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On *D*-Distance and *D*-Closed Graphs

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Abstract

This paper studies the concepts of D-boundary vertex, D-interior vertex, D-null vertex, D-closure of a graph and D-closed graph, based on the idea of D-distance. We investigate the structural properties of these concepts and determine whether some special classes of graphs are D-closed or not.

Keywords: D-distance, D-boundary vertex, D-interior vertex, D-null vertex, D-closure, D-closed graphs.

1. Introduction

The distance concept has become particularly essential in graph theory due to the extensive growth of networks. The notion of distance is studied by many authors and many innovative ideas have been arrived at. It has resulted in the formulation of a number of graph parameters.

In this paper, the idea of D-boundary vertex, D-interior vertex, and D-null vertex based on D-distance [3] are introduced. Also, we introduce the concepts of D-closure of a graph and D-closed graph based on D-boundary vertices and D-interior vertices. D-null vertex in a graph is a vertex which is neither a D-boundary vertex nor a D-

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interior vertex. The graph which contains a D-null vertex is not D-closed.

We investigate the structural properties of the D-boundary vertices, D-interior vertices and D-null vertices of graphs and make a study on some special classes of graphs and determine whether they are D-closed or not.

2. Preliminaries

Throughout this paper, we consider G as a connected graph.

Definition 2.1 [3,4] If x, y are vertices of a connected graph G and P is a $x - y$ path, then the D-length of the $x - y$ path P is defined as $l_D(P) = l(P) + deg(x) + deg(y) + \sum deg(z)$, where the sum runs over all intermediate vertices z of P and $l(P)$ is the length of the path.

Definition 2.2. [3,4] The D-distance $d_D(x, y)$ between x, y of the graph G is $d_D(x, y) = min\{l_D(P)\}$, if x and y are distinct and $d_D(x, y) = 0$, if $x = y$, where the minimum is taken over all $x - y$ paths P in G.

$$d_D(x, y) = \begin{cases} min\{l_D(P)\}, & if\ x \neq y \\ 0, & if\ x = y \end{cases}$$

Theorem 2.3. [3] In a connected graph G, the D-distance is a metric.

Definition 2.4. [3,6] The D-eccentricity of a vertex x in G is defined as $e_D(x) = max\{d_D(x, y);\ y \in V\}$. Each vertex at D-distance $e_D(x)$ from x is called a D- eccentric vertex of x.

Definition 2.5. [3] The minimum of the D-eccentricity of all the vertices in G is called D- radius $r_D(G)$ and maximum of the D-eccentricity of all the vertices in G is called D-diameter $d_D(G)$.

Definition 2.6. [3] A vertex x in G is a D-central vertex if $e_D(x) = r_D(G)$. A connected graph G is D-self-centred if each vertex is a D-central vertex.

Definition 2.7. [3] A vertex x in G is a D-peripheral vertex if $e_D(x) = d_D(G)$.

Theorem 2.8. [3] Let G be a connected graph. Then $r_D(G) \leq d_D(G) \leq 2r_D(G)$.

Readers may refer to Harary [2] and Chartrand. G, Zhang. P [1] for graph terminology and definitions that are not specifically stated here and refer to [5] for topological terms and concepts.

3. Main Results

Definition 3.1. A vertex y in G is a D-boundary vertex of a vertex x in G, if $d_D(x,y) \geq d_D(x,z)$, for each neighbour z of y. The D-boundary vertices of x is represented by x_D^b.

Definition 3.2. A vertex y in G is a D-boundary vertex of G, if y is a D-boundary vertex of some vertex of G.

Definition 3.3. A vertex z is a D-interior vertex of G, if for every vertex $x \neq z$, $\exists\, y \in G$ such that $d_D(x,y) = d_D(x,z) + d_D(z,y) - deg(z)$.

Definition 3.4. A vertex y in G is a D-null vertex if y is neither a D-boundary vertex nor a D-interior vertex.

Definition 3.5. The subgraph, $\delta_D(G)$ of G induced by its D-boundary vertices is called D-boundary of G and G is a D-boundary graph if $\delta_D(G) = G$

Definition 3.6. The subgraph, $int_D(G)$ of G induced by its D-interior vertices is called D-interior of G.

Definition 3.7. The subgraph, $cl_D(G)$ of G induced by the union of its D-boundary vertices and D-interior vertices is called D-closure of G and G is a D-closed graph if $cl_D(G) = G$.

Definition 3.8. A graph G which contains a D-null vertex is not D-closed.

Remark 3.9. Every D-boundary graph G is a D-closed graph.

Proposition 3.10. Let x be a vertex in a connected graph G with $deg(x) = 1$, then x is a D- boundary vertex.

Proof. Consider G with n vertices $x_1, x_2, \ldots, x_n$. Let x_1 be a vertex of degree 1; that is, $deg(x_1) = 1$. Then, it has only one neighbour say, x_2. Then, clearly $d_D(x_i, x_1) \geq d_D(x_i, x_2), 1 < i \leq n$; that is, x_1 is a D-boundary vertex of the other vertices x_i in G.

Theorem 3.11. A D-boundary vertex y in a connected graph G is not a D-interior vertex of G.

Proof. Let y be a D-boundary vertex of a vertex x in a connected graph G. Then $d_D(x,y) \geq d_D(x,z)$, for all neighbours z of y. If possible, let y be the D-interior vertex of G. Then, there exists a vertex z, where $x \neq z \neq y$ such that $x < y < z$. Let $P: x = y_1, y_2, \dots \dots . y = y_k, y_{k+1}, \dots \dots, y_m = z$ be a $x - z$ path, $1 < k < m$. Then, $y_{k+1} \in N(y)$, and this implies $d_D(x, y_{k+1}) > d_D(x,y)$, which is a contradiction. Hence y is not a D-interior vertex of G.

Theorem 3.12. The path graph P_n is D-closed.

Proof. Let P_n be the path graph with n vertices $x_1, x_2, \dots, x_n$ where $deg(x_1) = deg(x_n) = 1$ and for $2 \leq i \leq n-1$, $deg(x_i) = 2$. In P_n,

$d_D(x_1, x_i) = d_D(x_n, x_i) = 3k + 1$, when $d(x_1, x_i) = d(x_n, x_i) = k$, $k = 1,2, \dots (n-2)$,

$d_D(x_i, x_j) = 3k + 2$, when $d(x_i, x_j) = k$, $k = 1,2, \dots (n-3)$, $2 \leq j \leq n-1$,

$$d_D(x_1, x_n) = 3(n-1).$$

Since $deg(x_1) = deg(x_n) = 1$, the vertices x_1, x_n are D-boundary vertices by Proposition 3.10. Also, since $d_D(x_i, x_m) = d_D(x_i, x_j) + d_D(x_j, x_m) - deg(x_j)$, $i \neq j \neq m$, $1 \leq i, m \leq n$, the vertices $x_j, 2 \leq j \leq (n-1)$ are D-interior vertices of P_n. Thus $cl_D(P_n) = P_n$. Hence, P_n is D-closed.

Theorem 3.13. The star graph $K_{1,n}$ is D-closed.

Proof. Let $y, x_1, x_2, \dots, x_n$ be the vertices of the star graph $G = K_{1,n}$, such that $deg(x_i) = 1$, $1 \leq i \leq n$ and $deg(y) = n$. In $K_{1,n}$,

$d_D(y, x_i) = n + 2$, $1 \leq i \leq n$.

$d_D(x_i, x_j) = n + 4$, $\forall\, i, j, i \neq j$.

All the vertices x_i are D-boundary vertices by Proposition 3.10. Also, the shortest path joining any two vertices x_i, x_j, $1 \leq i, j \leq n$ must pass through the central vertex y, for which $deg(y) = n$. Then, clearly y is a D-interior vertex of G. Hence, $cl_D(K_{1,n}) = K_{1,n}$.

Theorem. 3.14. The complete graph K_n is D-boundary and D-closed.

Proof. Consider the complete graph $K_n, n \geq 3$, with n vertices x_i , $i = 1,2, \dots , n$,

$d_D\left(x_i, x_j\right) = 2n - 1,\ i \neq j$.

Since, $x_{i_D}^{\ b} = \{x_j, i \neq j\}$, $\delta_D(K_n) = cl_D(K_n) = K_n$. Hence, K_n is D-boundary and D-closed.

Theorem 3.15. The cycle graph C_n, $n \geq 3$, is D-boundary and D-closed.

Proof. Consider the cycle C_n with n vertices x_i, $1 \leq i \leq n$.

In C_n, when $d\left(x_i, x_j\right) = k$, $d_D\left(x_i, x_j\right) = 3k + 2$, $k = 1,2, \dots$., where $1 \leq i, j \leq n$ and $x_{i_D}^{\ b} = \{x_j\}$, where, x_i, x_j are non-adjacent vertices.

That is, all the vertices x_i are D-boundary vertices of C_n. Thus, $\delta_D(C_n) = cl_D(C_n) = C_n$.

Theorem 3.16. The Petersen graph G is D-boundary and D-closed.

Proof. Consider the Petersen graph G with vertices x_i, $1 \leq i \leq 10$.

Here,

$d_D\left(x_i, x_j\right) = 4k + 3$, for all i , j, when $d\left(x_i, x_j\right) = k, k = 1,2$.

$x_{i_D}^{\ b} = \{x_j\}$, where x_i,x_j are non-adjacent vertices.

That is, all the vertices x_i, $1 \leq i \leq 10$ are D-boundary vertices of G. So, $\delta_D(G) = cl_D(G) = G$.

Theorem 3.17. The bull graph G is not D-closed.

Proof. Let $x_i, 1 \leq i \leq 5$ be the vertices of the bull graph G with $deg(x_i) = 1, i = 1,2$, $deg\left(x_j\right) = 3$, $j = 3,4$ and $deg(x_5) = 2$. By Proposition 3.10, x_1, x_2 are D-boundary vertices. Also, x_3, x_4 are D-interior vertices and x_5 is a D-null vertex. Then, $cl_D(G) \neq G$. So, G is not D-closed.

Theorem 3.18. The butterfly graph G is D-closed.

Proof. Let x_i, $1 \leq i \leq 4$, be the vertices of the butterfly graph G with $deg(x_i) = 2$ and x_5 be the common vertex with $deg(x_5) = 4$. In G, for $1 \leq i \leq 4$,

$d_D\left(x_i, x_j\right) = 5k$, when $d\left(x_i, x_j\right) = k$, $k = 1,2$,

$d_D(x_i, x_5) = 7$.

Then, for $1 \leq i \leq 4$, $x_{i_D}^{b} = \{x_j, 1 \leq j \leq 5\}$, where, x_i, x_j are non-adjacent vertices.

$$x_{5_D}^{b} = \{x_j, 1 \leq j \leq 4\}$$

That is, $\{x_i, 1 \leq i \leq 4\}$ are D-boundary vertices and x_5 is a D-interior vertex. Hence $cl_D(G) = G$ and G is D-closed.

Theorem 3.19. The friendship graph F_n is D-closed.

Proof. Let x_i, $1 \leq i \leq 2n$, be the vertices of the friendship graph F_n with $deg(x_i) = 2$ and x_{2n+1} be the common vertex with $deg(x_{2n+1}) = 2n$.

In F_n, for $1 \leq i \leq 2n$, $d_D(x_i, x_j) = \begin{cases} 5, & when\ d(x_i, x_j) = 1 \\ 2n+6, & when\ d(x_i, x_j) = 2 \end{cases}$

$$d_D(x_i, x_{2n+1}) = 2n + 3.$$

Then, for $1 \leq i \leq 2n$, $x_{i_D}^{b} = \{x_j, 1 \leq j \leq 2n\}$, where, x_i, x_j are non-adjacent vertices.

$$x_{2n+1_D}^{b} = \{x_j, 1 \leq j \leq 2n\}$$

That is, $\{x_i, 1 \leq i \leq 2n\}$ are D-boundary vertices and x_{2n+1} is a D-interior vertex. Then, $cl_D(F_n) = F_n$. Hence, F_n is D-closed.

Theorem 3.20. The bistar graph $B_{p,q}$ is D-closed.

Proof. The bistar graph $B_{p,q}$ is the graph obtained from K_2 with vertices x, y with p pendant edges at x and q pendant edges at y. Let $x_i, 1 \leq i \leq p$ and $y_j, 1 \leq j \leq q$, be vertices of $B_{p,q}$ with $\deg(x_i) = \deg(y_j) = 1$. In $B_{p,q}$,

$$d_D(x_i, x) = p + 3,$$

$$d_D(x_i, y) = d_D(y_j, x) = p + q + 5,$$

$d_D(x_i, y_j) = p + q + 7$,

$d_D(y_j, y) = q + 3$.

Here, $\{x_i, 1 \leq i \leq p\}$ and $\{y_j, 1 \leq j \leq q\}$ are D-boundary vertices and x, y are D-interior vertices. Hence, $cl_D(B_{p,q}) = B_{p,q}$.

4. Conflict of Interests

The author(s) declare that there is no conflict of interests.

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