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Edge Italian Domination in some wheel related graphs

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Abstract

A function $f: E\,(G) \to \{0,1,2\}$ is an edge Italian dominating function (EIDF) if it satisfies the rule that every edge with weight 0 is either adjacent to an edge with weight 2 or adjacent to at least two edges with weight 1 each. The weight of an EIDF is $\sum_{e\,\in\,E(G)} f(e)$. The minimum $\sum_{e\,\in\,E(G)} f(e)$ is the edge Italian domination number (EIDN). The symbol $\gamma_I'(G)$ is used to denote the EIDN. In this paper, we obtain the EIDN of some wheel related graphs like gear graph, helm graph, flower graph, web graph etc.

Keywords: Italian Domination, Edge Italian Dominating function, Edge Italian domination number.

1. Introduction

Motivated by Stewart's article "Defend the Roman Empire" [7], Cockayne et al. introduced Roman dominating function (RDF) in [2]. Chellali et al. [1] initiated a study on a variant of RDF called Roman {2} domination. Henning and Klostermeyer [4] did more research on it and changed its name to Italian domination (ID). Let $G = (V, E)$ be a simple connected graph. A function $f: V(G) \to \{0,1,2\}$ is an ID function if it has the property that every vertex v with $f(v) = 0$ has a neighbour u with $f(u) = 2$ or has at least two neighbours

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x, y with $f(x) = f(y) = 1$. The Italian domination number, $\gamma_I(G)$, is the minimum $\sum_{v \in V(G)} f(v)$.

In [4] we have introduced the edge Italian domination in graphs as a function $f: E(G) \to \{0,1,2\}$ satisfying the rule that every edge with weight 0 is either adjacent to an edge with weight 2 or adjacent to at least two edges with weight 1 each. The weight of an EIDF is $\sum_{e \in E(G)} f(e)$. The minimum $\sum_{e \in E(G)} f(e)$ is the edge Italian domination number (EIDN). The symbol $\gamma_I'(G)$ is used to denote the EIDN.

Wheel graph W_n, $n \geq 3$, is the join of the graphs C_n and K_1. It is created by joining the only vertex of K_1 to every vertex of an n-cycle. The Gear graph G_n, n ≥ 3 is the graph obtained from the wheel graph W_n by putting an additional vertex between every pair of neighbouring vertices on the rim of W_n. Then G_n has $2n + 1$ vertices and $3n$ edges. Helm graph H_n, n ≥ 3 is constructed by adding a pendant edge at every vertex of C_n in W_n. Closed Helm graph CH_n, is constructed from H_n by inserting edges between the pendant vertices. Flower graph Fl_n, is constructed from H_n by connecting every pendant vertex to the single vertex at the center.

Web graph is obtained from H_n, by connecting the pendant vertices of H_n to form C_n and by inserting a pendant edge at every vertex of C_n. We can keep connecting the pendant vertices of this graph and insert a pendant edge at every vertex of this new cycle and so on. A web graph with t such cycles is denoted by $W_{t,n}$. Friendship graph, F_n can be constructed from W_{2n} by deleting every alternate edge of C_{2n} in W_{2n}. Sunflower graph, SF_n is constructed from W_n by inserting n additional vertices $v_1', v_2', v_3', \ldots, v_n'$ such that v_i' is adjacent to v_i and v_{i+1}, for $i = 1$ *to* $n - 1$ and v_n' is adjacent to v_n and v_1. For terms and definitions not explicitly defined here, the reader may refer to Harary [3].

The following results will be used in the sequel.

Theorem.1.1. [5] For the path graph, P_m, $\gamma_I'(P_m) = \left\lceil \frac{m}{2} \right\rceil$, $m \geq 2$.

Theorem.1.2. [5] For the cycle graph, C_n, $\gamma_I'(C_n) = \left\lceil \frac{n}{2} \right\rceil$, $n \geq 3$.

Theorem.1.3. [5] For the Wheel graph, W_n, $\gamma_I'(W_n) = \left\lceil \frac{n+2}{2} \right\rceil$, $n \geq 3$.

2. Edge Italian Domination in some wheel related graphs.

Theorem 2.1. For the Gear graph G_n, $\gamma_I'(G_n) = n + 1, \forall n \geq 3$.

Proof. Let the edge set E of G_n be partitioned into two sets X and Y, where $X = \{x_1, x_2, \dots, x_n\}$ are the central edges joining the apex vertex to the rim vertices and $Y = \{y_1, y_2, \dots, y_{2n}\}$ are the rim edges. Let the central edge x_1 be adjacent to the rim edges y_1 and y_2. Define $f: E(G_n) \rightarrow \{0,1,2\}$ such that $f(x_1) = 1$, $f(x_i) = 0$, $\forall i \neq 1$ and $f(y_i) = 0, \forall$ odd i and $f(y_i) = 1$, $\forall$ even i. Then $\Sigma f(e) \leq n + 1$.

Also, we can see that the set Y, of the rim edges of G_n form a cycle on $2n$ vertices and by Theorem 1.2, we have $\gamma_I'(C_{2n}) = \left\lceil \frac{2n}{2} \right\rceil = n$. Hence, for the remaining edges, if exactly one of the central edges is assigned the weight 1 and all other edges 0, we get a minimum EIDF of G_n. So, $\Sigma f(e) \geq n + 1$. Therefore, $\gamma_I'(G_n) = n + 1$.

Proposition 2.2. For the Helm graph H_n, $\gamma_I'(H_n) = n, n \geq 3$.

Proof. Consider the EIDF, f, defined on H_n by giving weight 1 to all the edges of the cycle and weight 0 to all other edges, then f gives a minimum EIDF. Hence, $\gamma_I'(H_n) = n$.

Proposition 2.3. For the Closed Helm graph CH_n, $\gamma'_I(CH_n) = n + 1$.

Proof. Consider an EIDF f on CH_n in which all the edges of 2 cycles in CH_n are given the weight 0 and the n edges connecting the 2 cycles are given weight 1. Also, give the weight 1 to one of the n central edges. Then f is minimum and $\gamma'_I(CH_n) = n + 1$.

Proposition 2.4. For the Flower graph Fl_n, $\gamma'_I(Fl_n) = n + 1$.

Proof . Consider the EIDF, f on Fl_n in which all the n pendant edges and one of the central edges joining the apex vertex to the rim vertices get the weight 1. Then, all other edges being adjacent to this central edge and one of the pendant edges can be given the weight 0. Clearly f is minimum. Hence, $\gamma_I'(Fl_n) = n + 1$.

Theorem.2.5. For the *Web graph* $W_{t,n}$, $\gamma_I'(W_{t,n}) = n \left\lceil \frac{(t+1)}{2} \right\rceil + 1$

Proof. Let $C_1, C_2, C_3, \dots, C_t$ be the t cycles of length n each.

Case.1. When t is even.

Define $f: E\left(W_{t,n}\right) \rightarrow \{0,1,2\}$ such that an edge joining the apex vertex to a rim vertex of the innermost cycle C_1 and all the $'n'$ pendant edges get the weight1. Also give the weight 1 to all the edges connecting C_1 *and* C_2, C_3 *and* C_4, …, C_{t-1} *and* C_t. There are $n\left(\frac{t}{2}\right)$ such edges. Give the weight 0 to all other edges. Then, f gives a minimum Edge Italian dominating function and $\sum f(e) = n\left(\frac{t}{2}\right) + n + 1 = \frac{1}{2}[n(t+2)+2] \leq \mathrm{n}\left\lceil\frac{(t+1)}{2}\right\rceil + 1$.

Case.2. When t is odd.

Define $f: E\left(W_{t,n}\right) \rightarrow \{0,1,2\}$ such that an edge joining the apex vertex to a rim vertex of the innermost cycle C_1 and all the $'n'$ pendant edges get the weight 1. Also give the weight 1 to all the edges connecting the corresponding vertices of C_1 *and* C_2, C_3 *and* C_4, …, C_{t-2} *and* C_{t-1}. There are $n\left(\frac{t-1}{2}\right)$ such edges. Give the weight 0 to all other edges. Then f gives a minimum Edge Italian dominating function and

$$\sum f(e) = n\left(\frac{t-1}{2}\right) + n + 1 = \frac{1}{2}[n(t+1)+2] \leq \mathrm{n}\left\lceil\frac{(t+1)}{2}\right\rceil + 1.$$

Thus, in either case, $\sum f(e) \leq \mathrm{n}\left\lceil\frac{(t+1)}{2}\right\rceil + 1$.

Now, consider the n paths P_{t+1} of length t from each pendant vertex to the innermost cycle. By Theorem 1.1, we have $\gamma_I'(P_{t+1}) = \left\lceil\frac{(t+1)}{2}\right\rceil$. Now assign the weight 1 to one of the central edges (edges joining the apex vertex to a rim vertex of the innermost cycle C_1) and weight 0 to all other edges except that of the induced subgraphs P_{t+1} . Then, we get a minimum EIDF defined on E and $\sum f(e) \geq \mathrm{n}\left\lceil\frac{(t+1)}{2}\right\rceil + 1$. Therefore, $\sum f(e) = \mathrm{n}\left\lceil\frac{(t+1)}{2}\right\rceil + 1$.

Theorem.2.6. For the friendship graph F_n, $\gamma'_I(F_n) = n + 1$.

Proof. In the friendship graph F_n, there are n copies of C_n. Let v be the apex vertex. For $i = 1$ *to* n, let v, v_i, v_i' be the three vertices on each copy of C_3. Also, for $i = 1$ *to* n, let $e_i = v_i v_i'$, $x_i = v v_i$ and $y_i = v v_i'$ be the edges of F_n. Then, e_i, $i = 1$ *to* n are the rim edges and x_i and y_i are the central edges.

Define $f: E\,(F_n) \rightarrow \{0,1,2\}$ such that $f(e_i) = 1$, $\forall\, i$, $f(x_i) = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases}$ and

$f(y_i) = 0$, $\forall\, i$. Then, $\sum f(e) \leq n + 1$.

Each of the rim edges of F_n is adjacent to exactly two central edges and all the central edges are adjacent to each other. So, in a minimum EIDF on F_n, each of the rim edges must be given minimum positive weight, which is 1. Now, if one of the central edges is also given the weight 1, then each of the remaining edges will be adjacent to two edges of weight 1 each. Hence, these edges can get the weight 0. So, $\sum f(e) \geq \mathrm{n} + 1$. Hence, $\gamma'_I(F_n) = n + 1$.

***Theorem* 2.7.** For the sunflower graph SF_n, $\gamma'_I(SF_n) = n$.

Proof. Let $e_1, e_2, e_3, \dots, e_n$ be the edges of C_n and $x_1, x_2, x_3, \dots, x_n$ be the central edges, where $x_i = vv_i$, $i = 1\ to\ n$. The edges $y_i = v_i v_i'$, $i = 1\ to\ n$, $z_i = v_i' v_{i+1}$, $i = 1\ to\ n - 1$ and $z_n = v_n' v_1$ together with the edges e_i , $i = 1\ to\ n$ form the petals of SF_n. Define $f: E\,(SF_n) \rightarrow \{0,1,2\}$ such that $f(e_i) = 1$, $\forall\, i$, $f(x_i) = 0$, $\forall\, i$, $f(y_i) = 0$, $\forall\, i$, $f(z_i) = 0$, $\forall\, i$. Then f is an EIDF on SF_n and $\sum f(e) \leq n$.

Each central edge and each edge forming the petals of SF_n are adjacent to exactly two edges of C_n. So, in a minimum EIDF on SF_n, all the edges of C_n can get the weight 1 and all central edges and all edges forming the petals can get the weight 0 . So, $\sum f(e) \geq n$. Therefore, $\gamma'_I(SF_n) = n$.

Conclusion

In this article we have determined the EIDN of some wheel related graphs. It can be observed that for all these graphs, there exists an EIDF independent of the weight 2.

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