Mapana – Journal of Sciences
2023, Vol. 22, Special Issue 1, *89-99*
ISSN 0975-3303 | https://doi.org/10.12723/mjs.sp1.8

On The Unit Group of Certain Finite Group Algebras

Suchi Bhatt* and Harish Chandra†

Abstract

In this paper, we have established the structure of unit group of group algebras for the abelian groups of order 40, over the finite field of odd characteristics p, having $q = p^n$ elements.

Keywords: Group Algebras, Wedderburn Structure Theorem, Augmentation ideal

1. Introduction

The group algebra of a group G over a field F is denoted by FG. If N is a normal subgroup of G, then obviously we have a natural homomorphism $g \to gN$ and this can be extended to another homomorphism of group algebra from $FG \to F[G/N]$ defined as $\Sigma_{g\in G} a_g \to \Sigma_{g\in G} a_g\, gN$ for $a_g \in F$. Also $\frac{FG}{\omega(N)} \cong F\left(\frac{G}{N}\right)$, where $\omega(N)$ is the kernal of this F-algebra homomorphism. Now $\frac{FG}{\omega(G)} \cong F$ implies $J(FG) \subseteq \omega(FG)$, where $J(FG)$ denotes the Jacobson radical of FG. Let I be an ideal such that $I \subseteq J(FG)$, then the natural homomorphism FG to FG/I induces an epimorphism from $U(FG)\ to\ U(FG/I)$ with kernel $1 + I$ and $\frac{U(FG)}{1+I} \cong U\left(\frac{FG}{I}\right)$.

We denote $V_1 = 1 + J(FG)$ as the kernel of epimorphism. For other basic notations, see [2]. The structure of unit group $U(FG)$ has

* Department of Mathematics and Scientific Computing, M. M. M. University of Technology, Gorakhpur, U.P. 1995suchibhatt@gmail.com

† Department of Mathematics and Scientific Computing, M. M. M. University of Technology, Gorakhpur, U.P. hcmsc@mmmut.ac.in

created a lot of interest in this area of research. Many publications have been appeared in this area, few of them are [4, 5, 6, 8, 9, 11]. In this direction, the structure of unit groups of group algebra for some non-abelian groups namely $G = A_4, S_3$ and S_4has been obtained by Sharma and Srivastava (see [13, 12, 7]). The characterization of unit group structure of the group algebra for D_{60} has been obtained by Bhatt and Chandra in [2]. Recently, the characterization of unit group structure of group algebras for the groups of order up to 32 can be seen in [3, 1, 14]. In the present paper, we have three abelian groups up to isomorphic of order 40, namely $C_{40}, C_4 \times C_{10}, C_2 \times C_2 \times C_{10}$ and classified the structure of unit group of group algebra for these abelian groups, over the field of odd characteristics $p > 2$. Throughout the paper, notations and symbols are same as discussed in [2, 3].

2. Main Results

Theorem 2.1 Let F be a field of finite characteristic $p > 0$ having $|F| = q = p^n$ and $G \cong C_{40}$.

For p = 5.

1. $U(FC_{40}) \cong C_5^{32} \times C_{p^n-1}^8$ $q \equiv 1$ mod 8;
2. $U(FC_{40}) \cong C_5^{32} \times C_{p^n-1}^2 \times C_{p^{2n}-1}^3, q \equiv -1{,}3$ mod 8;
3. $U(FC_{40}) \cong C_5^{32} \times C_{p^n-1}^4 \times C_{p^{2n}-1}^2$ $q \equiv -3$ mod 8.

For $p \neq 2$ and $p \neq 5$.

1. If $q \equiv 1 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^{40}$.
2. If $q \equiv -1 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^{19}$.
3. If $q \equiv 3 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^3 \times C_{p^{4n}-1}^8$.
4. If $q \equiv -3 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^4 \times C_{p^{2n}-1}^2 \times C_{p^{4n}-1}^8$.
5. If $q \equiv 7 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^3 \times C_{p^{4n}-1}^8$.
6. If $q \equiv -7 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^8 \times C_{p^{4n}-1}^8$.
7. If $q \equiv 11 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^{10} \times C_{p^{2n}-1}^{15}$.
8. If $q \equiv -11 \; mod \; 40$, then $U(FC_{40}) \cong C_{p^n-1}^4 \times C_{p^{2n}-1}^{18}$.

9. If $q \equiv 13 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^4 \times C_{p^{2n}-1}^2 \times C_{p^{4n}-1}^8$.
10. If $q \equiv -13 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^3 \times C_{p^{4n}-1}^8$.
11. If $q \equiv 17 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^8 \times C_{p^{4n}-1}^8$.
12. If $q \equiv -17 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^3 \times C_{p^{4n}-1}^8$.
13. If $q \equiv 19 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^{19}$.
14. If $q \equiv -19 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^{19}$.
15. If $q \equiv 9 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^8 \times C_{p^{2n}-1}^{16}$.
16. If $q \equiv -9 \, mod \, 40$, then $U(FC_{40}) \cong C_{p^n-1}^{10} \times C_{p^{2n}-1}^{15}$.

Proof. The Group C_{40} is given by:

$$C_{40} = < a \mid a^{40} = 1 > .$$

Let $p = 5$. If $K =< a^5 >$, then $\omega(K)$ is nilpotent and $(FC_{40}) = \omega(K)$, $FC_{40}/J(FC_{40}) = FC_8$ and $dim(J(FC_{40})) = 32$. Hence $U(FC_{40}) \cong V \times U(FC_8)$. Also, $J(FC_{40})^5 = 0$, implies that $V^5 = 1$. Hence $V \cong C_5^{16}$ and the structure of $U(FC_8)$ is given by (14, Theorem 3.3).

If $p \neq 2$ and $p \neq 5$, then p does not divide $|C_{40}|$, therefore FC_{40} is semi-simple over F. Applying the Wedderburn structure theorem and by (10, Proposition 3.6.11), we will compute the number of simple components of FC_{40}, and for all i, $n_i \geq 1$ and K_i's denotes the finite field extension of F. As G is abelian, we have $n_i = 1$, for every i due to dimension constraints. Now, C_{40} has 40 conjugacy classes. Here $x^{qk} = x$, for all $x \in Z(FC_{40})$, for any $k \in N$ only if $C_i^{qt} = C_i$, for every $1 \leq i \leq 40$, where C_i denotes the conjugacy class of C_{40}follows only if $40|q^s - 1$ or $40|q^s + 1$.

Now, if $k_i^* =< y_i >$, for all i , $1 \leq i \leq r$, then $x^{q^s} = x$, for all $x \in Z(FC_{40})$ if and only if $y_i^{q^s} = 1$, which satisfied if and only if $[Ki : F]|s$, for all $1 \leq i \leq r$. Therefore, the least number t, $t = l.c.m.\{[Ki : F]|1 \leq i \leq r\}$. Hence, all p −regular F classes are the conjugacy class of $C_{40}$and $m = 40$ as discussed in introduction. By simple calculations, we have the following possible values of q:

1. For $q \equiv 1 \, mod \, 40$, we have $t = 1$.

2. For $q \equiv -1 \, mod \, 40$, we have $t = 2$.
3. For $q \equiv 3 \, mod \, 40$, we have $t = 4$.
4. For $q \equiv -3 \, mod \, 40$, we have $t = 4$.
5. For $q \equiv 7 \, mod \, 40$, we have $t = 4$.
6. For $q \equiv -7 \, mod \, 40$, we have $t = 4$.
7. For $q \equiv 9 \, mod \, 40$, we have $t = 2$.
8. For $q \equiv -9 \, mod \, 40$, we have $t = 2$.
9. For $q \equiv 11 \, mod \, 40$, we have $t = 2$.
10. For $q \equiv -11 \, mod \, 40$, we have $t = 2$.
11. For $q \equiv 13 \, mod \, 40$, we have $t = 4$.
12. For $q \equiv -13 \, mod \, 40$, we have $t = 4$.
13. For $q \equiv 17 \, mod \, 40$, we have $t = 4$.
14. For $q \equiv -17 \, mod \, 40$, we have $t = 4$.
15. For $q \equiv 19 \, mod \, 40$, we have $t = 2$.
16. For $q \equiv -19 \, mod \, 40$, we have $t = 2$.

Next, we calculate T and p-regular $F-$ conjugacy classes. Let c denote the number of p-regular $F-$conjugacy classes. Using (10, Theorem 3.6.2), we have $dim(Z(FC_{40})) = 40$, thus $\Sigma_{i=1}^{r} | K_i : F | = 40$ and we have the cases as follows:

1. Let $q \equiv 1 \, mod \, 40$. This implies $T = \{1\} mod \, 40$. So, $p-$ regular and $F-$ conjugacy classes are same as the conjugacy class of C_{40}. Thus $c = 40$ and $FC_{40} \cong F^{40}$.
2. Let $q \equiv -1 \, mod \, 40$. This implies $T = \{1, -1\} mod \, 40$. So,$p-$ regular and $F-$ conjugacy classes will be $\{a^{\pm i}\}$, for $1 \leq i \leq 19$, $\{a^{20}\}$. Thus, $c = 21$ and $FC_{40} \cong F^2 \oplus F_2^{19}$.

3. Let $q \equiv 3 \, mod \, 40$. This implies $T = \{1,3,9,27\} \, mod \, 40$. So, $p-$ regular and $F-$ conjugacy classes will be $\{1\}, \{a, a^3, a^9, a^{27}\}$, $\{a^2, a^6, a^{18}, a^{14}\}, \{a^4, a^{12}, a^{36}, a^{28}\}$, $\{a^5, a^{15}\}$, $\{a^7, a^{21}, a^{23}, a^{29}\}$, $\{a^8, a^{24}, a^{32}, a^{16}\}$, $\{a^{10}, a^{30}\}$, $\{a^{11}, a^{33}, a^{19}, a^{17}\}$, $\{a^{13}, a^{39}, a^{37}, a^{31}\}$,

$\{a^{20}\}, \{a^{22}, a^{26}, a^{38}, a^{34}\}, \{a^{25}, a^{35}\}$. Thus, $c = 13$ and $FC_{40} \cong \mathrm{F}^2 \oplus F_2^3 \oplus F_4^8$.

4. Let $q \equiv -3$ mod 40. This implies $T = \{1,9,13,37\} mod\ 40$. So, $p-$ regular and $F-$ conjugacy classes will be {1},
5. $\{a, a^9, a^{13}, a^{37}\}, \{a^2, a^{18}, a^{26}, a^{34}\}, \{a^3, a^{27}, a^{39}, a^{31}\}, \{a^5, a^{25}\}$, $\{a^4, a^{36}, a^{12}, a^{28}\}, \{a^6, a^{14}, a^{38}, a^{22}\}, \{a^7, a^{23}, a^{11}, a^{19}\}, \{a^8, a^{32}, a^{24}, a^{16}\}$,
6. $\{a^{15}, a^{35}\}, \{a^{10}\}, \{a^{17}, a^{33}, a^{21}, a^{29}\}, \{a^{20}\}, \{a^{30}\}$. Thus, $c = 13$ and $FC_{40} \cong \mathrm{F}^4 \oplus F_2^2 \oplus F_4^8$.
7. Let $q \equiv 7 mod\ 40$. This implies $T = \{1,7,9,23\} mod\ 40$. So, $p-$regular and $F-$ conjugacy classes will be {1}, $\{a, a^7, a^9, a^{23}\}$, $\{a^2, a^{14}, a^{18}, a^6\}, \{a^3, a^{21}, a^{27}, a^{29}\}, \{a^4, a^{28}, a^{36}, a^{12}\}, \{a^5, a^{35}\}$, $\{a^8, a^{16}, a^{32}, a^{24}\}, \{a^{10}, a^{30}\}, \{a^{11}, a^{37}, a^{19}, a^{13}\}, \{a^{15}, a^{25}\}$ $\{a^{26}, a^{22}, a^{34}, a^{38}\}, \{a^{20}\}, \{a^{17}, a^{39}, a^{33}, a^{31}\}$. Thus, $c = 13$ and $FC_{40} \cong F^2 \oplus F_2^3 \oplus F_4^8$.
8. Let $q \equiv -7\ mod\ 40$. This implies $T = \{1,9,17,33\} mod\ 40$. So,- $p-$ regular and $F-$ conjugacy classes will be $\{1\}, \{a, a^9, a^{17}, a^{33}\}, \{a^2, a^{18}, a^{34}, a^{26}\}, \{a^3, a^{27}, a^{11}, a^{19}\}$, $\{a^4, a^{36}, a^{28}, a^{12}\}$, $\{a^5\}$, $\{a^6, a^{14}, a^{22}, a^{38}\}$, $\{a^7, a^{23}, a^{39}, a^{31}\}$, $\{a^8, a^{32}, a^{16}, a^{24}\}$, $\{a^{10}\}$, $\{a^{13}, a^{37}, a^{21}, a^{29}\}$, $\{a^{15}\}$, $\{a^{20}\}$, $\{a^{25}\}, \{a^{30}\}, \{a^{35}\}$. Thus, $c = 16$ and $FC_{40} \cong F^8 \oplus F_4^8$.
9. Let $q \equiv 9\ mod\ 40$. This implies $T = \{1,9\}\ mod\ 40$. So, $p -$ regular and $F -$ conjugacy classes will be $\{1\}, \{a, a^9\}, \{a^2, a^{18}\}, \{a^3, a^{27}\}, \{a^4, a^{36}\}$, $\{a^5\}, \{a^6, a^{14}\}, \{a^7, a^{23}\}$, $\{a^8, a^{32}\}$, $\{a^{10}\}$, $\{a^{11}, a^{19}\}$, $\{a^{12}, a^{28}\}, \{a^{13}, a^{37}\}$, $\{a^{15}\}, \{a^{16}, a^{24}\}$, $\{a^{17}, a^{33}\}, \{a^{20}\}$, $\{a^{21}, a^{29}\}, \{a^{22}, a^{38}\}$, $\{a^{25}\}, \{a^{26}, a^{34}\}$, $\{a^{30}\}$, $\{a^{31}, a^{39}\}$, $\{a^{35}\}$. Thus, $c = 24$ and $FC_{40} \cong F^8 \oplus F_2^{16}$.
10. Let $q \equiv -9\ mod\ 40$. This implies $T = \{1,31\} mod\ 40$. So $p-$ regular and $F-$conjugacy classes will be $\{1\}, \{a, a^{34}\}, \{a^2, a^{22}\}$, $\{a^3, a^{13}\}, \{a^4\}, \{a^5, a^{35}\}, \{a^6, a^{26}\}, \{a^7, a^{17}\}, \{a^8\}, \{a^{10}, a^{30}\}$, $\{a^{11}, a^{21}\}$, $\{a^{12}\}$, $\{a^{15}, a^{25}\}$, $\{a^{16}\}, \{a^{32}\}, \{a^{36}\}$, $\{a^9, a^{39}\}$, $\{a^{14}, a^{34}\}, \{a^{18}, a^{38}\}$, $\{a^{28}\}, \{a^{19}, a^{29}\}, \{a^{20}\}$, $\{a^{23}, a^{33}\}$, $\{a^{24}\}, \{a^{27}, a^{37}\}$. Thus, $c = 25$ and $FC_{40} \cong F^{10} \oplus F_2^{15}$.
    11. Let $q \equiv 11\ mod\ 40$. So, the number c of $p-$regular and $F-$conjugacy classes of FC_{40} is 25 and thus $FC_{40} \cong F^{10} \oplus F_2^{15}$.

12. Let $q \equiv -11 \, mod \, 40$. So, the number c of $p-$regular and $F-$conjugacy classes of FC_{40} will be 22 and thus $FC_{40} \cong F^4 \oplus F_2^{18}$.

13. Let $q \equiv 13 \, mod \, 40$. This implies $T = \{1,9,13,37\} \, mod \, 40$. So, $p-$ regular and $F-$ conjugacy classes will be $\{1\}, \{a, a^9, a^{13}, a^{37}\}, \{a^2, a^{18}, a^{26}, a^{34}\}, \{a^3, a^{27}, a^{39}, a^{31}\}, \{a^5, a^{25}\}$, $\{a^4, a^{36}, a^{12}, a^{28}\}, \{a^6, a^{14}, a^{38}, a^{22}\}, \{a^7, a^{23}, a^{11}, a^{19}\}, \{a^8, a^{32}, a^{24}, a^{16}\}$, $\{a^{15}, a^{35}\}, \{a^{10}\}, \{a^{17}, a^{33}, a^{21}, a^{29}\}, \{a^{20}\}, \{a^{30}\}$. Thus, $c = 14$ and $FC_{40} \cong F^4 \oplus F_2^2 \oplus F_4^8$.

14. Let $q \equiv -13 \, mod \, 40$. This implies $T = \{1,3,9,27\} \, mod \, 40$. So $p-$ regular and $F-$ conjugacy classes will be $\{1\}$, $\{a, a^3, a^9, a^{27}\}, \{a^2, a^6, a^{18}, a^{14}\}, \{a^4, a^{12}, a^{36}, a^{28}\}, \{a^5, a^{15}\}, \{a^7, a^{21}, a^{23}, a^{29}\}$,

15. $\{a^8, a^{24}, a^{32}, a^{16}\}, \{a^{10}, a^{30}\}, \{a^{11}, a^{33}, a^{19}, a^{17}\}, \{a^{13}, a^{39}, a^{37}, a^{31}\}, \{a^{20}\}$, $\{a^{22}, a^{26}, a^{38}, a^{34}\}, \{a^{25}, a^{35}\}$. Thus, $c = 13$ and $FC_{40} \cong F^2 \oplus F_2^3 \oplus F_4^8$.

16. Let $q \equiv 17 \, mod \, 40$. This implies $T = \{1,9,17,33\} \, mod \, 40$. So, $p-$ regular and $F-$conjugacy classes will be $\{1\}$, $\{a, a^9, a^{17}, a^{33}\}, \{a^2, a^{18}, a^{34}, a^{26}\}, \{a^3, a^{27}, a^{11}, a^{19}\}, \{a^4, a^{36}, a^{28}, a^{12}\}$, $\{a^5\}, \{a^6, a^{14}, a^{22}, a^{38}\}, \{a^7, a^{23}, a^{39}, a^{31}\}, \{a^8, a^{32}, a^{16}, a^{24}\}, \{a^{10}\}$, $\{a^{13}, a^{37}, a^{21}, a^{29}\}, \{a^{15}\}, \{a^{20}\}, \{a^{25}\}, \{a^{30}\}, \{a^{35}\}$. Thus, $c = 16$ and $FC_{40} \cong F^8 \oplus F_4^8$.

17. Let $q \equiv -17 \, mod \, 40$. This implies $T = \{1,7,9,23\} \, mod \, 40$. So, $p-$ regular and $F-$conjugacy classes are $\{1\}, \{a, a^7, a^9, a^{23}\}$, $\{a^2, a^{14}, a^{18}, a^6\}, \{a^3, a^{21}, a^{27}, a^{29}\}, \{a^4, a^{28}, a^{36}, a^{12}\}, \{a^5, a^{35}\}$, $\{a^8, a^{16}, a^{32}, a^{24}\}, \{a^{10}, a^{30}\}, \{a^{11}, a^{37}, a^{19}, a^{13}\}, \{a^{15}, a^{25}\}$, $\{a^{26}, a^{22}, a^{34}, a^{38}\}, \{a^{20}\}, \{a^{17}, a^{39}, a^{33}, a^{31}\}$. Thus, $c = 13$ and $FC_{40} \cong F^2 \oplus F_2^3 \oplus F_4^8$.

18. Let $q \equiv 19 \, mod \, 40$. So, the number c of $p-$ regular and $F-$ conjugacy classes of C_{40} will be $c = 21$ and thus $FC_{40} \cong F^2 \oplus F_2^{19}$.

19. Let $q \equiv -19 \, mod \, 40$. So, the number c of $p-$ regular and $F-$ conjugacy classes of C_{40} will be $c = 21$ and thus $FC_{40} \cong F^2 \oplus F_2^{19}$.

Hence, the above result follows.

Theorem 2.2 Let F is a field of finite characteristic $p > 0$ with $|F| = q = p^n$ and $G \cong C_4 \times C_{10}$.

For $p = 5$.

1. $U(F[C_4 \times C_{10}]) \cong C_5^{32} \times C_{5^k-1}^{8}$, $q \equiv 1 mod\ 4$;
2. $U(F[C_4 \times C_{10}]) \cong C_5^{32} \times C_{5^k-1}^{4} \times C_{5^{2k}-1}^{2}, q \equiv -1\ mod\ 4$.

For $p \neq 2$ and 5.

1. If $q \equiv 1\ mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{40}$.
2. If $q \equiv -1\ mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{4} \times C_{p^{2n}-1}^{18}$.
3. If $q \equiv 3\ mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{4} \times C_{p^{2n}-1}^{2} \times C_{p^{4n}-1}^{8}$.
4. If $q \equiv -3\ mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{8} \times C_{p^{4n}-1}^{8}$.
5. If $q \equiv 7\ mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{4} \times C_{p^{2n}-1}^{2} \times C_{p^{4n}-1}^{8}$.
6. If $q \equiv -7 mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{8} \times C_{p^{4n}-1}^{8}$.
7. If $q \equiv 9\ mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{8} \times C_{p^{2n}-1}^{16}$.
8. If $q \equiv -9\ mod\ 20$, then $U(F[C_4 \times C_{10}]) \cong C_{p^n-1}^{20} \times C_{p^{2n}-1}^{10}$.

Proof. The Group $C_4 \times C_{10}$ is given by

$$C_4 \times C_{10} =< a, b \mid a^4 = b^{10} = 1 >.$$

a) Let $p = 5$. If $K =< a, b^5 >$, then $\omega(K)$ is nilpotent and $(F[C_4 \times C_{10}]) = \omega(K)$, $F[C_4 \times C_{10}]/J(F[C_4 \times C_{10}]) = F[C_2 \times C_4]$ and $dim(J(F[C_4 \times C_{10}])) = 32$. Hence, $U(F[C_4 \times C_{10}]) \cong V \times U(F[C_2 \times C_4])$. Also, $J\ (F\ [C_4 \times C_{10}])^5 = 0$, implies that $V^5 = 1$. Hence, $V \cong C_5^{32}$ and the structure of $U\ (F\ [C_2 \times C_4])$ is given by (14, Theorem 3.4).

b) Let $p \neq 2$ and 5, then p does not divide $|C_4 \times C_{10}|$, therefore $F\ [C_4 \times C_{10}]$ is semisimple over F. Now, using the same arguments as in Theorem 2.1, we have $m = 20$. By simple calculations, we have the following values of t depends on q.

1. For $q \equiv 1 \, mod \, 20$, we have $t = 1$.
2. For $q \equiv -1 \, mod \, 20$, we have $t = 2$.
3. For $q \equiv 3 \, mod \, 20$, we have $t = 4$.
4. For $q \equiv -3 \, mod \, 20$, we have $t = 4$.
5. For $q \equiv 7 \, mod \, 20$, we have $t = 4$.
6. For $q \equiv -7 \, mod \, 20$, we have $t = 4$.
7. For $q \equiv 9 \, mod \, 20$, we have $t = 2$.
8. For $q \equiv -9 \, mod \, 20$, we have $t = 2$.

Next, we calculate T and $p-$regular $F-$ conjugacy classes. Let c denote the number of $p-$regular $F-$ conjugacy classes. Using (10, Theorem 3.6.2), we have $dim(Z(F[C_4 \times C_{10}])) = 40$. Thus, $\Sigma_{i=1}^{r} \mid K_i : F \mid = 40$ and we have the cases as follows:

1. Let $q \equiv 1 \, mod \, 20$. This implies $T = \{1\} mod 20$. So, the number of $p-$ regular and $F-$conjugacy classes are same as conjugacy classes of $C_4 \times C_{10}$. Thus, $c = 40$ and $F[C_4 \times C_{10}] \cong F^{40}$.
2. Let $q \equiv -1$ mod 20. This implies $T = \{1, -1\} mod 20$. So, $p-$ regular and $F-$ conjugacy classes will be $\{1\}, \{a, a^{-1}\}, \{a^2\}, \{b, a^{-1}\}, \{b^2, b^{-2}\}, \{b^3, b^{-3}\}, \{b^4, b^{-4}\}, \{b^5\}, \{ab, a^3b^9\}, \{ab^2, a^3b^8\}, \{ab^3, a^3b^7\}, \{ab^4, a^3b^6\}, \{ab^5, a^3b^5\}, \{ab^6, a^3b^4\}, \{ab^7, a^3b^3\}, \{ab^8, a^3b^2\}, \{ab^9, a^3b\}, \{a^2b, a^2b^9\}, \{a^2b^2, a^2b^8\}, \{a^2b^3, a^2b^7\}, \{a^2b^4, a^2b^6\}, \{a^2b^5\}$. Thus, $c = 22$ and $F[C_4 \times C_{10}] \cong F^4 \oplus F_2^{18}$.
3. Let $q \equiv 3,7 \, mod \, 20$. This implies $T = \{1,3,7,9\} mod 20$. So, $p-$ regular and $F-$ conjugacy classes will be $\{1\}, \{a, a^3\}, \{a^2\}, \{b, b^9, b^3, b^7\}, \{b^2, b^8, b^6, b^4\}, \{b^5\}, \{ab, ab^9, a^3b^3, a^3b^7\}, \{a^2b^2, a^2b^8, a^2b^6, a^2b^4\}, \{ab^2, a^3b^6, a^3b^4, ab^8\}, \{ab^3, a^3b^9, ab^7, a^3b\}, \{ab^4, a^3b^2, a^3b^8, ab^6\}, \{ab^5, a^3b^5\}, \{a^2b^3, a^2b^7, a^2b^9\}, \{a^2b^5\}$. Thus, $c = 14$ and $F[C_4 \times C_{10}] \cong F^4 \oplus F_2^2 \oplus F_4^8$.
4. Let $q \equiv -3, -7 \, mod \, 20$. This implies $T = \{1,9,13,17\} mod 20$. So, $p-$ regular and $F-$ conjugacy classes will be $\{1\}, \{a\}, \{a^2\}, \{a^3\}, \{b, b^9, b^3, b^7\}, \{b^2, b^8, b^6, b^4\}, \{b^5\}, \{ab, ab^9, ab^3, ab^7\}, \{ab^2, ab^4, ab^6, ab^8\}, \{ab^5\}, \{a^2b, a^2b^9, a^2b^3, a^2b^7\}, \{a^2b^2, a^2b^8, a^2b^6, a^2b^4\}, \{a^2b^5\}, \{a^3b^2, a^3b^8, a^3b^6, a^3b^4\}, \{a^3b^5\}$. Thus, $c = 16 \; and \; F[C_4 \times C_{10}] \cong F^8 \oplus F_4^8$.

5. Let $q \equiv 9 \ mod \ 20$. This implies $T = \{1,9\}mod20$. So, $p-$ regular and $F-$ conjugacy classes will be $\{1\},\{a\},\{a^2\},\{a^3\},\{b,b^9\},\{b^3,b^7\},\{b^2,b^8\},\{b^6,b^4\},\{b^5\},\{ab,ab^9\},$ $\{ab^3,ab^7\},\{ab^2,ab^8\},\{ab^6,ab^4\},\{ab^5\},\{a^2b,a^2b^9\},\{a^2b^3,a^2b^7\},$ $\{a^2b^2,a^2b^8\},\{a^2b^6,a^2b^4\},\{a^2b^5\},\{a^3b,a^3b^9\},\{a^3b^3,a^3b^7\},$ $\{a^3b^2,a^3b^8\},\{a^3b^6,a^3b^4\},\{a^3b^5\}$. Thus, $c = 24$ and $F[C_4 \times C_{10}] \cong F^8 \oplus F_2^{16}$.
6. Let $q \equiv -9$ mod 20. This implies $T = \{1,11\}mod$ 20. So, $p-$ regular and $F-$ conjugacy classes will be $\{1\},\{a,a^3\},\{a^2\},$ $\{b\},\{b^9\},\{b^3\},\{b^7\},\{b^2\},\{b^8\},\{b^6\},\{b^4\},\{b^5\},\{ab,a^3b\},\{a^2b\},$ $\{ab^2,a^3b^2\},\{a^2b^2\},\{ab^3,a^3b^3\},\{a^2b^3\},\{ab^4,a^3b^4\},\{a^2b^4\},$ $\{ab^5,a^3b^5\},\{a^2b^5\},\{ab^6,a^3b^6\},\{a^2b^6\},\{ab^7,a^3b^7\},\{a^2b^7\},$ $\{ab^8,a^3b^8\},\{a^2b^8\}$. Thus, $c = 30$ and $F[C_4 \times C_{10}] \cong F^{20} \oplus F_2^{10}$.

Hence, the above result follows. □

Theorem 2.3 Let F is a field of finite characteristic $p > 0$ having $|F| = q = p^n$ and $G \cong C_2 \times C_2 \times C_{10}$.

a) For $p = 5, U(F[C_2 \times C_2 \times C_{10}]) \cong C_5^{32} \times C_{2^n-1}^8$.

b) For $p \neq 2$ and 5.

1. If $q \equiv 1 \ mod \ 10$, then $U(F[C_2 \times C_2 \times C_{10}]) \cong C_{p^n-1}^{40}$.
2. If $q \equiv -1 \ mod \ 10$, then $U(F[C_2 \times C_2 \times C_{10}]) \cong C_{p^n-1}^{8} \times C_{p^{2n}-1}^{16}$.
3. If $q \equiv 3 \ mod \ 10$, then $U(F[C_2 \times C_2 \times C_{10}]) \cong C_{p^n-1}^{8} \times C_{p^{4n}-1}^{8}$.
4. If $q \equiv -3 \ mod \ 10$, then $U(F[C_2 \times C_2 \times C_{10}]) \cong C_{p^n-1}^{8} \times C_{p^{4n}-1}^{8}$.

Proof. The Group $C_2 \times C_2 \times C_{10}$ is given by:

$C_2 \times C_2 \times C_{10} =< a,b,c \mid a^2 = b^2 = c^{10} = 1 >$.

a) Let $p = 5$. If $K =< a,b,c^5 >$, then $\omega(K)$ is nilpotent and $U(F[C_2 \times C_2 \times C_{10}]) = \omega(K), \frac{F[C_2 \times C_2 \times C_{10}]}{J(F[C_2 \times C_2 \times C_{10}])} = FC_2^3$ and $dim(J(F[C_2 \times C_2 \times C_{10}])) = 32$. Hence, $U(F[C_2 \times C_2 \times C_{10}]) \cong V \times U(FC_2^3)$. Also, $J(F[C_2 \times C_2 \times C_{10}])^5 = 0$, implies that $V^5 = 1$. Hence $V = C_5^{32}$ and the structure of $U(FC_2^3)$ is given by (14, Theorem 3.5).

b) Let $p \neq 2$ and 5, then p does not divide $|C_2 \times C_2 \times C_{10}|$, therefore $F[C_2 \times C_2 \times C_{10}]$ is semisimple over F. Now using the same arguments as in Theorem 2.1, we have $m = 10$. By simple calculations, we have following values of t depends on q:

1. For $q \equiv 1 \; mod \; 10$, we have $t = 1$.
2. For $q \equiv -1 \; mod \; 10$, we have $t = 2$.
3. For $q \equiv 3 \; mod \; 10$, we have $t = 4$.
4. For $q \equiv -3 \; mod \; 10$, we have $t = 4$.

Next, we calculate T and p −regular $F-$ conjugacy classes. Let c denotes the number of p −regular F −conjugacy classes. Using (10, Theorem 3.6.2), we have $dim(Z(F[C_2 \times C_2 \times C_{10}]) = 40$, thus $\Sigma_{i=1}^{r} | K_i : F| = 4$ and we have the cases as follows:

1. Let $q \equiv 1 \; mod \; 10$. So, we have p −regular $F-$ conjugacy classes are same as the conjugacy classes of $C_2 \times C_2 \times C_{10}$. Thus, $c = 40$ and $(F[C_2 \times C_2 \times C_{10}]) \cong F^{40}$.
2. Let $q \equiv -1 \; mod \; 10$. This implies $T = \{1, -1\} mod \; 10$. So, p −regular F −conjugacy classes will be $\{1\}, \{a\}, \{b\}, \{c, c^9\}, \{c^2, c^8\}, \{c^3, c^7\}, \{c^4, c^6\}, \{c^5\}, \{ac, ac^9\}, \{bc, bc^9\}$, $\{ac^2, ac^8\}, \{bc^2, bc^8\}, \{ac^3, ac^7\}, \{ac^4, ac^6\}, \{ac^5\}, \{bc^3, bc^7\}, \{bc^4, bc^6\}$, $\{bc^5\}, \{ab\}, \{abc, abc^9\}, \{abc^2, abc^8\}, \{abc^3, abc^{97}\}, \{abc^4, abc^6\}, \{abc^5\}$. Thus, $c = 24$ and $F[C_2 \times C_2 \times C_{10}] \cong F^8 \oplus F_2^{16}$.
3. Let $q \equiv \pm 3 \; mod \; 10$. This implies $T = \{1, -1\}$ mod 10. So, p −regular $F-$ conjugacy classes will be $\{1\}, \{a\}, \{b\}, \{c^2, c^4, c^6, c^8\}, \{c, c^3, c^7, c^9\}, \{c^5\}, \{ac^5\}, \{bc^5\}$, $\{ac^2, ac^4, ac^6, ac^8\}, \{ac, ac^3, ac^7, ac^9\}, \{bc^2, bc^4, bc^6, bc^8\}$, $\{bc, bc^3, bc^7, bc^9\}, \{ab\}, \{abc^2, abc^4, abc^6, abc^8\}, \{abc, abc^3, abc^7, abc^9\}$, $\{abc^5\}$. Thus, $c = 16$ and $F[C_2 \times C_2 \times C_{10}] \cong F^8 \oplus F_4^8$.

Hence, we have the desired result. □

References

1. S. F. Ansari and M. Sahai, Unit groups of group algebras of groups of order 20, *Quaes. Math.* DOI: https://doi.org/10.2989/16073606.2020.1727583.

2. S. Bhatt and H. Chandra, Structure of unit group of $F_p nD_{60}$, *Asi. Eu. J. Math.* https://doi.org/10.1142/S1793557121500753 (2020).
3. S. Bhatt and H. Chandra, Unit groups of group algebras of abelian groups of order 32, *Proy. Jr. Math.* 40(5) (2021).
4. L. Creedon, The unit group of small group algebras and the minimum counter example to the isomorphism problem. *Int. J. Pure. Appl. Math*. 2008; 49: 531-537.
5. L. Creedon, J. Gildea, The Structure of the unit group of the group algebra F_3kD_6. *Int. J. Pure. Appl. Math*. 2008; 45(2): 315-320.
6. M. Khan, Structure of the unit group of FD_{10}. Serdica Math. J. 2009; 35(1): 15-24.
7. M. Khan, R. K. Sharma, J.B. Srivastava, The Unit Group of FS_4. *Acta Math. Hungar.* 2008; (118): 105-113.
8. N. Makhijani, R. K. Sharma, Srivastava JB. A Note on Units in $F_pmD_{2p}n$. *Acta Math. Acad. Paedagog. Nyi'r* 2014; 30: 17-25.
9. N. Makhijani, R. K. Sharma, J. B. Srivastava, Unit in F_2kD_{2n}. Int. J. Group Theory 2014; 3(3): 25-34.
10. C. P. Milies, S. K. Sehgal, An introduction to Group Rings. *Algebras and Applications*, Bol. Soc. Brasil. Mat. **1** (2002).
11. C. P. Milies, The Unit of the integral Group ring ZD_4. *Bol.Soc. Brasil.* Mat. 1973; 4(2): 85-92.
12. R. K. Sharma RK, J. B. Srivastava, M. Khan, The unit group of FA_4. *Pub. Math. Debrecen* 2007; 71: 21-26.
13. R. K. Sharma, J. B. Srivastava, M. Khan, The unit group of FS_3. *Acta Math. Acad. Paedagog, Nyhazi (N. S.)* 2007; 23(2): 129-142.
14. M. Sahai, M. and S. F. Ansari, Unit groups of finite group algebras of abelian groups of order at most 16, *Asi. Eur. J. Math.*, (2020) 2150030.