only if $n \geq 7$ and $\varrho(C_n) = 3$.

Remark 1.3. Since the M distance patterns of every vertex of a distance pattern distinguishing graph are distinct, every distance pattern distinguishing graph is distance pattern colourable. But the converse need not be true. For example, consider the cycle C_4, which is distance pattern colourable by taking M as any two of its alternating vertices. But C_4 is not a distance pattern distinguishing graph by Theorem 1.2. Cycle C_5 is neither a distance pattern distinguishing graph nor a distance pattern colourable graph as none of the subsets of $V(C_5)$ gives distance pattern colouring to C_5.

Remark 1.4. For a graph G, $\chi_d(G) \leq \varrho(G)$ by the injective property of f_M in distance pattern distinguishing graphs. The bound is sharp and attained for the graphs like paths, odd cycles, etc.

Theorem 1.5. For a tree T, $\chi_d(T) = 1$.

Proof. Let T be a tree and let $M = \{v_0\}$ be the center vertex of T. Then for all $v_i \in V(T)$, $f_M(v_i) = \{d_i\}$, where $d_i = d(v_0, v_i)$. Since T is acyclic, no two adjacent vertices of T have same $f_M$and hence, trees are distance pattern colourable with $\chi_d(T) = 1$.

Theorem 1.6. Complete graphs are distance pattern colourable if $n = 2$.

Proof. K1 is distance pattern colourable since it has only one vertex. K2 is distance pattern colourable by taking M as any of its vertex. Consider a complete graph Kn, $n \geq 3$. If

$|M| = 1$, then $f_M(u) = 1$ for all the vertices in V (G) \ M. If $|M| \geq 2$ then $f_M(u) = f_M(v) = \{0, 1\}$; $u, v \in M$. Hence, Kn, $n \geq 3$ is not distance pattern colourable.

Theorem 1.7. For a cycle Cn, $\chi_d(C_n) = 2$ when n is even and $\chi_d(C_n) = 3$ when $n \geq 7$ is odd.

Proof. Let $C_n = v_1, v_2, \dots, v_n$ be an even cycle with diameter d.

Case 1: n is even

Let M contain any two alternative vertices in Cn say, v1 and v3. Then, $f_M(v_1) = f_M(v_3) = \{0, 2\}$, $f_M(v_2) = \{1\}$, $f_M(v_4) = f_M(v_n) = \{1, 3\}$, $f_M(v_5) = f_M(v_{n-1}) = \{2,4\}, \dots, f_M\left(v_{\frac{n}{2}+1}\right) = \{d-2, d\}$ and $f_M(v_{\frac{n}{2}})) = \{d - 1\}$.

As none of the adjacent vertices have same distance pattern, $M = \{v_1, v_3\}$ gives a distance pattern colouring to C_n.

Case 2: n is odd

By Theorem1.6 and by Remark1.3, $\chi_d(C_n) = 3$ when $n \geq 7$

Theorem 1.8. Distance pattern colouring number of a bipartite graph and complete bipartite graph is 1.

Proof. Consider a bipartite graph $B_{m,n}$with partition P_1 and P_2. Let M be a set that contains a singleton vertex u of $B_{m,n}$. Without loss of generality, let $u \in P_1$ Then,

$$f_M(\mathrm{v}) = \begin{cases} \{0\} & if\ u = v \\ \{2\} & if\ v \in P_1 \backslash M \\ \{0,2\} & if\ v \in P_2\ and\ (u,v) \notin E(B_{m,n}) \\ \{1\} & if\ v \in P_2\ and\ (u,v) \in E(B_{m,n}) \end{cases}$$

Then $B_{m,n}$is distance pattern colourable with $\chi_d(B_{m,n}) = 1$ as none of the adjacent have same distance pattern.

$K_{m,n}$is also distance pattern colourable with $\chi_d(K_{m,n}) = 1$ by taking M as in the case of bipartite graph and we get

$$f_M(\mathrm{v}) = \begin{cases} \{0\} & if\ u = v \\ \{2\} & if\ v \in P_1 \backslash M \\ \{1\} & if\ v \in P_2 \end{cases}$$

Theorem 1.9. Wheel W_n, is distance pattern colourable only if n is odd and $\chi_d(W_n) = \frac{n-1}{2}$.

Proof. Consider a wheel W_n, with vertex set {w1, w2, w3..., wn−1, wn} where, wn is the hub of the wheel.

Case 1: n is odd

Let $M = \{w_1, w_3, w_5, \dots, w_{n-2}\}$ be the set of all alternating vertices. Then $f_M(w_i) = \{0,2\}$; $w_i \in M$, $f_M(w_j) = \{1,2\}, w_j \notin M$ and $f_M(w_n) = 1$. Since no two adjacent vertices have same distance pattern, W_n is distance pattern colourable.

Case 2: n is even

If any two adjacent vertices of $\{w_n\}$ contained in M, then they will have the same distance pattern as {0, 1, 2}. Similarly if any two of them are not in M then they will have the distance pattern as {1, 2}. Thus in both the cases distance pattern colouring is not possible.

2. Distance pattern distinguishing colouring of a graph

A distance pattern distinguishing colouring of a graph G is an M − distance pattern colouring of G for which both $f_M(G)$ and $f_M^{\oplus}(G)$ are injective functions. A graph is called a distance pattern distinguishing colourable graph if it admits a distance pattern distinguishing colouring. A distance pattern distinguishing colouring is called a sequential distance pattern distinguishing colouring if $f(G)$ and $f^{\oplus}(G)$ are disjoint subsets of X and, further form a partition of Y (X). If G admits such a colouring, then G is a sequentially distance pattern distinguishing colourable graph. A distance pattern distinguishing colouring is called a graceful distance pattern distinguishing colouring if $f^{\oplus}(G) = Y$ (X). If G admits such a colouring then G is a gracefully distance pattern distinguishing colourable graph.

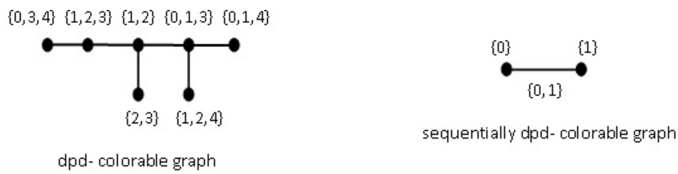

Figure 1

Remark 2.1. Since the distance patterns of each vertex in a distance pattern distinguishing colourable graph are distinct, distance pattern distinguishing colourable graphs are distance pattern distinguishing graphs. But the converse need not be true. For example, consider the graph given in Figure 2. Let $M = \{a, c, d, g\}$ then $f_M(a) = \{0,1,2\}$, $f_M(b) = \{1,2\}$, $f_M(c) = \{0,2,3\}$, $f_M(d) = \{0,1,2,3\}$, $f_M(e) = \{1,2,3\}$, $f_M(f) = \{1,3\}$, $f_M(g) = \{0,2\}$. But both $f_M^{\oplus}(ab) = f_M^{\oplus}(de) = \{0\}$ and hence, G is not distance pattern distinguishing colourable.

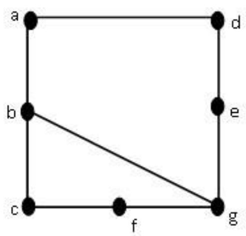

Figure 2:

***Remark* 2.2.** Since all the nonempty subsets have to appear in any sequential distance pattern distinguishing colouring of a $(n, m)-$ graph G, a necessary condition for G to besequentially distance pattern distinguishing colourable is that $n + m = 2^{d+1} - 1$, where d is the diameter of G. This necessary condition yields that no cycle is sequentially distance pattern distinguishing colourable. Also, the above condition is not sufficient for saying that a graph G is sequentially distance pattern distinguishing colourable. For, consider the graph G given in Figure 2, for which $d = 3$ and $M = \{a, c, d, g\}$. Here, G satisfies the condition but one can verify that it is not sequentially distance pattern distinguishing colourable.

Theorem 2.3. All paths are distance pattern distinguishing colourable.

Proof. Let $P_n = v_1, v_2, \ldots, v_n$ be a path on n vertices. Let $M = \{v_1\}$. Then, $f(v_i) = \{i-1\}$ and hence, $(f^{\oplus}(v_i, v_{i+1}) = \{i-1, i\}$ for $1 \leq i \leq n$. Since both $f(G)$ and $f^{\oplus}(G)$ are injective, P_n is distance pattern distinguishing colourable.

Remark 2.4. Path P_n satisfies the necessary condition for the sequential distance pattern distinguishing colouring given in Remarks 2.7 only if $n = 1, 2$. When $n = 1$, graph is trivial, which is sequentially distance pattern distinguishing colourable. When $n = 2$, P_2 is sequentially distance pattern distinguishing colourable by taking M as one of its vertices. Thus, path P_n is sequentially distance pattern distinguishing colourable if and only if n ≤ 2.

***Theorem* 2.5.** Complete graph K_n is sequentially distance pattern distinguishing col- orable if and only if $n = 2$.

Proof. Let the complete graph K_n be sequentially distance pattern distinguishing col- orable. Since K_n is sequentially distance pattern distinguishing colourable graph of diameter one, n+nC2 which implies that $n = 2$. K_2 is sequentially distance pattern distinguishing colourable by taking one of its pendant vertices.

Theorem 2.6. [3] For any graph G , there exists no distance pattern distinguishing set M of cardinality 2.

Theorem 2.7. [5] Path is the only graph which possesses a distance pattern distinguishing set M of cardinality 1.

Theorem 2.8. [6] P_3 is the only distance pattern distinguishing graph of diameter two.

Remark 2.9. By Theorem 2.6, there exists no distance pattern distinguishing colouring set M of cardinality 2. By Theorem 2.7, path is the only graph which possesses a distance pattern distinguishing colouring set M of cardinality 1 and by Theorem 2.8, P_3 is the only distance pattern distinguishing colourable graph of diameter two.

Theorem 2.10. [5] A uniform binary tree T is a distance pattern distinguishing tree if and only if $O(T) = 2^m - 1$ where $m = 1, 2, 3$.

Theorem 2.11. A uniform binary tree T is distance pattern distinguishing colourable if and only if $O(T) = 2^m - 1$where m = 1, 2, 3...

Proof. When $m = 1$, $T \cong K_1$, obviously distance pattern distinguishing colourable. When $m = 2$, $T \cong P_3$, distance pattern distinguishing colourable by Theorem 2.3. When $m = 3$, let w be the central vertex of T, $\{u, v\}$ be the vertices adjacent to w and let $\{u_1, u_2\}$ and $\{v_1\ v_2\}$ be the set of pendant vertices adjacent to u and v respectively. Let M = {u, u1, v2}. Then, both $f\ (G)$ and $f^{\oplus}(G)$ are injective and hence, $T$is distance patterndistinguishing colourable. Conversely, let T be a distance pattern distinguishing colourable uniform binary tree. Then, by Theorem 2.10, T is a distance pattern distinguishing tree, which implies that $O(T) = 2^m - 1$ where $m = 1, 2, 3$.

Theorem 2.12. There is no sequentially distance pattern distinguishing set colourable graph with diameter two.

Proof. By Theorem 2.8, P_3 is the only distance pattern distinguishing graph of diameter two. But by Remark 2.4, P_3 is not sequentially distance pattern distinguishing colourable. Hence, There is no sequentially distance pattern distinguishing set colourable graph of diameter 2.

The closure (M) of a set M of vertices consists of the vertices in M together with all vertices on geodesics between any two vertices of M. In [7], it is proved that if G $\ncong$ Pn be a graph of diameter 3 with distance pattern distinguishing set M then the distance patterns of the vertices in M are $\{0,2\}, \{0,1,2\}, \{0,2,3\}$ and $\{0,1,2,3\}$ and the corresponding induced subgraph $< (M) >$ is one of the four graphs given in Figure 3. But $f^{\oplus}$ is not injective for any of the graphs in Figure 3. Hence, the following theorem.

Theorem 2.13. There is no sequentially distance pattern distinguishing colourable graph of diameter three.

Theorem 2.14. A graph G is sequentially distance pattern distinguishing colourable if and only if $G + K_1$ with $V(K_1) = \{v\}$ has a graceful distance pattern distinguishing colouring f ′ such that $f'(v) = \emptyset$.

Proof. Let f be a sequential distance pattern distinguishing colouring of G. Then extend f to the vertices of $G + K_1$ to a function f' so that the restriction map $f'|G$ of f' to $V(G)$ is f and $f'(v) = \emptyset$. Since f is a sequential distance pattern distinguishing colouring of G, the edges of $G + K_1$ having the form uv where $u \in V(G)$ will receive $f(u)$. So f' turns out to be a required graceful distance pattern distinguishing colouring of $G + K_1$. Conversely, if $G + K_1$ has a graceful distance pattern distinguishing colouring f ′ with $f'(v) = \emptyset$. Then the removal of v from $G + K_1$ results in a sequential distance pattern distinguishing colouring of G.

Theorem 2.15. If a graph G with diameter d has a sequential distance pattern distinguishing colouring f, there exists a partition of the vertex set V in to two sets V_1 and V_2 such that the number of edges joining the vertices of V_1 with those of V_2 is exactly $2^d - |V_2|$

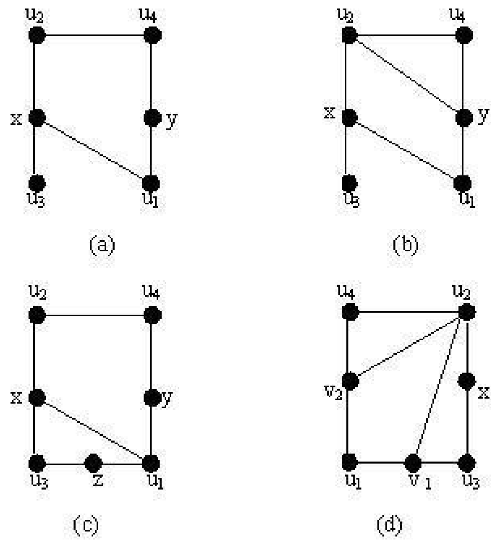

Figure 3

Proof. Suppose that G is sequentially distance pattern distinguishing colourable graph with diameter $d \geq 2$. Then $|X| = d + 1$. Consider a partition of V in to two subsets V_1andV_2 such that $V_2 = \{u \in V : |f(u)|\ is\ even\}$ and $V_2 = \{v \in V : |f(v)|\ is\ odd\}$. We can obtain other odd subsets of X which are not the distance patterns of the vertices, by taking the symmetric differences between the vertices of V1 with those of V2. Since there are exactly 2^d subsets of each parity for a set X of cardinality $d + 1$, we get the proof.

Theorem 2.16. If a graph $G(p > 2)$ has:

exactly one or two vertices of even degree or

exactly three vertices of even degree and any two of them are adjacent or

exactly four vertices of even degree, say, v1, v2, v3, v4 such that v1v2 and v3v4 are edges in G, then G is not sequentially distance pattern distinguishing colourable.

Proof. Let G be a graph of diameter d with a sequential distance pattern distinguishing colouring f. Let $v_1, v_2, \ldots, v_p$ be the vertices of G such that $f(v_i) = A_i,\ 1 \leq i \leq p,$ and $A_i \in Y(X)$. Then

$f(G) \cup f^{\oplus}(G) = \{A_1, A_2, \ldots, A_p\} \cup \{A_i \oplus A_j : v_i v_j \in E\} = Y(X)$. (1)

As the symmetric difference of all the nonempty subsets of any set is the empty set, the symmetric difference of all elements of $f(G) \cup f^{\oplus}(G)$ in equation (1) is $\emptyset$.

If the degree of a vertex vi is even then the set Ai appears an odd number of times and the degree of a vertex v_j is odd the the set Aj appears an even number of times in equation (1).

Suppose that G has exactly one vertex of even degree, say, v_1. Then A1 appears an odd number of times and all other sets appear an even number of times in equation (1). Also, $\oplus$ is a commutative binary operator and hence, all the sets assigned to the vertices of odd degree will vanish and therefore, A1 = $\emptyset$, a contradiction to the definition of sequentially distance pattern distinguishing colourable graph. Hence, G is not sequentially distance pattern distinguishing colourable if G has exactly one vertex of even degree. If G has exactly two vertices of even degree, say, v_1 and v_2 then, by the similar argument given above, we obtained that $A_1 \oplus A_2 = \emptyset$, which implies that $A_1 = A_2$, a contradiction to the injectivity of f. Hence, G is not sequentially distance pattern distinguishing colourable if G has exactly two vertices of even degree.

(ii) Suppose that G has exactly three vertices, say, v_1, v_2, v_3 of even degree such that $v_1v_2 \in E(G)$. Then by arguments similar to those for (i) and from (1) we get $A_1 \oplus A_2 \oplus A_3 = \emptyset$. That is, $A_1 = A_2 \oplus A_3$ or $A_2 = A_1 \oplus A_3$ or $A_3 = A_1 \oplus A_2$, a contradiction to the definition of sequentially distance pattern distinguishing colourable graph. Hence, if G has exactly three vertices of even degree such that any two of them are adjacent then G is not sequentially distance pattern distinguishing colourable.

(iii)Suppose that G has exactly four vertices of even degree, say, v_1, v_2, v_3, v_4 such that v_1v_2 and v_3v_4 are edges in G. Then by arguments similar to those for (i) and from (1) we get $A_1 \oplus A_2 \oplus A_3 \oplus A_4 = \emptyset$.That is, $A_1 \oplus A_2 = A_3 \oplus A_4$ or $A_2 \oplus A_4 = A_1 \oplus A_3$ or $A_2 \oplus A_3 = A_1 \oplus A_4$, a contradiction to the injectivity of $f^{\oplus}$. Hence, the proof.

References

1. F. Buckley and F. Harary (1990), Distance in graphs, Addison Wesley Publishing Company, Advanced Book Programme, Redwood City, CA.
2. Joseph A. Gallian (2014), A dynamic survey of graph labeling, The electronic journal of combinatorics, 17.
3. Germina K.A., Alphy Joseph and Sona Jose (2010), Distance neighbourhood pattern matrices, European journal of Pure and Applied Mathematics, Vol.3 (4), 748-764.
4. P. N. Balister, E. Gyori and R. H. Schelp (2011), Coloring vertices and edges of a graph by nonempty subset of a set, European Journal of Combinatorics, Vol.32, 533-537.
5. Germina K.A. and Sona Jose (2011), Distance neighbourhood pattern matrices of trees, International Mathematical Forum, Vol.6 (12), 591-604.
6. Germina K.A. and Alphy Joseph (2011), Some general results on distance pattern distinguishing graphs, International Journal of Contemporary Mathematical Sci- ences, Vol.6 (15), 713-720.
7. Sona Jose and Germina K.A. (2017), A characterization of self-complementary distance pattern distinguishing graphs, Indian Journal of Discrete Mathematics, Vol.3 (1), 37-47.
8. F.Harary (1969), Graph Theory, Addison Wesley Publishing Company, Reading, Massachusetts.
9. Sona Jose and Germina K A (2014), On the distance pattern distinguishing num- ber of graphs, Journal of Applied Mathematics, Hindawi Publications, Article ID: 328703.